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Fermions in an AdS₃ black hole background and the gauge-gravity dualityC. D. Fosco,¹ E. F. Moreno,² and F. A. Schaposnik^{3,*}¹*Centro Atómico Bariloche and Instituto Balseiro, Comisión Nacional de Energía Atómica, R8402AGP Bariloche, Argentina*²*Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA*³*Departamento de Física, Universidad Nacional de La Plata, Instituto de Física La Plata, C.C. 67, 1900 La Plata, Argentina*

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We study a model whose dynamics is determined by a Maxwell Lagrangian coupled to a complex scalar and a Dirac fermion field, in an AdS₃ black hole background. Our study is performed within the context of the Euclidean formalism, in terms of an effective action S^{eff} that results from integrating out the fermion field. In particular, S^{eff} includes an induced parity breaking part which reduces, in the weak coupling limit, to Chern-Simons terms for both the gauge and spin connections, with temperature dependent coefficients. We find numerically the effective action minimum and, applying the AdS/CFT correspondence, we discuss the properties of the dual quantum field theory defined on the boundary. We show that, in contrast with what happens in the absence of fermions, the system does not undergo a phase transition at any finite temperature.

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I. INTRODUCTION

A startling consequence of the parity anomaly [1] for three-dimensional gauge theories coupled to fermion matter fields is that, because of the virtual fluctuations the latter, a Chern-Simons (CS) action $S_{\text{CS}}[A]$ for the gauge field connection A [2] may be induced. This result has been obtained by following different approaches; among them functional methods [3], by evaluating the Atiyah-Patodi-Singer- η invariant associated to the Dirac operator [4], and by using a ζ -function regularization for its determinant [5]. In an entirely similar fashion, a 2 + 1-dimensional Dirac field coupled to a gravitational field background induces a gravitational Chern-Simons term, an object which may be written in terms of the spin connection ω , $S_{\text{CS}}[\omega]$ or, alternatively, as $S_{\text{CS}}[\Gamma]$ for the corresponding Christoffel connection Γ [4,6–8].

The finite temperature extension of these calculations is more involved and leads to a nonlocal effective action [9–11]. Nevertheless, in the weak coupling limit one still does get, for gauge theories, a CS term with a temperature dependent coefficient.

The parity anomaly in odd-dimensional space has been exactly evaluated for situations where both gauge and gravitational background fields are present [4]. However, to our knowledge, the form of the full (anomalous) parity-odd contribution to the fermion determinant has not been calculated, even at zero temperature. As we shall show, this term becomes just the sum of two CS terms, one for the gauge field and the other for the spin connection:

$$\log \det(\not{D}[A, \omega] + M)|_{\text{odd}} = iS_{\text{CS}}^{\text{gauge}}[A] + iS_{\text{CS}}^{\text{grav}}[\omega]. \quad (1)$$

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This result is relevant in connection with applications of the AdS/CFT holographic correspondence to the study of condensed matter systems (for a complete list of references, see, for example, the reviews [12–15]). In fact, much of the work on this subject has been restricted to leading order in semiclassical calculations in the bulk, namely, to evaluating the path integral that defines the partition function Z^{bulk} on its Euclidean saddle point, so that $Z^{\text{bulk}} \approx \exp(-S_E)$ with S_E the on-shell action connected to the free energy of the field theory through the usual relation $F = -T \log Z^{\text{bulk}}$. When fermionic matter is present, its contribution is usually considered within the Thomas-Fermi approximation [15]. More recently, studies of the effects of fermion loops and in general quantum fermions have been presented [16–18].

Using the results on the $d = 3$ fermion determinant described above we shall study, within the gauge/gravity framework, a gravity system consisting of an AdS₃ black hole coupled to a $U(1)$ gauge field, a complex scalar field, and a massive Dirac fermion. After integrating out the fermions we obtain a (bulk) Euclidean effective action S_E^{eff} , which incorporates the full parity-odd fermionic contribution to the bulk partition function. Afterwards, using an appropriate ansatz, we shall numerically solve the equations of motion associated to S_E^{eff} and then use the AdS/CFT correspondence to study the resulting dual theory defined on the boundary. This will lead to the main result of our work, concerning the thermodynamical properties of the corresponding dual 2-dimensional theory.

The paper is organized as follows. We introduce in Sec. II the model in the bulk with dynamics governed by a Maxwell Lagrangian coupled to a charged scalar and a Dirac fermion in an AdS₃ black hole background. Then we show, in Sec. III, that when both gauge and spin connections are taken into account there are no mixing terms and the complete result is just the sum of the two respective CS

terms, as written in Eq. (1). Having obtained the effective action resulting from integration of fermions, we analyze in Sec. IV its classical equations of motion and determine the appropriate boundary conditions to apply the gauge/gravity correspondence. We then proceed to solve (numerically) the coupled system of equations and, from the asymptotic behavior of the solutions, we analyze the critical behavior of the quantum field theory defined in the boundary. Our results are discussed in Sec. V.

II. THE MODEL

A. Notation

The most relevant degrees of freedom we shall consider correspond to a $U(1)$ gauge field connection

$$A \equiv A_\mu dx^\mu \quad (2)$$

defined on a $d = 3$ manifold \mathcal{M} (the bulk) with $\mu = 1, 2, 3$. The 2-form field strength F is then given by

$$F = dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu. \quad (3)$$

The dreibein fields $\{e^a\}$ ($a = 1, 2, 3$) are an orthonormal basis of 1-forms on \mathcal{M} , locally given by

$$e^a = e^a_\mu dx^\mu. \quad (4)$$

The spin connection, ω_{ab} , is an $SO(3)$ -valued 1-form on \mathcal{M} satisfying

$$de_a + \omega_{ab}e_b = 0, \quad \omega_{ab} = -\omega_{ba}. \quad (5)$$

In local coordinates

$$\omega_{ab} = \omega_{ab}^\mu dx_\mu = \omega_{abc}e_c, \quad (6)$$

where

$$\omega_{ab\mu} = e_{a\nu}e^\nu_{b;\mu}, \quad e_b^\nu = g^{\nu\rho}e_{b\rho}. \quad (7)$$

The semicolon refers to differentiation using the Christoffel symbol $\Gamma_{\nu\alpha}^\mu$,

$$v^\mu_{;\alpha} = \partial_\alpha v^\mu + \Gamma_{\nu\alpha}^\mu v^\nu, \quad (8)$$

and, finally, the curvature 2-form R associated to ω is given by

$$R_{ab} = d\omega_{ab} + \omega_{ac}\omega_{cb}. \quad (9)$$

Regarding Dirac field related conventions, we use an irreducible representation of Dirac algebra, based on 2×2 γ matrices which satisfy

$$\begin{aligned} \gamma^\mu(x) &= e_a^\mu(x)\gamma_a, \\ \gamma_a^\dagger &= -\gamma_a, \quad \{\gamma_a, \gamma_b\} = 2\delta_{ab}, \\ \sigma^{ab} &= \frac{1}{4i}[\gamma^a, \gamma^b], \end{aligned}$$

while for the spin connection we have

$$\omega_\mu(x) = \frac{1}{2}\omega_\mu^{ab}\sigma_{ab}. \quad (10)$$

B. The black hole background

We consider an anti-de Sitter (2 + 1)-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole [18,19] in Poincaré coordinates with Lorentzian signature $(-1, 1, 1, 1)$, such that

$$ds^2 = \frac{L^2}{z^2}(-f(z)dt^2 + f^{-1}(z)dz^2 + dx^2), \quad (11)$$

$$f(z) = 1 - \frac{z^2}{z_h^2}. \quad (12)$$

Here, L denotes the AdS radius, and $L^2 = -(1/\Lambda)$, with Λ the negative cosmological constant. The boundary is at $z = 0$ and the horizon at $z = z_h$. The Hawking temperature becomes

$$T = \frac{|f'(z_H)|}{4\pi} = \frac{1}{2\pi z_h}. \quad (13)$$

C. The bulk action

We start from a $U(1)$ gauge field with Maxwell Lagrangian coupled to a charged scalar and a Dirac fermion in the background of the black hole metric (11) and (12). The action takes the form

$$\begin{aligned} S = \int d^3x \sqrt{|g|} &\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}D^\mu\Phi^*D_\mu\Phi + \bar{\psi}\not{D}\psi \right. \\ &\left. + M\bar{\psi}\psi - V(|\Phi|) \right). \end{aligned} \quad (14)$$

Here, e denotes the gauge coupling, Φ is a complex scalar, and ψ a two-component Dirac fermion. The derivatives are defined, respectively, as

$$D_\mu\Phi = (\partial_\mu - ieA_\mu)\Phi, \quad (15)$$

$$\not{D}\psi = \gamma^\mu(\partial_\mu + eA_\mu(x) + \omega_\mu(x))\psi.$$

Since in (2 + 1)-dimensional space e^2 has the dimensions of a mass, in order to work with dimensionless quantities, we redefine $e^2 \rightarrow e^2L$ (a similar rescaling is done with all dimensionful quantities). The form of $V(|\Phi|)$ will be made explicit below.

The partition function for the theory with dynamics governed by action (14), defined in the black hole background (11) and (12), is then written as a functional integral

$$Z^{\text{bulk}} = \int DA_\mu D\Phi D\bar{\psi} D\psi e^{iS}. \quad (16)$$

This partition function plays a central role in the gauge-gravity duality: its weak coupling limit will give information on the strong coupling behavior of the dual quantum field theory defined on the boundary.

Note that, within the gauge/gravity framework, one usually works in the classical approximation, writing

$Z \sim \exp(iS_{cl})$ where “ cl ” indicates that the bulk classical action is taken on shell, with the appropriate boundary conditions for the classical solution. As discussed in the introduction, we shall instead proceed to integrate over fermions thus generating an effective action that already includes fermionic quantum effects in the bulk. That is, we path integrate over the fermions, obtaining

$$Z = \int DA_\mu D\Phi e^{iS^{\text{eff}}}, \quad (17)$$

where

$$S^{\text{eff}} = \int d^3x \sqrt{|g|} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} D_\mu \Phi^* D_\mu \Phi - V(|\Phi|) \right) - i \log \det(\not{D} + M). \quad (18)$$

III. THE FERMION DETERMINANT

A. The fermion determinant at zero temperature

To our knowledge, explicit results for the fermion determinant in (18) have been reported when just one of the two connections is present. We shall discuss here the case in which fermions are coupled to both A and ω (or its associated Γ) showing that the result becomes just the sum of the two separate CS terms.

We are interested in the induced parity-violating term, understanding by such the parity-violating 3-form in the effective action due to the fermion loop contribution.

Although we are interested in the $U(1)$ gauge theory, we shall also consider in this section the case of an $SU(N)$ gauge group since the demonstration does not turn out to be more complicated than in the Abelian case. Concerning the gravitational CS term, as already mentioned, it can be expressed either in terms of the spin or the associated Christoffel connection depending on the anomaly calculation one chooses (no coordinate anomaly or no frame anomaly, respectively). For definiteness we shall choose the Christoffel connection Γ to write the induced parity-odd term.

We start by separating out the parity odd and even contributions to the fermion determinant,

$$\log \det(\not{D} + M) = \log \det(\not{D} + M)|_{\text{even}} + \log \det(\not{D} + M)|_{\text{odd}}. \quad (19)$$

For the case of purely $U(1)$ gauge theory (no spin connection), and keeping the leading terms in the IR limit, the even part takes the form [1–6]

$$\begin{aligned} & \log \det(\not{D} + M)|_{\text{even}}[A] \\ &= \frac{ie^2}{48\pi|M|} \int \sqrt{|g|} d^3x \left(F_{\mu\nu}^2 + \frac{i}{60M^2} (18F^{\nu\lambda} F_{\nu\lambda, \mu}{}^\mu \right. \\ & \quad \left. + 7F^{\nu\lambda, \mu} F_{\nu\lambda, \mu} - 2F_\lambda^{\nu\lambda} F_{\nu\mu, \mu}) \right) \end{aligned} \quad (20)$$

We shall however discard this even contribution, since we will consider from now on the large M limit, the same holds true for the gravitational contribution.

By a perturbative expansion argument, it is evident that S_{odd} will contain—at least—the purely gauge and the purely gravitational CS terms found in [2,4,7,8],

$$\log \det(\not{D} + M)|_{\text{odd}} = iS_{\text{CS}}^{\text{gauge}} + iS_{\text{CS}}^{\text{grav}} + iS^{\text{mixed}}, \quad (21)$$

where

$$S_{\text{CS}}^{\text{gauge}} = \frac{\alpha e^2}{16\pi|M|} \int d^3x \text{tr} \left(A \wedge dA + \frac{2e}{3} A \wedge A \wedge A \right), \quad (22)$$

with $\alpha = 2, 1$ for $U(1), SU(N)$, respectively, and

$$S_{\text{CS}}^{\text{grav}} = \frac{1}{32\pi|M|} \int d^3x \text{tr} \left(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right). \quad (23)$$

The term S^{mixed} represents possible contributions mixing A and Γ . We will show that it vanishes. We use the standard 1-form matrix notation for both terms; for example, $\Gamma = \Gamma_\mu dx^\mu$, where Γ_μ is, for each μ , a 3×3 matrix: $(\Gamma_\mu)_\beta^\alpha = \Gamma_{\mu\beta}^\alpha$. For the gauge field, we have, in the case of a gauge group which can be $SU(N)$ or $U(1)$ (with generators t^A), $A = A_\mu dx^\mu$, where $A_\mu = A_\mu^B t^B$. The t_r traces refer, in each case, to the corresponding product of matrices.

Using again perturbative arguments, let us note that

$$\begin{aligned} S_{\text{odd}} &= i \lim_{M \rightarrow \infty} \text{Im}[\log \det(\not{D} + M)] \\ &= i \lim_{M \rightarrow \infty} \text{Im} \left\{ -\frac{1}{2} \text{Tr}[(\not{D} + M)^{-1}(\not{A} + \omega)]^2 \right. \\ & \quad \left. + \frac{1}{3} \text{Tr}[(\not{D} + M)^{-1}(\not{A} + \omega)]^3 \right\}, \end{aligned} \quad (24)$$

where the T_r is a functional and internal space trace and we have discarded terms that cannot produce 3-forms.

Because of the one to one relation between ω and Γ , the expression above shows explicitly that S_{mixed} can only be constructed in terms of the 1-forms A, Γ , and the exterior derivative d . Since we are taking the large M limit, the leading order terms will have only three of them. There are two possible cases:

- (1) Terms involving one exterior derivative, d . Any 3-form must, in this case, contain one Γ and one A (otherwise it would not be part of the mixed term). Thus, it should be the integral of a term of the form $\text{tr}(\Gamma \wedge dA)$ or $\text{tr}(A \wedge d\Gamma)$. We should distinguish between the non-Abelian and Abelian cases:
 - (a) Non-Abelian gauge field case: it is not possible to build such a term, since the trace of A is zero.
 - (b) Abelian gauge field case: there is no trace over the Lie algebra, and a careful examination shows that, in a torsion-free spacetime, the only possible nonvanishing term is of the form

$$\Gamma_{\mu\alpha}^\alpha \epsilon_{\mu\nu\rho} \partial_\nu A_\rho$$

which is a total derivative, since

$$\Gamma_{\mu\alpha}^\alpha = \frac{\partial \log(\sqrt{g})}{\partial x_\mu}, \quad (25)$$

and we assume $\epsilon_{\mu\nu\rho} \partial_\mu \partial_\nu A_\rho = 0$.

(2) Terms without any exterior derivative. In this case 3-forms contributing to the mixed term have to be of the form $\Gamma \wedge A \wedge A$ or $A \wedge \Gamma \wedge \Gamma$ (or permutations of them). Now

(a) The first kind of term (two A 's) vanishes, because after contracting the \mathcal{G} -algebra indices, we end up with the three objects: $A_\mu^B A_\nu^B$, $\epsilon^{\alpha\beta\gamma}$, and $\Gamma_{\rho\nu}^\lambda$, which should be contracted and balanced. The two A 's form a symmetric tensor in μ, ν . Thus, they cannot be contracted both with ϵ . One of them has to be contracted with ϵ , though (otherwise one should contract two indices in ϵ).

Then we are left with two indices in ϵ to be contracted with $\Gamma_{\rho\nu}^\lambda$. This yields zero, in a torsion-free spacetime.

The argument above, with a minimal modification, holds true in the Abelian case.

(b) The second kind of term (two Γ 's) vanishes since the trace over the Lie algebra is zero (there is only one A). In the Abelian case, it is possible to balance the indices; however, such an object would break Abelian gauge invariance, since it would be necessary have the form: $\epsilon^{\mu\nu\lambda} \Gamma_{\mu\alpha}^\beta \Gamma_{\nu\beta}^\alpha A_\lambda$.

Having enumerated all the possible nontrivial options, we conclude that

$$S_{\text{mixed}} = 0. \quad (26)$$

Remark: strictly speaking, it is sufficient to prove just 1 (absence of terms with a derivative), since it is not possible to construct a general covariant scalar just involving connections.

In summary, we see from Eq. (21) that the fermion determinant is just the product of the gauge and gravitational determinants:

$$\log \det(\not{D} + M)|_{\text{odd}} = i(S_{\text{CS}}^{\text{gauge}} + S_{\text{CS}}^{\text{grav}}). \quad (27)$$

B. The fermion determinant at finite temperature

The parity-violating part of the finite temperature fermion determinant for the case in which there is only a $U(1)$ gauge connection has been computed in Refs. [9–11]. The answer may be put in the form

$$S_{\text{odd}}[A] = i \frac{e}{2\pi} \int d^2x \epsilon_{jk} \partial_j A_k \times \arctan \left(\tanh \left(\frac{\beta M}{2} \right) \tan \left(\frac{e}{2} \int_0^\beta d\tau A_3 \right) \right), \quad (28)$$

$j, k = 1, 2.$

Here, the odd-parity action S_{odd} is defined in Euclidean time $\tau \in (0, \beta = 1/T)$.

An expansion of (28) in powers of e yields the perturbative result

$$S_{\text{odd}}[A] = \frac{i}{2} \tanh \left(\frac{\beta M}{2} \right) \frac{e^2}{4\pi} \int d^3x \epsilon_{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha + O(e^4). \quad (29)$$

Following the approach in [10,11], one has an analogous result for the case of the spin connection. The arguments leading to a zero mixed term still hold so that the result for the complete odd-parity contribution to the finite temperature fermion determinant is

$$S_{\text{odd}}[A, \Gamma; T] = i \tanh \left(\frac{\beta M}{2} \right) (S_{\text{CS}}^{\text{gauge}}[A] + S_{\text{CS}}^{\text{grav}}[\omega] + O(e^4)), \quad (30)$$

where the Chern-Simons actions adopts the form

$$S_{\text{CS}}^{\text{gauge}}[A] = \frac{e^2}{8\pi} \int d^3x \epsilon_{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha$$

$$S_{\text{CS}}^{\text{grav}}[\omega] = \frac{1}{32\pi} \int d^3x \epsilon_{\mu\nu\alpha} \left((\partial_\mu \omega_{\alpha\nu}^b) \omega_{b\alpha}^a + \frac{2}{3} \omega_{\alpha\nu}^b \omega_{b\mu}^c \omega_{a\alpha}^c \right). \quad (31)$$

IV. THE EFFECTIVE ACTION AND THE AdS₃/CFT₂ CORRESPONDENCE

We shall follow the AdS/CFT approach to the study of a strongly coupled system defined on the boundary of an asymptotically AdS space (the bulk). If the bulk metric corresponds to a black hole with Hawking temperature T then, according to the AdS/CFT correspondence, the dual theory in the boundary is a quantum field theory at finite temperature T and its properties should be studied by performing a semiclassical approximation of the (Euclidean) partition function in the bulk, Z^{bulk} .

The general form of the AdS/CFT conjecture reads

$$Z^{\text{bulk}}[\Sigma \rightarrow \Sigma_0] = \left\langle \exp \left(- \int_{\text{boundary}} d^2x \Sigma_0 \mathcal{O} \right) \right\rangle, \quad (32)$$

where

$$Z^{\text{bulk}}[\Sigma \rightarrow \Sigma_0] \equiv \int_{\Sigma=\Sigma_0} D\Sigma \exp(-S[\Phi]). \quad (33)$$

Here, Z_{bulk} represents the generating functional for the theory in the bulk, which in the present case corresponds to asymptotically AdS₃ spacetime. We denote with Σ all the fields in the bulk which tend on the boundary to the value Σ_0 up to an overall power of z . Concerning \mathcal{O} , it denotes the operators for the quantum field theory in the boundary associated to each field in Σ .

Usually, one takes the classical limit in the right-hand side of (33) in which case the path integral defining Z^{bulk} is approximated by $\exp(-S_{\text{on shell}})$. In the present case we have gone one step further and integrated out fermions, what lead us to the effective action (35). This yields an

effective action with a parity violation term which, in the weak coupling regime corresponds to a Chern-Simons term. Now, this term arises as a phase both with Euclidean and Lorentzian metric signature (a well-known fact resulting from the topological character of CS terms). As a result, the Euclidean effective action has complex saddle points which, nevertheless, have proven to be of relevance in problems like, for example, that of confinement in compact QED₃ [20,21]. Here we shall look for complex saddle points of S_E^{eff} leading to a real on-shell

Euclidean action and, consequently, to a sensible free energy $F = -TS_E^{\text{eff}}$.

We shall then start, for the study of the correspondence, from

$$Z^{\text{bulk}} = \int DA_\mu D\Phi \exp(-S_E^{\text{eff}}), \quad (34)$$

where the Euclidean version of the effective action S_E^{eff} is obtained inserting (30) and (31) in the effective action S^{eff} (18) and then Wick rotating,

$$S_E^{\text{eff}} = \int d^3x \sqrt{|g|} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} D_\mu \Phi^* D^\mu \Phi - V(|\Phi|) \right) - \frac{i}{2} \tanh\left(\frac{\beta M}{2}\right) \left(\frac{e^2}{4\pi} \int d^3x \epsilon_{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha + \frac{1}{32\pi} \int d^3x \epsilon_{\mu\nu\alpha} (\partial_\mu \omega_{\alpha\nu}^b) \omega_{b\alpha}^a + \frac{2}{3} \omega_{\alpha\nu}^b \omega_{b\mu}^c \omega_{c\alpha}^a \right). \quad (35)$$

One then has

$$Z^{\text{bulk}} = \exp\left(-\int d^3x \sqrt{|g|} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} D_\mu \Phi^* D^\mu \Phi - V(|\Phi|) \right)\right) \exp\left(-\frac{i}{2} \tanh\left(\frac{\beta M}{2}\right) \left(\frac{e^2}{4\pi} \int d^3x \epsilon_{\mu\nu\alpha} A_\mu \partial_\nu A_\alpha \right)\right). \quad (36)$$

We are not including the contribution of the spin connection Chern-Simons action since we are taking the AdS black hole as a background metric so that its contribution is just an irrelevant factor both for computing the vacuum expectation value (v.e.v.) of operators for the theory on the boundary and also for differences of free energies (see below).

Taking an AdS₃ black hole as background metric and making for the gauge and scalar field the ansatz: $A_0 = \phi(z)$, $A_x = iA(z)$, and $\Phi = \psi(z)$, the equations of motion associated to the effective action (35) become

$$\begin{aligned} \psi'' - \frac{(1+z^2)}{(1-z^2)} \frac{1}{z} \psi' + \left(-\frac{z_h^2 \phi^2}{(1-z^2)} - \frac{m^2}{z^2} + z_h^2 A^2 \right) \frac{\psi}{1-z^2} &= 0 \\ \phi'' + \frac{\phi'}{z} - 2 \frac{\psi^2}{z^2(1-z^2)} \phi - \frac{2\kappa}{z} A' &= 0 \\ A'' + \frac{1-3z^2}{1-z^2} \frac{1}{z} A' - 2 \frac{\psi^2}{z^2(1-z^2)} A - \frac{2\kappa}{z(1-z^2)} \phi' &= 0 \end{aligned} \quad (37)$$

with

$$\kappa = \frac{e^2}{8\pi} \tanh\left(\frac{M}{2T}\right),$$

where we have rescaled $z \rightarrow zz_h$ so the horizon is located at $z = 1$ and we have taken $V(|\Phi|) = m^2 |\Phi|^2$.

Concerning the plain symmetric ansatz chosen for the scalar and gauge fields, it is the simplest one leading to nontrivial results in models like the one we are discussing. Indeed, just by considering that the field dependence on coordinates reduces to the z variable which goes from the horizon to the boundary allows one to establish the holographic correspondence without the complication introduced by considering more involved ansatz (see [12–15] for a more extended discussion on this issue).

Concerning the scalar mass m^2 , stability studies of Maxwell-scalar field models in an AdS black hole background have shown that, for all masses satisfying the Breitenlohner-Freedman (BF) bound $m \geq m_{\text{BF}}$, there is

no drastic change in the behavior of the system [22]. The main feature concerning mass dependence is that as the mass grows, it is harder for the scalar hair to form, as it is to be expected. From now on we take the scalar mass m in the Breitenlohner-Freedman bound, $m^2 = -1$, a choice that will allow comparison with the results in [23] obtained for the case in which fermions—and hence the Chern-Simons term—are absent.

Concerning boundary conditions, for $z \sim 0$ we have

$$\psi = D_1 z \log z + D_2 z \quad (38)$$

$$\phi = B_1 + B_2 z^{2\kappa} + B_3 z^{-2\kappa} \quad (39)$$

$$A = C_1 + B_2 z^{2\kappa} - B_3 z^{-2\kappa} \quad (40)$$

with D_i, B_i, C_1 real constants. As for the solution at the horizon $z = 1$,

$$\phi(1) = 0 \quad (41)$$

$$2\psi'(1) - (A(1)^2 z_h^2 - m^2)\psi(1) = 0 \quad (42)$$

$$A'(1) + \psi(1)^2 A(1) + \kappa \phi'(1) = 0. \quad (43)$$

We assume that the solutions are analytic at the horizon so that ϕ should vanish as $1 - z$.

Let us concentrate in the scalar field $\Phi = \psi(z)$ with boundary conditions (38). Within the AdS/CFT approach there are in principle two possibilities to relate the associated scalar operator O_i ($i = 1, 2$) for the quantum field theory on the boundary with the asymptotic values of ψ :

$$\langle O_i \rangle = D_i, \quad \varepsilon^{ij} D_j = 0. \quad (44)$$

One has to decide which of the two possible boundaries should be set to zero, and we shall choose the most divergent one, which means that we set $D_1 = 0$ so that D_2 will give the expectation value. One can see that the alternative choice (that is, taking the coefficient of the $z \log z$ term as the expectation value) leads to a negative Euclidean action so that it should be discarded as can be seen by calculating the associated free energy (see the discussion in the next section).

A condensate phase would correspond to a nontrivial v.e.v., $\langle O_2 \rangle = D_2 \neq 0$. This can be tested by finding, numerically, solutions to the boundary value problem (44), thus determining the coefficients D_2 as a function of T and the parameters of the theory. Concerning the boundary conditions for the gauge fields, the choices $B_2 = 0$ and $B_3 = 0$ give rise to virtually the same Higgs configuration, so the condensate behavior is insensible to this election (see Fig. 1). For definiteness we choose the condition $B_2 = 0$ so that the boundary conditions (38)–(40) reduce to

$$\psi \sim D_2 z \quad (45)$$

$$\phi \sim B_1 + B_3 z^{-2\kappa} \quad (46)$$

$$A \sim C_1 - B_3 z^{-2\kappa}. \quad (47)$$

Here B_3 should be identified with the chemical potential and B_1 with the charge density.

Regarding the differential equations, we have employed a shooting method to numerically solve them. The equations are integrated back from the horizon $z = 1$ with the

initial conditions (41)–(43). The solutions depend on three parameters, that can be chosen to be $\psi(1)$, $\phi'(1)$, and $A(1)$, respectively. These parameters are then adjusted (using, for example, Newton-Fourier's method) so that the solutions satisfy the conditions (45)–(47) at the boundary $z = 0$.

We numerically computed the D_2 coefficient in the scalar field ψ asymptotic expansion as a function of the temperature T for various values of the CS coupling α . In contrast with what happens when the CS term is absent (i.e., when fermions are not present), the nontrivial solution exists at all temperatures. In the absence of fermions it has been shown in [24,25] that there exists a critical temperature such that as $T \rightarrow T_c$ the condensed phase approaches the normal (disordered phase) phase with $\psi = 0$. The presence of the Chern-Simons term in the effective action (35) prevents this from happening since, due to the Chern-Simons coupling between A_0 and A_x , Eqs. (37) had no nontrivial solution for a normal (disordered) phase which would correspond to an $A_0(z) = \phi(z)$ with asymptotic behavior (46) together with $\psi(z) = 0$ and $A_x(z) = 0$.

In connection with the fact that in the normal phase, the presence of the Chern-Simons term forces the gauge field to vanish, it is interesting to recall the results presented in [26] concerning charged black holes in topologically massive electrodynamics. In particular, in this work it is shown that when a Chern-Simons term is present the theory cannot support a local electromagnetic field so that the solution of the Einstein-Maxwell-Chern-Simons equations corresponds to an uncharged (BTZ) black hole, thus forcing $A(z)$ and $\phi(z)$ to be zero. Interestingly enough this is true for any value of the Chern-Simons coefficient $\alpha = e^2/4\pi M \neq 0$ while for $\alpha = 0$ there is a nontrivial solution of the (2+1) Einstein-Maxwell equations corresponding to a charged black hole and a charged gauge field, $\phi(z) \propto \log z$.

Remarkably, our numerical results go in the same direction as those in [26] in the sense that they show that the absence of a critical temperature seems to be present for any value of the CS parameter α , so the transition to $\alpha \rightarrow 0$ (absence of CS term) is singular. We show in Fig. 2 the behavior of D_2 as a function of T for three values of $\alpha \neq 0$ and we see that even for α as small as 0.001 the behavior is radically different from what one finds for α strictly zero.

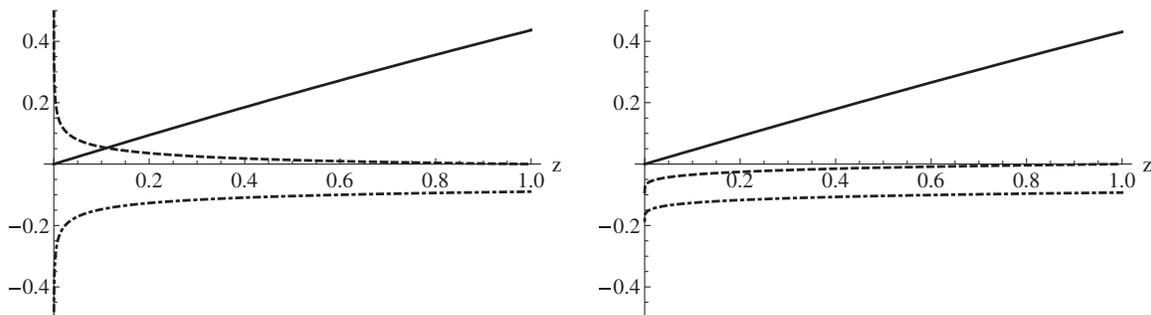


FIG. 1. Field profiles (ψ —solid line, ϕ —dashed line, A —dot-dashed line) for boundary conditions $B_2 = 0$ (left plot) and $B_3 = 0$ (right plot). Notice that, even though the gauge fields significantly differ in both cases, the Higgs profiles are essentially identical.

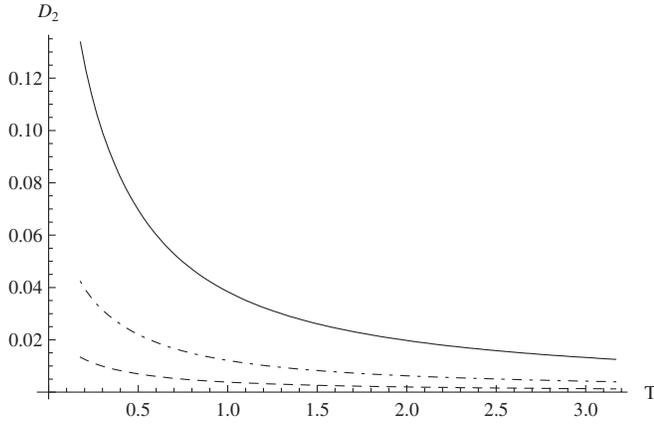


FIG. 2. The D_2 coefficient in the scalar field ψ asymptotic expansion (with $D_1 = 0$) as a function of the temperature T (measure in units of M) for $\alpha = (e^2/4\pi M) = 0.1$ (solid line), $\alpha = 0.01$ (dot-dashed line), and $\alpha = 0.001$ (dashed line). In all cases $B_3 = 0.1$.

As usual for this kind of system, the differential equations with boundary conditions (38)–(43) support several solutions, with an increasing number of nodes. The free energy of each solution increases with the number of nodes so it is the solution with no nodes the thermodynamically favored one. This behavior can be understood by noting that, within the gauge/gravity approach, the equation for the scalar field in the bulk can be viewed as a Schrödinger equation in one spatial dimension where the well-known node theorem holds [27].

V. FREE ENERGY

Within the gauge/gravity duality the free energy F of the theory defined on the boundary is related to the on-shell (Euclidean) action for the solution in the bulk. Using our conventions one has

$$F = TS_E^{\text{eff}}|_{\text{on shell}}. \quad (48)$$

In the present case, after Wick rotating action (35) and using the proposed ansatz, one has

$$S_E^{\text{eff}} = \beta L \int dz \left(-\frac{z}{2} (\phi')^2 + \frac{zf(z)}{2} (A')^2 - \frac{f(z)}{z_h^2 z} (\psi')^2 - \frac{1}{zf(z)} \phi^2 \psi^2 + \frac{1}{z} A^2 \psi^2 - \frac{m^2}{z_h^2 z^3} \psi^2 - \kappa (\phi A' - A \phi') \right), \quad (49)$$

where Euclidean time has been integrated and L is the length of the x interval. Note that we have not included the purely gravitational last term in (35) since being the metric a background, it will cancel out once the difference of free energies between different phases is considered.

After integrating by parts and using the equation of motion, one gets from (49)

$$TS_\epsilon = L \left[-\frac{z}{2} \phi' \phi + \frac{zf}{2} A' A - \frac{f}{z_h^2 z} \psi' \psi \right]_\epsilon^1 + L \int_\epsilon^1 dz \left(\frac{\phi^2}{f} - A^2 \right) \frac{\psi^2}{z}, \quad (50)$$

where we have introduced a cutoff ϵ in order to handle the usual action divergencies. Using the boundary conditions (38)–(40), one finds for the boundary term in the small κ perturbative regime

$$TS_{\epsilon \text{ boundary}} = \frac{L}{z_h^2} (D_1 D_2 + D_2^2) + \text{divergent terms in } \log \epsilon. \quad (51)$$

Once divergent terms are subtracted the $\epsilon \rightarrow 0$ limit can be safely taken and the free energy per unit length takes the form

$$\frac{F}{L} = - \int_0^1 dz \left(\frac{\phi^2}{f} - A^2 \right) \frac{\psi^2}{z} - \frac{1}{z_h^2} (D_1 D_2 + D_2^2). \quad (52)$$

Using this formula we have calculated the free energy for the $D_1 = 0$ solution found in the previous section. We find that the free energy is negative definite in all the temperature range, opposite to what happens in the case $D_2 = 0$. These results are plotted in Fig. 3.

From Eq. (52), one can see that the alternative $D_2 = 0$ choice of boundary conditions causes the boundary contribution to the free energy to vanish. As for the bulk contribution, we have found numerically that, in this case, it turns out to be positive definite in all the temperature range (positive definite Euclidean action). This is due to the A^2 negative term in Eq. (52) and directly related to the necessity, when a Chern-Simons term is present, to include an A_x component to solve Eqs. (37). In view of these results, only the $D_1 = 0$ boundary condition

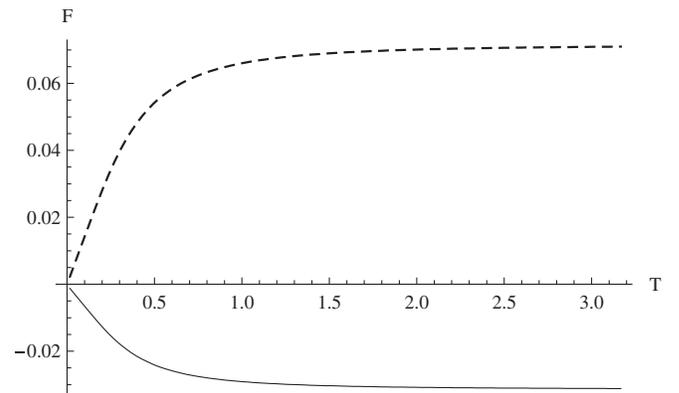


FIG. 3. The free energy F as a function of the temperature T for the solution with $D_1 = 0$ (solid line) and $D_2 = 0$ (dashed line) for the same choice of parameters as in Fig. 1. The $D_2 = 0$ curve has positive free energy so is not thermodynamically favored.

should be considered when applying the AdS/CFT correspondence to the theory defined by effective action (35).

VI. SUMMARY AND DISCUSSION

The existence of a parity anomaly in odd-dimensional fermionic theories coupled to gauge and gravitational fields and the consequent emergence of Chern-Simons terms in the effective action induces a variety of phenomena that are of interest both in high energy physics and in condensed matter physics. Recent results of the application of the AdS/CFT correspondence to study strongly coupled field theories using a weakly coupled gravity dual motivated the investigation presented in this paper where an Abelian gauge theory coupled to a complex scalar and massive fermions in an asymptotically AdS₃ spacetime was discussed. The main idea of our work was to integrate out fermions in the path integral defining the bulk partition function, so that finite temperature quantum fermion effects were incorporated in the effective action. After solving the associated equations of motion, the AdS/CFT correspondence was applied to investigate the strong coupling behavior of the dual quantum theory defined in the boundary.

Although the explicit form of the effective action for pure gauge and pure gravitational theories has been known for a long time [2–8], the case in which both gauge and gravitational fields are present seems not to be studied so the first step in our investigation was to determine whether terms mixing gauge and spin connections were induced by the parity anomaly. Our result, Eq. (27) shows that no mixing term arises and the odd-parity part of the effective action is just the addition of two Chern-Simons terms, one for each connection. This result extends to the finite temperature case leading to an effective action S_E^{eff} which in the weak coupling regime is given by Eq. (35).

The next step in our analysis consisted in finding solutions to the equations of motion for A_0, A_x and ψ in an AdS₃ black hole background. Once the appropriate boundary conditions were imposed, the solution can be used to saturate the bulk partition function and, using the gauge-gravity recipe, to analyze the fate of symmetry breaking in the dual quantum field theory. We worked in the probe limit taking as a gravitation background an AdS₃ black hole and the main result of our numerical analysis can be summarized by Figs. 1 and 2. In the first one, one can see that a dual system does not show a phase transition to a normal (disordered) phase but remains in the ordered (symmetry breaking) state in the whole range of temperatures. One can clearly see that such a result derives from the presence of the Chern-Simons term and will be

present for all nonzero values of the gauge-fermion coupling constant. Concerning the free energy represented in Fig. 2, its behavior shows that the solution remains stable in the whole range of temperatures. Whether these results can be associated to the infinite order phase transition found, for example, for Ising models defined on networks where the critical temperature becomes infinite [28,29] would require worthwhile analysis including the study of backreaction of matter fields on the geometry.

The fact that the presence of the fermions and their contribution to the effective action through the Chern-Simons term drastically changes the behavior of the system at the boundary could be seen as a striking result. One should however recall that the addition of a Chern-Simons term to a 2 + 1 Maxwell-Einstein model radically changes the solution: without the Chern-Simons term the solution of the coupled system is a charged black hole and a non-trivial gauge field potential A_0 . Once the Chern-Simons term is introduced, no matter the strength of its coupling, the gauge field decouples and the black hole becomes uncharged [26]. We have numerically confirmed this result finding that the only solution in the uncondensed phase is the trivial one. There are other domains where the introduction of a Chern-Simons term completely changes the properties of the theory. At the level of classical equations of motion it is well known that existence of Bogomol'nyi-Prasad-Sommerfield (BPS) equations for the Chern-Simons-Higgs model requires a sixth order symmetry breaking potential instead of the fourth-order one required in the Maxwell-Higgs model. This implies the nonexistence of BPS equations for the complete Maxwell-Chern-Simons-Higgs system, no matter the strength of each coupling. The origin of these properties can be traced back to the relation between electric charged and magnetic field imposed by the presence of the Chern-Simons term. Through the connection between BPS equations and supersymmetry algebra, a similar condition holds concerning the possibility of supersymmetric extensions of these models [30–35].

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