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Fermion mass generation in $SO(10)$ with a unified Higgs sector

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An analysis of generating fermion masses via cubic couplings in $SO(10)$ grand unification with a unified Higgs sector is given. The new framework utilizes a single pair of vector-spinor representation $144 + \overline{144}$ to break the gauge symmetry all the way to $SU(3)_C \times U(1)_{\rm em}$. Typically the matter-Higgs couplings in this framework are quartic and lead to naturally suppressed Yukawa couplings for the first two generations. Here we show that much larger third generation couplings naturally arise at the cubic level with additional matter in 10-plet and 45-plet representations of $SO(10)$. Thus the physical third generation is a mixture of 16, 10, and 45-plet representations while the remaining components become superheavy and are removed from the low energy spectrum. In this scenario it is possible to understand the heaviness of the top in a natural way since the analysis generates a hierarchy in the Yukawa couplings so that $h_t/h_b \gg 1$ where $h_t(h_b)$ are the top (bottom) Yukawa couplings. It is then possible to realize values of $\tan\beta$ as low as 2, which also helps to stabilize the proton.

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I. INTRODUCTION

$SO(10)$ is a desirable group as it leads to the unification of the electroweak and strong interactions, contains a full one generation of quarks and leptons in one irreducible representation, and allows for the generation of neutrino masses in a natural way [1]. One drawback of the model is the lack of a unique minimal model due to the many possibilities for the choice of the Higgs sector necessary to break the $SO(10)$ gauge group to the group $SU(3)_C \times U(1)_{\rm em}$. Thus typically one needs 45 or 210 plus either a $16 + \overline{16}$ or a $126 + \overline{126}$ to break $SO(10)$ down to $SU(3)_C \times SU(2)_L \times U(1)_Y$, and further one needs a 10 representation to break the electroweak symmetry. The fact that the electroweak symmetry is broken by the 10-plet representation then naturally leads to a $b - \tau - t$ unification of the Yukawa couplings at the grand unification (GUT) scale and consequently to a large value of $\tan\beta$ (ratio of the two Higgs vacuum expectation values), as large as $\tan\beta \sim 50$ to get compatibility with the experiment [2]. It should be noted that the preceding result can be modified significantly by a small perturbation such as by the inclusion of $SU(2)_L$ doublets H_u and H_d from the $16 + \overline{16}$ spinor Higgs. In this case the physical Higgs doublets responsible for giving masses to the up and down quarks and leptons are linear combinations of the Higgs doublets from the 10 and $16 + \overline{16}$ and $\tan\beta$ depends on an additional parameter and is thus arbitrary.

Recently a new framework for symmetry breaking was introduced where a single vector spinor and its conjugate, i.e., $144 + \overline{144}$ can break $SO(10)$ down to $SU(3)_C \times U(1)_{\rm em}$ [3]. In this model it is possible to get an understanding of the heaviness of the top with low $\tan\beta$ values, as we shall show here. The couplings of the 16-plet of quarks and leptons with the Higgs fields are quartic in nature, i.e., one has couplings of the type 16.16.144.144 etc. which are suppressed by a heavy scale, presumably the string scale $M_{\rm st}$. After spontaneous breaking of the GUT symmetry where $\langle 144\rangle \sim M_G$, one will have the Yukawa couplings of size $M_G/M_{\rm st}$ and thus naturally small which may explain the smallness of the first and the second generation Yukawa couplings. However, the third generation Yukawa couplings are not that small, and so one needs an alternative solution to the couplings of the third generation fermions.

In this paper we propose a possible solution to the puzzle within the unified symmetry breaking framework. We postulate the existence of 10-plet and 45-plet matter representations which couple with the 16-plet of matter at the cubic level via 144 and $\overline{144}$ and the cubic couplings can be naturally $O(1)$ and not suppressed. Further after spontaneous breaking one expects that the matter fields will be mixtures of the fields from the 16-plet, the 10-plet, and the 45-plet. The reason for including both a 10-plet and a 45-plet of matter is this. The 10-plet couplings give mass only to the down quark and the lepton, and not to the up quark. For generating mass for the up quark one needs couplings of the 45-plet of matter. While the 45-plet of matter couplings also give a contribution to the down quark mass, they alone do not lead to a satisfactory solution to the $b - \tau - t$ unification. In the following we will assume that only the third generation matter has cubic couplings and at the end we will discuss inclusion of the other generations.

II. BREAKING OF $SO(10)$

For one step breaking of $SO(10)$ GUT group down to the standard model group and for doublet-triplet splitting, we consider the following quartic couplings

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$$\mathsf{W} = \mathsf{W}^{(\overline{144}_H\times 144_H)} + \mathsf{W}^{(\overline{144}_H\times 144_H)_{45_1}(\overline{144}_H\times 144_H)_{45_1}} + \mathsf{W}^{(\overline{144}_H\times 144_H)_{45_2}(\overline{144}_H\times 144_H)_{45_2}} + \mathsf{W}^{(\overline{144}_H\times 144_H)_{210}(\overline{144}_H\times 144_H)_{210}}. \quad (1)$$

Computation of these couplings require special techniques [4] using the oscillator expansion [5]. Details of the technique including notation and explicit forms can be found in Appendices A and B and in Refs. [3,6]. We begin by collecting terms in the quartic interactions of Eq. (1) that contribute to symmetry breaking. We have

$$\mathsf{W}_{\rm SB} = M\hat{\mathbf{Q}}^i_j\hat{\mathbf{P}}^j_i + \frac{1}{M'}\left[-\lambda_{45_1} + \frac{1}{6}\lambda_{210}\right]\hat{\mathbf{Q}}^i_j\hat{\mathbf{P}}^j_i\hat{\mathbf{Q}}^k_l\hat{\mathbf{P}}^l_k + \frac{1}{M'}\left[-4\lambda_{45_1} - \frac{1}{2}\lambda_{45_2} - \lambda_{210}\right]\hat{\mathbf{Q}}^i_k\hat{\mathbf{P}}^k_j\hat{\mathbf{Q}}^j_l\hat{\mathbf{P}}^l_i. \quad (2)$$

Here fields with a ˆ stand for chiral superfields, while the ones without a "hat" represents the charge scalar component of the corresponding superfield. The indices i, j, k here and later are the $SU(5)$ indices which take on the values 1–5. For symmetry breaking we invoke the following vacuum expectation values (VEV's)

$$\langle\mathbf{Q}^i_j\rangle = q\,\mathrm{diag}(2, 2, 2, -3, -3), \qquad \langle\mathbf{P}^i_j\rangle = p\,\mathrm{diag}(2, 2, 2, -3, -3) \quad (3)$$

and together with the minimization of $\mathsf{W}_{\rm SB}$, we find

$$\frac{MM'}{qp} = 116\lambda_{45_1} + 7\lambda_{45_2} + 4\lambda_{210}. \quad (4)$$

The D-flatness condition $\langle 144\rangle = \langle\overline{144}\rangle$ gives $q = p$. With the above VEV, spontaneous breaking occurs so that $SO(10) \to SU(3)_C \times SU(2)_L \times U(1)_Y$.

To achieve the further breaking of $SU(2)_L \times U(1)_Y$ one needs light Higgs doublets. It is achieved in this scenario by a fine tuning which is justifiable within the framework of string landscapes. The terms that enter in the analysis of doublet-triplet splitting are

$$\begin{aligned}\mathsf{W}_{\rm DT} = {} & \left\{\frac{4}{5}M + \frac{1}{M'}\left(\frac{24}{5}\lambda_{45_1} - \frac{4}{15}\lambda_{210}\right)\langle\hat{\mathbf{Q}}^m_n\rangle\langle\hat{\mathbf{P}}^n_m\rangle\right\}\hat{\mathbf{Q}}_i\hat{\mathbf{P}}^i + \left\{\frac{1}{M'}\left[-\frac{4}{5}\lambda_{45_2} - \frac{32}{15}\lambda_{210}\right]\langle\hat{\mathbf{Q}}^m_j\rangle\langle\hat{\mathbf{P}}^i_m\rangle\right\}\hat{\mathbf{Q}}_i\hat{\mathbf{P}}^j \\ & + \left\{M + \frac{1}{M'}\left(6\lambda_{45_1} - \frac{1}{3}\lambda_{210}\right)\langle\hat{\mathbf{Q}}^m_n\rangle\langle\hat{\mathbf{P}}^n_m\rangle\right\}\hat{\mathbf{Q}}^i\hat{\mathbf{P}}_i + \left\{\frac{1}{M'}(\lambda_{45_2})\langle\hat{\mathbf{Q}}^m_i\rangle\langle\hat{\mathbf{P}}^j_m\rangle\right\}\hat{\mathbf{Q}}^i\hat{\mathbf{P}}_j \\ & + \left\{-\frac{1}{2}M + \frac{1}{M'}\left(\lambda_{45_1} - \frac{1}{6}\lambda_{210}\right)\langle\hat{\mathbf{Q}}^m_n\rangle\langle\hat{\mathbf{P}}^n_m\rangle\right\}\left[\hat{\mathbf{Q}}^k_{ij} + \frac{1}{2\sqrt{5}}(\delta^k_i\hat{\mathbf{Q}}_j - \delta^k_j\hat{\mathbf{Q}}_i)\right]\left[\hat{\mathbf{P}}^{ij}_k + \frac{1}{2\sqrt{5}}(\delta^i_k\hat{\mathbf{P}}^j - \delta^j_k\hat{\mathbf{P}}^i)\right] \\ & + \left\{\frac{1}{M'}\left(-\frac{1}{2}\lambda_{45_2}\right)\langle\hat{\mathbf{Q}}^m_i\rangle\langle\hat{\mathbf{P}}^j_m\rangle\right\}\left[\hat{\mathbf{Q}}^i_{kl} + \frac{1}{2\sqrt{5}}(\delta^i_k\hat{\mathbf{Q}}_l - \delta^i_l\hat{\mathbf{Q}}_k)\right]\left[\hat{\mathbf{P}}^{kl}_j + \frac{1}{2\sqrt{5}}(\delta^k_j\hat{\mathbf{P}}^l - \delta^l_j\hat{\mathbf{P}}^k)\right] \\ & + \left\{\frac{1}{M'}\left[\left(8\lambda_{45_1} - \frac{2}{3}\lambda_{210}\right)\langle\hat{\mathbf{Q}}^i_m\rangle\langle\hat{\mathbf{P}}^m_j\rangle\right]\right\}\left[\hat{\mathbf{Q}}^k_{il} + \frac{1}{2\sqrt{5}}(\delta^k_i\hat{\mathbf{Q}}_l - \delta^k_l\hat{\mathbf{Q}}_i)\right]\left[\hat{\mathbf{P}}^{lj}_k + \frac{1}{2\sqrt{5}}(\delta^l_k\hat{\mathbf{P}}^j - \delta^j_k\hat{\mathbf{P}}^l)\right]. \end{aligned} \quad (5)$$

We note that in addition to the pairs of doublets: $(\mathbf{Q}_a, \mathbf{P}^a)$, $(\mathbf{Q}^a, \mathbf{P}_a)$ $(a, b, c = 4, 5)$ there are also pairs of $SU(2)$ doublets that reside in $\mathbf{Q}^k_{ij}$ and $\mathbf{P}^{ij}_k$. We denote them by $(\tilde{\mathbf{Q}}_a, \tilde{\mathbf{P}}^a)$ (see Appendix B). The mass matrix of the Higgs doublets is given by

$$\begin{array}{c} \\ \mathbf{P}^a \\ \tilde{\mathbf{P}}^a \\ \mathbf{Q}^a \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{Q}_a & \tilde{\mathbf{Q}}_a & \mathbf{P}_a \end{array} \\ \begin{bmatrix} \frac{3}{5}M + \frac{qp}{M'}\left(\frac{666}{5}\lambda_{45_1} - \frac{33}{4}\lambda_{45_2} - \frac{273}{10}\lambda_{210}\right) & \sqrt{\frac{3}{5}}\frac{qp}{M'}\left(10\lambda_{45_1} + \frac{5}{4}\lambda_{45_2} - \frac{5}{6}\lambda_{210}\right) & 0 \\ \sqrt{\frac{3}{5}}\frac{qp}{M'}(10\lambda_{45_1} + \frac{5}{4}\lambda_{45_2} - \frac{5}{6}\lambda_{210}) & -\frac{1}{2}M + \frac{qp}{M'}\left(-74\lambda_{45_1} - \frac{31}{4}\lambda_{45_2} + \frac{7}{6}\lambda_{210}\right) & 0 \\ 0 & 0 & M + \frac{qp}{M'}(180\lambda_{45_1} + 9\lambda_{45_2} - 10\lambda_{210}) \end{bmatrix} \end{array}. \quad (6)$$

We diagonalize in the Higgs doublet subsectors $(\tilde{\mathbf{Q}}_a, \tilde{\mathbf{P}}^a)$ and $(\mathbf{Q}_a, \mathbf{P}^a)$. After, diagonalization we have the following pairs of doublets:

$$\mathsf{D}_1\colon (\mathbf{Q}^a, \mathbf{P}_a), \qquad \mathsf{D}_2\colon (\mathbf{Q}'_a, \mathbf{P}'^a), \qquad \mathsf{D}_3\colon (\tilde{\mathbf{Q}}'_a, \tilde{\mathbf{P}}'^a). \quad (7)$$

The rotated fields above are expressed in terms of the primitive ones through the following transformation matrices

$$\begin{bmatrix} (\mathbf{Q}'_a, \mathbf{P}'^a) \\ (\tilde{\mathbf{Q}}'_a, \tilde{\mathbf{P}}'^a) \end{bmatrix} = \begin{bmatrix} \cos\vartheta_{\mathsf{D}} & \sin\vartheta_{\mathsf{D}} \\ -\sin\vartheta_{\mathsf{D}} & \cos\vartheta_{\mathsf{D}} \end{bmatrix}\begin{bmatrix} (\mathbf{Q}_a, \mathbf{P}^a) \\ (\tilde{\mathbf{Q}}_a, \tilde{\mathbf{P}}^a) \end{bmatrix}, \quad (8)$$

where

$$\tan\vartheta_{\mathsf{D}} = \frac{1}{\mathsf{d}_3}(\mathsf{d}_2 + \sqrt{\mathsf{d}_2{}^2 + \mathsf{d}_3{}^2}) \quad (9)$$

and that

$$\mathsf{d}_1 = -\frac{2}{5}M + \frac{qp}{M'}\left(\frac{296}{5}\lambda_{45_1} - 16\lambda_{45_2} - \frac{392}{15}\lambda_{210}\right),$$
$$\mathsf{d}_2 = -\frac{8}{5}M + \frac{qp}{M'}\left(-\frac{1036}{5}\lambda_{45_1} + \frac{1}{2}\lambda_{45_2} + \frac{427}{15}\lambda_{210}\right),$$
$$\mathsf{d}_3 = \sqrt{\frac{12}{5}}\frac{qp}{M'}\left(10\lambda_{45_1} + \frac{5}{4}\lambda_{45_2} - \frac{5}{6}\lambda_{210}\right). \tag{10}$$

The mass eigenvalues are found to be

$$M_{\mathsf{D}_2,\mathsf{D}_3} = \tfrac{1}{2}(\mathsf{d}_1 \pm \sqrt{\mathsf{d}_2{}^2 + \mathsf{d}_3{}^2}) \tag{11}$$

and

$$M_{\mathsf{D}_1} = M + \frac{qp}{M'}(180\lambda_{45_1} + 9\lambda_{45_2} - 10\lambda_{210}). \tag{12}$$

We note in passing that the condition for achieving a light Higgs doublet while keeping the triplet heavy depends sensitively on the number of quartic couplings. In general the number of allowed terms is quite large and thus inclusion of additional couplings will modify the results by order one. However, the qualitative features of the results discussed in the paper would not be affected.

III. ANALYSIS OF $16_M \cdot \overline{144}_H \cdot 45_M$ AND $16_M \cdot 144_H \cdot 10_M$ COUPLINGS

The superpotential involving the 45_M of matter fields which is quadratic and cubic in the fields is

$$\mathsf{W}^{16\times\overline{144}\times 45} = \frac{1}{2!}h^{(45)}_{\acute{a}\acute{b}}\langle\hat{\Psi}^*_{(+)\acute{a}}|B\Gamma_{[\mu}|\hat{\Upsilon}_{(+)\nu]}\rangle\hat{\mathbf{F}}^{(45)}_{\acute{b}\mu\nu},$$
$$\mathsf{W}^{(45)}_{\text{mass}} = m^{(45)}_F\hat{\mathbf{F}}^{(45)}_{\mu\nu}\hat{\mathbf{F}}^{(45)}_{\mu\nu} \tag{13}$$

and similarly the superpotential involving the 10_M of matter fields which is quadratic and cubic in the fields is

$$\mathsf{W}^{16\times 144\times 10} = h^{(10)}_{\acute{a}\acute{b}}\langle\hat{\Psi}^*_{(+)\acute{a}}|B|\hat{\Upsilon}_{(-)\mu}\rangle\hat{\mathbf{F}}^{(10)}_{\acute{b}\mu},$$
$$\mathsf{W}^{(10)}_{\text{mass}} = m^{(10)}_F\hat{\mathbf{F}}^{(10)}_{\mu}\hat{\mathbf{F}}^{(10)}_{\mu}. \tag{14}$$

Here $\acute{a}$, $\acute{b}$ are generation indices and take on the values 1, 2, 3 and $\mu, \nu, \rho, \sigma = 1, 2, .., 10$ represent the $SO(10)$ indices. Using the technique discussed in Appendices A and B and in Refs. [3,4,6] one may expand Eqs. (13) and (14) to obtain

$$\begin{aligned}\mathsf{W}^{16\times\overline{144}\times 45} = 2h^{(45)}_{\acute{a}\acute{b}}\Big\{&\langle\hat{\Psi}^*_{(+)\acute{a}}|Bb_i|\hat{\Upsilon}_{(+)c_j}\rangle\hat{\mathbf{F}}^{(45)}_{\acute{b}ij}\\ &+\langle\hat{\Psi}^*_{(+)\acute{a}}|Bb^\dagger_i|\hat{\Upsilon}_{(+)\bar{c}_j}\rangle\hat{\mathbf{F}}^{(45)ij}_{\acute{b}}\\ &+\langle\hat{\Psi}^*_{(+)\acute{a}}|Bb^\dagger_i|\hat{\Upsilon}_{(+)c_j}\rangle\hat{\mathbf{F}}^{(45)i}_{\acute{b}j}\\ &-\langle\hat{\Psi}^*_{(+)\acute{a}}|Bb_i|\hat{\Upsilon}_{(+)\bar{c}_j}\rangle\hat{\mathbf{F}}^{(45)j}_{\acute{b}i}\\ &+\tfrac{1}{5}[\langle\hat{\Psi}^*_{(+)\acute{a}}|Bb^\dagger_n|\hat{\Upsilon}_{(+)c_n}\rangle\\ &-\langle\hat{\Psi}^*_{(+)\acute{a}}|Bb_n|\hat{\Upsilon}_{(+)\bar{c}_n}\rangle]\hat{\mathbf{F}}^{(45)}_{\acute{b}}\Big\},\end{aligned} \tag{15}$$

$$\mathsf{W}^{(45)}_{\text{mass}} = m^{(45)}_F[\hat{\mathbf{F}}^{(45)ij}\hat{\mathbf{F}}^{(45)}_{ij} - \hat{\mathbf{F}}^{(45)i}_{j}\hat{\mathbf{F}}^{(45)j}_{i} - \hat{\mathbf{F}}^{(45)}\hat{\mathbf{F}}^{(45)}], \tag{16}$$

and

$$\begin{aligned}\mathsf{W}^{16\times 144\times 10} = h^{(10)}_{\acute{a}\acute{b}}\{&\langle\hat{\Psi}^*_{(+)\acute{a}}|B|\hat{\Upsilon}_{(-)c_i}\rangle\hat{\mathbf{F}}^{(10)}_{\acute{b}i}\\ &+\langle\hat{\Psi}^*_{(+)\acute{a}}|B|\hat{\Upsilon}_{(-)\bar{c}_i}\rangle\hat{\mathbf{F}}^{(10)i}_{\acute{b}}\},\end{aligned} \tag{17}$$

$$\mathsf{W}^{(10)}_{\text{mass}} = m^{(10)}_F\hat{\mathbf{F}}^{(10)}_{i}\hat{\mathbf{F}}^{(10)i}. \tag{18}$$

Defining

$$f^{(.)}_{\acute{a}\acute{b}} \equiv ih^{(.)}_{\acute{a}\acute{b}}; \qquad f^{(.)} \text{ real} \tag{19}$$

a detailed analysis of the couplings in the superpotential that can contribute to the top, bottom, and τ Yukawa couplings is given by

$$\mathsf{W}^{(45)} = \sum_{i=1}^{5}\mathsf{W}^{(45)}_i, \qquad \mathsf{W}^{(10)} = \sum_{i=1}^{4}\mathsf{W}^{(45)}_i, \tag{20}$$

where

$$\begin{aligned}\mathsf{W}^{(45)}_1 &= \frac{1}{\sqrt{10}}f^{(45)}_{\acute{a}\acute{b}}\epsilon_{ijklm}\hat{\mathbf{M}}^{ij}_{\acute{a}}\hat{\mathbf{P}}^k\hat{\mathbf{F}}^{(45)lm}_{\acute{b}},\\ \mathsf{W}^{(45)}_2 &= 2\sqrt{2}f^{(45)}_{\acute{a}\acute{b}}\mathbf{M}^{ij}_{\acute{a}}\hat{\mathbf{P}}^k_i\hat{\mathbf{F}}^{(45)}_{\acute{b}jk},\\ \mathsf{W}^{(45)}_3 &= \frac{1}{\sqrt{2}}f^{(45)}_{\acute{a}\acute{b}}\epsilon_{ijklm}\hat{\mathbf{M}}^{ij}_{\acute{a}}\hat{\mathbf{P}}^{kl}_n\hat{\mathbf{F}}^{(45)mn}_{\acute{b}},\\ \mathsf{W}^{(45)}_4 &= m^{(45)}_F\hat{\mathbf{F}}^{(45)ij}\hat{\mathbf{F}}^{(45)}_{ij},\\ \mathsf{W}^{(45)}_5 &= -2\sqrt{2}f^{(45)}_{\acute{a}\acute{b}}\hat{\mathbf{M}}_{\acute{a}i}\hat{\mathbf{P}}_j\hat{\mathbf{F}}^{(45)ij}_{\acute{b}},\end{aligned} \tag{21}$$

and where

$$\begin{aligned}\mathsf{W}^{(10)}_1 &= -\frac{1}{\sqrt{2}}f^{(10)}_{\acute{a}\acute{b}}\hat{\mathbf{M}}_{\acute{a}i}\hat{\mathbf{Q}}^i_j\hat{\mathbf{F}}^{(10)j}_{\acute{b}},\\ \mathsf{W}^{(10)}_2 &= -\frac{1}{2\sqrt{10}}f^{(10)}_{\acute{a}\acute{b}}\hat{\mathbf{M}}^{ij}_{\acute{a}}\hat{\mathbf{Q}}_j\hat{\mathbf{F}}^{(10)}_{\acute{b}i},\\ \mathsf{W}^{(10)}_3 &= \frac{1}{2\sqrt{2}}f^{(10)}_{\acute{a}\acute{b}}\hat{\mathbf{M}}^{ij}_{\acute{a}}\hat{\mathbf{Q}}^k_{ij}\hat{\mathbf{F}}^{(10)}_{\acute{b}k},\\ \mathsf{W}^{(10)}_4 &= m^{(10)}_F\hat{\mathbf{F}}^{(10)i}\hat{\mathbf{F}}^{(10)}_i.\end{aligned} \tag{22}$$

Yukawa interactions in the Lagrangian are constructed in the usual way:

$$\mathsf{L}_{\text{Yukawa}} = -\frac{1}{2}\frac{\partial^2\mathsf{W}(A_1, A_2, \ldots)}{\partial A_r\partial A_s}\psi^{\mathbf{T}}_r C\psi_s + \text{H.c.},$$

where A_r's and ψ_r's represent the charged scalar and four-component Dirac Fields, respectively, while C is the Dirac charge conjugation matrix. From now on Dirac fields with a bar are defined so that $\bar{\psi} = \psi^\dagger\gamma^0$ while its left and right components are given by $\psi_{\substack{R\\L}} = \frac{1}{2}(1 \pm \gamma_5)\psi$. The quark and lepton masses will arise after the scalar fields develop vacuum expectation values. The relevant vacuum expecta-

tion values that will contribute to the quark and lepton masses are

$$(\langle \mathbf{Q}_5\rangle, \langle \mathbf{P}^5\rangle); \qquad (\langle \tilde{\mathbf{Q}}_5\rangle, \langle \tilde{\mathbf{P}}^5\rangle),$$
$$\begin{pmatrix} \langle \mathbf{Q}^i_j\rangle \\ \langle \mathbf{P}^i_j\rangle \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix} \mathrm{diag}(2, 2, 2, -3, -3). \tag{23}$$

$(\mathbf{Q}^a, \mathbf{P}_a)$ is a pair of spectator doublets which does not mix with others and cannot be made massless and hence does not enter in the analysis. However, for completeness we include its contribution and later on set: $\langle \mathbf{Q}^5\rangle = 0 = \langle \mathbf{P}_5\rangle$.

IV. BOTTOM QUARK AND TAU LEPTON MASSES

We discuss first the mass growth for the bottom quark and for the tau lepton. The interactions that contribute to the bottom quark mass can be gotten from Eqs. (21) and (22). One has

$$\begin{aligned}
\mathsf{L}^{(45)}_{2,b} &= -2\sqrt{2} f^{(45)}_{33} p[{}^{(\overline{10}_{45})}\overline{b}_{R\alpha}{}^{(10_{16})}b^{\alpha}_L] + \text{H.c.},\\
\mathsf{L}^{(45)}_{4,b} &= -2m^{(45)}_F[{}^{(\overline{10}_{45})}\overline{b}_{R\alpha}{}^{(10_{45})}b^{\alpha}_L] + \text{H.c.},\\
\mathsf{L}^{(45)}_{5,b} &= -2\sqrt{2} f^{(45)}_{33}\langle \mathbf{P}_5\rangle[{}^{(\bar{5}_{16})}\overline{b}_{R\alpha}{}^{(10_{45})}b^{\alpha}_L] + \text{H.c.},\\
\mathsf{L}^{(10)}_{1,b} &= \sqrt{2} f^{(10)}_{33} q[{}^{(\bar{5}_{16})}\overline{b}_{R\alpha}{}^{(5_{10})}b^{\alpha}_L] + \text{H.c.},\\
\mathsf{L}^{(10)}_{2,b} &= \frac{1}{2\sqrt{10}} f^{(10)}_{33}\langle \mathbf{Q}_5\rangle[{}^{(\bar{5}_{10})}\overline{b}_{R\alpha}{}^{(10_{16})}b^{\alpha}_L] + \text{H.c.},\\
\mathsf{L}^{(10)}_{3,b} &= \frac{1}{2\sqrt{6}} f^{(10)}_{33}\langle \tilde{\mathbf{Q}}_5\rangle[{}^{(\bar{5}_{10})}\overline{b}_{R\alpha}{}^{(10_{16})}b^{\alpha}_L] + \text{H.c.},\\
\mathsf{L}^{(10)}_{4,b} &= -2m^{(10)}_F[{}^{(\bar{5}_{10})}\overline{b}_{R\alpha}{}^{(5_{10})}b^{\alpha}_L] + \text{H.c.}
\end{aligned} \tag{24}$$

In the above our notation is as follows: The fields are labeled with a left superscript such as 5_{10}, 10_{16}, etc. which identifies the $SU(5)$ multiplet within the $SO(10)$ multiplet from where these fields arise. Thus the left superscript 5_{10} means that the field we are considering arises from the 5-plet of $SU(5)$ within a 10-plet of $SO(10)$. A similar notation is used in the rest of the analysis in this paper.

Returning to Eq. (24) the mass matrix for the b-quark is given by

$$M_b = \begin{array}{c} {}^{(\bar{5}_{16})}\bar{b}_{R\alpha} \\ {}^{(\bar{5}_{10})}\bar{b}_{R\alpha} \\ {}^{(\overline{10}_{45})}\bar{b}_{R\alpha} \end{array} \begin{pmatrix} {}^{(10_{16})}b^{\alpha}_L & {}^{(10_{45})}b^{\alpha}_L & {}^{(5_{10})}b^{\alpha}_L \\ 0 & m_b{}'' & m^{(10)}_D \\ m_b{}' & 0 & -2m^{(10)}_F \\ m^{(45)}_D & -2m^{(45)}_F & 0 \end{pmatrix}, \tag{25}$$

where

$$m_b{}' = \frac{1}{2} f^{(10)}_{33}\left[\frac{\langle \mathbf{Q}_5\rangle}{\sqrt{10}} + \frac{\langle \tilde{\mathbf{Q}}_5\rangle}{\sqrt{6}}\right]; \qquad m_b{}'' = -2\sqrt{2} f^{(45)}_{33}\langle \mathbf{P}_5\rangle,$$
$$m^{(45)}_D = -2\sqrt{2} f^{(45)}_{33} p; \qquad m^{(10)}_D = \sqrt{2} f^{(10)}_{33} q. \tag{26}$$

Note that M_b is asymmetric, hence it is diagonalized by two 3×3 orthogonal matrices, U_b and V_b satisfying

$$U_b \mathsf{M}_b V^{\mathbf{T}}_b = \mathrm{diag}(\lambda_{b_1}, \lambda_{b_2}, \lambda_{b_3}).$$

The matrices U_b and V_b are such that their columns are eigenvectors of matrices $\mathsf{M}_b\mathsf{M}^{\mathbf{T}}_b$ and $\mathsf{M}^{\mathbf{T}}_b\mathsf{M}_b$, respectively:

$$U_b[\mathsf{M}_b\mathsf{M}^{\mathbf{T}}_b]U^{\mathbf{T}}_b = \mathrm{diag}(\lambda^2_{b_1}, \lambda^2_{b_2}, \lambda^2_{b_3}) = V_b[\mathsf{M}^{\mathbf{T}}_b\mathsf{M}_b]V^{\mathbf{T}}_b.$$

Further, the rotated (prime) fields can be expressed in terms of the original fields through

$$\begin{aligned}
\begin{pmatrix} {}^{(\bar{5}_{16})}\bar{b}'_{R\alpha} \\ {}^{(\bar{5}_{10})}\bar{b}'_{R\alpha} \\ {}^{(\overline{10}_{45})}\bar{b}'_{R\alpha} \end{pmatrix} &= U_b \begin{pmatrix} {}^{(\bar{5}_{16})}\bar{b}_{R\alpha} \\ {}^{(\bar{5}_{10})}\bar{b}_{R\alpha} \\ {}^{(\overline{10}_{45})}\bar{b}_{R\alpha} \end{pmatrix};\\
\begin{pmatrix} {}^{(10_{16})}b'^{\alpha}_L \\ {}^{(10_{45})}b'^{\alpha}_L \\ {}^{(5_{10})}b'^{\alpha}_L \end{pmatrix} &= V_b \begin{pmatrix} {}^{(10_{16})}b^{\alpha}_L \\ {}^{(10_{45})}b^{\alpha}_L \\ {}^{(5_{10})}b^{\alpha}_L \end{pmatrix}.
\end{aligned} \tag{27}$$

The mass terms in the Lagrangian are then given by

$$\begin{aligned}
&\lambda_{b_1}{}^{(\bar{5}_{16})}\bar{b}'_{R\alpha}{}^{(10_{16})}b'^{\alpha}_L + \lambda_{b_2}{}^{(\bar{5}_{10})}\bar{b}'_{R\alpha}{}^{(10_{45})}b'^{\alpha}_L\\
&\quad + \lambda_{b_3}{}^{(\overline{10}_{45})}\bar{b}'_{R\alpha}{}^{(5_{10})}b'^{\alpha}_L + \text{H.c.}
\end{aligned} \tag{28}$$

Note that $\det(\mathsf{M}_b\mathsf{M}^{\mathbf{T}}_b - \lambda^2_b\mathbf{1})$ gives a cubic equation in λ^2_b and is not very illuminating. Instead in the limit $m_b{}'$ small, the light eigenvalue squared $\lambda^2_{b_1}$ can be calculated from

$$\lambda^2_{b_1} \approx \frac{\det(\mathsf{M}_b\mathsf{M}^{\mathbf{T}}_b)}{\Lambda^2_{b_2}\Lambda^2_{b_3}},$$

where $\Lambda^2_{b_2}$ and $\Lambda^2_{b_3}$ are the exact square of the eigenvalues of the matrix $\mathsf{M}_b\mathsf{M}^{\mathbf{T}}_b|_{m_b{}'=0}$. The results are

$$\begin{aligned}
m^2_b \equiv \lambda^2_{b_1} &\approx \frac{1}{4}\frac{y^2_{45}}{[1+y^2_{45}][1+(2y_{10})^2]}\left(f^{(10)}_{33}\left[\frac{\langle \mathbf{Q}_5\rangle}{\sqrt{10}} + \frac{\langle \tilde{\mathbf{Q}}_5\rangle}{\sqrt{6}}\right]\right)^2,\\
\lambda^2_{b_2} &\approx \Lambda^2_{b_2} = 8[1+y^2_{45}](f^{(45)}_{33}p)^2,\\
\lambda^2_{b_3} &\approx \Lambda^2_{b_3} = 2[1+(2y_{10})^2](f^{(10)}_{33}q)^2,\\
y_{45} &\equiv \frac{m^{(45)}_F}{\sqrt{2}f^{(45)}_{33}p}; \quad y_{10} \equiv \frac{m^{(10)}_F}{\sqrt{2}f^{(10)}_{33}q}.
\end{aligned} \tag{29}$$

The transformation matrices take the form

$$\begin{aligned}
U_b &= \begin{pmatrix} \cos\theta_{ub} & -\sin\theta_{ub} & 0 \\ \sin\theta_{ub} & \cos\theta_{ub} & 0 \\ 0 & 0 & 1 \end{pmatrix};\\
V_b &= \begin{pmatrix} \cos\theta_{vb} & -\sin\theta_{vb} & 0 \\ \sin\theta_{vb} & \cos\theta_{vb} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{30}$$

where

$$\tan\theta_{ub} = -\frac{1}{2y_{10}}; \qquad \tan\theta_{vb} = \frac{1}{y_{45}}. \tag{31}$$

Next we focus on the τ lepton mass growth. From Eqs. (21) and (22) one has

$$
\begin{aligned}
\mathsf{L}^{(45)}_{2,\tau} &= -12\sqrt{2} f^{(45)}_{33} p[{}^{(10_{16})}\overline{\tau^{(-)}}_R (\bar{10}_{45}) \tau^{(-)}{}_L] + \text{H.c.},\\
\mathsf{L}^{(45)}_{4,\tau} &= -2 m^{(45)}_F [{}^{(10_{45})}\overline{\tau^{(-)}}_R (\bar{10}_{45}) \tau^{(-)}{}_L] + \text{H.c.},\\
\mathsf{L}^{(45)}_{5,\tau} &= 2\sqrt{2} f^{(45)}_{33} \langle \mathbf{P}_5 \rangle [{}^{(10_{45})}\overline{\tau^{(-)}}_R (\bar{5}_{16}) \tau^{(-)}{}_L] + \text{H.c.},\\
\mathsf{L}^{(10)}_{1,\tau} &= -\frac{3}{\sqrt{2}} f^{(10)}_{33} q [{}^{(5_{10})}\overline{\tau^{(-)}}_R (\bar{5}_{16}) \tau^{(-)}{}_L] + \text{H.c.},\\
\mathsf{L}^{(10)}_{2,\tau} &= -\frac{1}{2\sqrt{10}} f^{(10)}_{33} \langle \mathbf{Q}_5 \rangle [{}^{(10_{16})}\overline{\tau^{(-)}}_R (\bar{5}_{10}) \tau^{(-)}{}_L] + \text{H.c.},\\
\mathsf{L}^{(10)}_{3,\tau} &= -\frac{1}{2}\sqrt{\frac{3}{2}} f^{(10)}_{33} \langle \tilde{\mathbf{Q}}_5 \rangle [{}^{(10_{16})}\overline{\tau^{(-)}}_R (\bar{5}_{10}) \tau^{(-)}{}_L] + \text{H.c.},\\
\mathsf{L}^{(10)}_{4,\tau} &= -2 m^{(10)}_F [{}^{(5_{10})}\overline{\tau^{(-)}}_R (\bar{5}_{10}) \tau^{(-)}{}_L] + \text{H.c.}
\end{aligned}
\tag{32}
$$

In this case the mass matrix takes the form

$$
M_\tau = \begin{array}{c} {}^{(10_{16})}\overline{\tau^{(-)}} \\ {}^{(10_{45})}\overline{\tau^{(-)}} \\ {}^{(5_{10})}\overline{\tau^{(-)}} \end{array}
\begin{array}{c} \begin{array}{ccc} {}^{(\bar{5}_{16})}\tau^{(-)}{}_L & {}^{(\bar{5}_{10})}\tau^{(-)}{}_L & {}^{(\overline{10}_{45})}\tau^{(-)}{}_L \end{array} \\ \begin{pmatrix} 0 & m'_\tau & m^{(45)}_E \\ m''_\tau & 0 & -2m^{(45)}_F \\ m^{(10)}_E & -2m^{(10)}_F & 0 \end{pmatrix} \end{array},
\tag{33}
$$

where

$$
\begin{aligned}
m'_\tau &= \frac{1}{2} f^{(10)}_{33} \left[-\frac{\langle \mathbf{Q}_5 \rangle}{\sqrt{10}} + \sqrt{\frac{3}{2}} \langle \tilde{\mathbf{Q}}_5 \rangle \right]; \qquad m''_\tau = 2\sqrt{2} f^{(45)}_{33} \langle \mathbf{P}_5 \rangle,\\
m^{(45)}_E &= -12\sqrt{2} f^{(45)}_{33} p; \qquad m^{(10)}_E = -\frac{3}{\sqrt{2}} f^{(10)}_{33} q.
\end{aligned}
\tag{34}
$$

Again in the limit $m'_\tau \to 0$, the eigenvalues take the form

$$
\begin{aligned}
m^2_\tau \equiv \lambda^2_{\tau_1} &\approx \frac{1}{144} \frac{y^2_{45}}{[1 + (\frac{y_{45}}{6})^2][1 + (\frac{4y_{10}}{3})^2]}\\
&\quad \times \left(f^{(10)}_{33} \left[-\frac{\langle \mathbf{Q}_5 \rangle}{\sqrt{10}} + \sqrt{\frac{3}{2}} \langle \tilde{\mathbf{Q}}_5 \rangle \right] \right)^2,\\
\lambda^2_{\tau_2} &\approx \Lambda^2_{\tau_2} = 288 \left[1 + \left(\frac{y_{45}}{6} \right)^2 \right] (f^{(45)}_{33} p)^2,\\
\lambda^2_{\tau_3} &\approx \Lambda^2_{\tau_3} = \frac{9}{2} \left[1 + \left(\frac{4y_{10}}{3} \right)^2 \right] (f^{(10)}_{33} q)^2.
\end{aligned}
\tag{35}
$$

The rotated fields can now be expressed in terms of the primitive ones

$$
\begin{aligned}
\begin{pmatrix} {}^{(10_{16})}\overline{\tau'^{(-)}} \\ {}^{(10_{45})}\overline{\tau'^{(-)}} \\ {}^{(5_{10})}\overline{\tau'^{(-)}} \end{pmatrix}_R &= U_\tau \begin{pmatrix} {}^{(10_{16})}\overline{\tau^{(-)}} \\ {}^{(10_{45})}\overline{\tau^{(-)}} \\ {}^{(5_{10})}\overline{\tau^{(-)}} \end{pmatrix}_R;\\
\begin{pmatrix} {}^{(\bar{5}_{16})}\tau'^{(-)}{}_L \\ {}^{(\bar{5}_{10})}\tau'^{(-)}{}_L \\ {}^{(\overline{10}_{45})}\tau'^{(-)}{}_L \end{pmatrix} &= V_\tau \begin{pmatrix} {}^{(\bar{5}_{16})}\tau^{(-)}{}_L \\ {}^{(\bar{5}_{10})}\tau^{(-)}{}_L \\ {}^{(\overline{10}_{45})}\tau^{(-)}{}_L \end{pmatrix}
\end{aligned}
\tag{36}
$$

and the mass terms in the Lagrangian are

$$
\begin{aligned}
&\lambda_{\tau_1} {}^{(10_{16})}\overline{\tau'^{(-)}}_R (\bar{5}_{16}) \tau'^{(-)}_L + \lambda_{\tau_2} {}^{(10_{45})}\overline{\tau'^{(-)}}_R (\bar{5}_{10}) \tau'^{(-)}_L\\
&\quad + \lambda_{\tau_3} {}^{(5_{10})}\overline{\tau'^{(-)}}_R (\overline{10}_{45}) \tau'^{(-)}_L + \text{H.c.}
\end{aligned}
\tag{37}
$$

Rotation matrices in this case are given by

$$
\begin{aligned}
U_\tau &= \begin{pmatrix} \cos\theta_{u\tau} & -\sin\theta_{u\tau} & 0 \\ \sin\theta_{u\tau} & \cos\theta_{u\tau} & 0 \\ 0 & 0 & 1 \end{pmatrix};\\
V_\tau &= \begin{pmatrix} \cos\theta_{v\tau} & -\sin\theta_{v\tau} & 0 \\ \sin\theta_{v\tau} & \cos\theta_{v\tau} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}
\tag{38}
$$

where

$$
\tan\theta_{u\tau} = \frac{6}{y_{45}}; \qquad \tan\theta_{v\tau} = \frac{3}{4y_{10}}.
\tag{39}
$$

One finds now that $b - \tau$ unification emerges under the constraints

$$
y^2_{10}, y^2_{45} \ll 1
\tag{40}
$$

which are easily arranged and lead to

$$
\frac{\langle \mathbf{Q}'_5 \rangle \sin\vartheta_{\mathsf{D}} + \langle \tilde{\mathbf{Q}}'_5 \rangle \cos\vartheta_{\mathsf{D}}}{\langle \mathbf{Q}'_5 \rangle \cos\vartheta_{\mathsf{D}} - \langle \tilde{\mathbf{Q}}'_5 \rangle \sin\vartheta_{\mathsf{D}}} = \left\{ \begin{array}{ll} -\frac{7}{\sqrt{15}} & \text{for } m_b = m_\tau \\ -\frac{1}{3}\sqrt{\frac{5}{3}} & \text{for } m_b = -m_\tau \end{array} \right\},
\tag{41}
$$

where we have used Eq. (8).

In the analysis we choose D_2 to be the light Higgs doublet and together with the $b - \tau$ condition, Eq. (41), and symmetry breaking condition, Eq. (4), we get the following relationship among the coupling constants:

$$
\left\{ \begin{array}{llll} \lambda_{45_2} = -14.58\lambda_{45_1}, & \lambda_{210} = 11.52\lambda_{45_1}, & \frac{MM'}{qp} = 60.04\lambda_{45_1}, & \tan\vartheta_{\mathsf{D}} = -1.81 \\ & \text{or} & & \\ \lambda_{45_2} = -13.06\lambda_{45_1}, & \lambda_{210} = 10.47\lambda_{45_1}, & \frac{MM'}{qp} = 66.49\lambda_{45_1}, & \tan\vartheta_{\mathsf{D}} = -0.43 \end{array} \right\}.
\tag{42}
$$

V. TOP QUARK MASS

The term in the Lagrangian arising from Eqs. (21) and (22) and contributing to the top quark mass is

$$\begin{aligned}
\mathsf{L}_{1,t}^{(45)} &= 2\sqrt{\tfrac{2}{5}} f_{33}^{(45)} \langle \mathbf{P}^5 \rangle [{}^{(10_{16})}\bar{t}_{R\alpha}{}^{(10_{45})}t_L^\alpha + {}^{(10_{45})}\bar{t}_{R\alpha}{}^{(10_{16})}t_L^\alpha] \\
&\quad + \text{H.c.}, \\
\mathsf{L}_{2,t}^{(45)} &= 2\sqrt{2} f_{33}^{(45)} p [4{}^{(10_{16})}\bar{t}_{R\alpha}{}^{(\overline{10}_{45})}t_L^\alpha - {}^{(\overline{10}_{45})}\bar{t}_{R\alpha}{}^{(10_{16})}t_L^\alpha] \\
&\quad + \text{H.c.}, \\
\mathsf{L}_{3,t}^{(45)} &= 2\sqrt{\tfrac{2}{3}} f_{33}^{(45)} \langle \tilde{\mathbf{P}}^5 \rangle [{}^{(10_{16})}\bar{t}_{R\alpha}{}^{(10_{45})}t_L^\alpha + {}^{(10_{45})}\bar{t}_{R\alpha}{}^{(10_{16})}t_L^\alpha] \\
&\quad + \text{H.c.}, \\
\mathsf{L}_{4,t}^{(45)} &= -2m_F^{(45)} [{}^{(10_{45})}\bar{t}_{R\alpha}{}^{(\overline{10}_{45})}t_L^\alpha + {}^{(\overline{10}_{45})}\bar{t}_{R\alpha}{}^{(10_{45})}t_L^\alpha] + \text{H.c.}
\end{aligned} \tag{43}$$

The corresponding mass matrix is given by

$$M_t = \begin{array}{c} {}^{(10_{16})}\bar{t}_{R\alpha} \\ {}^{(10_{45})}\bar{t}_{R\alpha} \\ {}^{(\overline{10}_{45})}\bar{t}_{R\alpha} \end{array} \begin{array}{c} \begin{array}{ccc} {}^{(10_{16})}t_L^\alpha & {}^{(10_{45})}t_L^\alpha & {}^{(\overline{10}_{45})}t_L^\alpha \end{array} \\ \begin{pmatrix} 0 & m_t' & 4m_U^{(45)} \\ m_t' & 0 & -2m_F^{(45)} \\ -m_U^{(45)} & -2m_F^{(45)} & 0 \end{pmatrix} \end{array}, \tag{44}$$

where we have defined

$$m_t' = 4f_{33}^{(45)} \left[\frac{\langle \mathbf{P}^5 \rangle}{\sqrt{10}} + \frac{\langle \tilde{\mathbf{P}}^5 \rangle}{\sqrt{6}} \right], \qquad m_U^{(45)} = 2\sqrt{2} f_{33}^{(45)} p. \tag{45}$$

In the limit $m_t' \to 0$, the approximate eigenvalues are

$$\begin{aligned}
m_t^2 \equiv \lambda_{t_1}^2 &\approx 9 \frac{y_{45}^2}{[1 + (\frac{y_{45}}{4})^2][1 + y_{45}^2]} \left(f_{33}^{(45)} \left[\frac{\langle \mathbf{P}^5 \rangle}{\sqrt{10}} + \frac{\langle \tilde{\mathbf{P}}^5 \rangle}{\sqrt{6}} \right] \right)^2, \\
\lambda_{t_2}^2 &\approx \Lambda_{t_2}^2 = 128 \left[1 + \left(\frac{y_{45}}{4} \right)^2 \right] (f_{33}^{(45)} p)^2, \\
\lambda_{t_3}^2 &\approx \Lambda_{t_3}^2 = 8[1 + y_{45}^2](f_{33}^{(45)} p)^2.
\end{aligned} \tag{46}$$

The rotated fields can now be obtained from

$$\begin{aligned}
\begin{pmatrix} {}^{(10_{16})}\bar{t}_{R\alpha}' \\ {}^{(10_{45})}\bar{t}_{R\alpha}' \\ {}^{(\overline{10}_{45})}\bar{t}_{R\alpha}' \end{pmatrix} &= U_t \begin{pmatrix} {}^{(10_{16})}\bar{t}_{R\alpha} \\ {}^{(10_{45})}\bar{t}_{R\alpha} \\ {}^{(\overline{10}_{45})}\bar{t}_{R\alpha} \end{pmatrix}; \\
\begin{pmatrix} {}^{(10_{16})}t_L'^\alpha \\ {}^{(10_{45})}t_L'^\alpha \\ {}^{(\overline{10}_{45})}t_L'^\alpha \end{pmatrix} &= V_t \begin{pmatrix} {}^{(10_{16})}t_L^\alpha \\ {}^{(10_{45})}t_L^\alpha \\ {}^{(\overline{10}_{45})}t_L^\alpha \end{pmatrix}.
\end{aligned} \tag{47}$$

The mass terms in the Lagrangian are then given by

$$\begin{aligned}
&\lambda_{t_1} {}^{(10_{16})}\bar{t}_{R\alpha}' {}^{(10_{16})}t_L'^\alpha + \lambda_{t_2} {}^{(10_{45})}\bar{t}_{R\alpha}' {}^{(10_{45})}t_L'^\alpha \\
&\quad + \lambda_{t_3} {}^{(\overline{10}_{45})}\bar{t}_{R\alpha}' {}^{(\overline{10}_{45})}t_L^\alpha + \text{H.c.}
\end{aligned} \tag{48}$$

In this case the approximate U_t and V_t matrices are given by

$$\begin{aligned}
U_t &= \begin{pmatrix} \cos\theta_{ut} & -\sin\theta_{ut} & 0 \\ \sin\theta_{ut} & \cos\theta_{ut} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\
V_t &= \begin{pmatrix} \cos\theta_{vt} & -\sin\theta_{vt} & 0 \\ \sin\theta_{vt} & \cos\theta_{vt} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{49}$$

where

$$\tan\theta_{ut} = -\frac{4}{y_{45}}; \qquad \tan\theta_{vt} = \frac{1}{y_{45}}. \tag{50}$$

Now the $b - \tau - t$ unification can be achieved with the further constraint

$$\frac{m_t}{m_b} = 6 \left(\frac{f_{33}^{(45)}}{f_{33}^{(10)}} \right) \tan\beta. \tag{51}$$

We will consider the case when D_2 is massless and for this case one has

$$\tan\beta \equiv \frac{\langle \mathbf{P}'^5 \rangle}{\langle \mathbf{Q}_5' \rangle}. \tag{52}$$

VI. LIGHT HIGGS-MATTER INTERACTIONS

From the preceding analysis we can write the following effective superpotential for interactions of the doublet of light Higgs H_1, H_2 with quarks and leptons as follows

$$W_{\text{eff}} = \epsilon_{\alpha\beta} [hb \hat{H}_1^\alpha \hat{Q}^\beta \hat{D} + h_\tau \hat{H}_1^\alpha \hat{L}^\beta \hat{E} + h_t \hat{H}_2^\alpha \hat{Q}^\beta \hat{U}], \tag{53}$$

where hb, h_τ, and h_t are given by

$$\begin{aligned}
h_b &= \frac{y_{45}}{\sqrt{[1 + y_{45}^2][1 + (2y_{10})^2]}} f_{33}^{(10)} \left[\frac{\cos\vartheta_{\mathsf{D}}}{2\sqrt{10}} + \frac{\sin\vartheta_{\mathsf{D}}}{2\sqrt{6}} \right], \\
h_\tau &= \frac{y_{45}}{\sqrt{[1 + (\frac{y_{45}}{6})^2][1 + (\frac{4y_{10}}{3})^2]}} f_{33}^{(10)} \left[-\frac{\cos\vartheta_{\mathsf{D}}}{12\sqrt{10}} + \frac{\sqrt{3}\sin\vartheta_{\mathsf{D}}}{12\sqrt{2}} \right], \\
h_t &= \frac{6y_{45}}{\sqrt{[1 + (\frac{y_{45}}{4})^2][1 + y_{45}^2]}} f_{33}^{(45)} \left[\frac{\cos\vartheta_{\mathsf{D}}}{2\sqrt{10}} + \frac{\sin\vartheta_{\mathsf{D}}}{2\sqrt{6}} \right].
\end{aligned} \tag{54}$$

Now it is seen that under the constraint that $y_{10}^2, y_{45}^2 \ll 1$ one gets a $b - \tau$ unification of Yukawa couplings just below the GUT scale, i.e., $|h_b| = |h_\tau|$ when

$$\tan\vartheta_{\mathsf{D}} = -\tfrac{7}{\sqrt{15}} \quad \text{or} \quad -\tfrac{1}{3}\sqrt{\tfrac{5}{3}}. \tag{55}$$

Under the same conditions one then finds that

$$|h_t| \approx 6 \left(\frac{f_{33}^{(45)}}{f_{33}^{(10)}} \right) |h_b|. \tag{56}$$

With $f_{33}^{(45)} \approx f_{33}^{(10)}$ one finds $|h_t| \approx 6|h_b|$. Thus instead of the relation

$$h_\tau = h_t = h_b \tag{57}$$

that holds in the $SO(10)$ models where the electroweak symmetry is broken by a 10-plet of Higgs one finds in the present model a relation of the type

$$|h_\tau| \approx |h_b| = z\frac{|h_t|}{6}, \tag{58}$$

where $z = \frac{f_{33}^{(45)}}{f_{33}^{(10)}}$ and z is $O(1)$. Here low values of $\tan\beta$ of size 10 can be realized consistent with $b - \tau - t$ unification.

It is also interesting to discuss the case when $y_{10}^2, y_{45}^2 \ll 1$. In this case the $|h_b| = |h_\tau|$ unification requires

$$\tan\vartheta_{\rm D} = \tfrac{\sqrt{15}}{7} \quad \text{or} \quad \tfrac{1}{11}\sqrt{\tfrac{3}{5}}. \tag{59}$$

In the same limit one has

$$|h_t| \approx 48\left(\frac{f_{33}^{(45)}}{f_{33}^{(10)}}\right)\left(\frac{y_{10}}{y_{45}}\right)|h_b|. \tag{60}$$

Here the heaviness of the top relative to the bottom is even more easy to understand since a value of $\tan\beta$ of order one may be realized naturally. Thus remarkably in two opposite limits, i.e., small and large y_{10}, y_{45} cases one finds that $\tan\beta$ much smaller than 50 can be realized while achieving $b - \tau - t$ unification.

We discuss now the inclusion of all the generations. To begin with suppose for the moment we forbid in our model nonrenormalizable coupling in the Yukawa sector. In this case all fermions are massless and we have additional 3 chiral unbroken $U(1)$ symmetries. Next we introduce a heavy fermionic 45-plet, which has cubic couplings with all the three generations. In this case we will break only one $U(1)$ chiral symmetry and the remaining two generations will remain massless. Essentially what it means is that in the above example we can choose a basis[1] where only one generation, which we assume is the third generation, will couple with the 45-plet and will lead to a heavy top. One problem, however, is that the fermionic 10-plet would couple to all generations even in the new basis and thus spoil the neat separation between the third generation and the first two generations. This problem can be overcome by the assumption that the generation dependence of the 10-plet and the 45-plet are "parallel," i.e., $h_{\acute{a}\acute{b}}^{(10)} = \lambda h_{\acute{a}\acute{b}}^{(45)}$ so they are proportional up to a scaling factor. In this case in the new basis both the 45-plet and the 10-plets will couple only with the third generation. The masses of the first two generations will arise via quartic couplings of the type $16.16.144.144$ and $16.16.\overline{144}.\overline{144}$ and are thus naturally suppressed by a factor of $M_{\rm GUT}/M_P$. The inclusion of quartic couplings will also affect the b quark and τ lepton masses and thus a more complete analysis should take into account the full set of couplings both cubic and quartic. Inclusion of such couplings would make the determination of $\tan\beta$ somewhat more arbitrary due to its dependence on the parameters of the quartic interactions.

The cubic couplings involving the 10-plet of matter also help in the generation of a right size tau neutrino mass. Specifically, the coupling $f_{33}^{(10)}(16_M.10_M.144_H)$ gives a Dirac mass $m_D = f_{33}^{(10)}\langle \mathbf{Q}^5\rangle$. However, a much larger Majorana mass arises from the quartic coupling

$$\Lambda(16.\overline{144})_{45}(16.\overline{144})_{45}, \tag{61}$$

where the subindex 45 means mediation by a 45 of $SO(10)$. An analytical analysis of this coupling gives a Majorana mass of $M_R = 30\Lambda p^2$ where p is as defined in Eq. (3). Typically the numerical size of $M_R \sim 10^{14}$ GeV. Thus the seesaw gives a tau neutrino mass $m_{\nu_\tau} \sim m_D^2/M_R \sim .1(f_{33}^{(10)})^2$ eV which is the right size with $f_{33}^{(10)} \sim 1$.

VII. CONCLUSION

In this paper we have presented a scenario for generating Yukawa couplings for the third generation quarks and leptons which lead to naturally heavy third generation masses. The analysis is done within a unified symmetry breaking framework where the breaking of $SO(10)$ can be accomplished with just one pair of $144 + \overline{144}$ of Higgs. Typically the Yukawa couplings of matter with Higgs arise from quartic couplings, i.e., couplings of the type $16.16.144.144$ and $16.16.\overline{144}.\overline{144}$. Such couplings will be suppressed by a heavy mass, e.g., by the string scale M_{st}. Thus one expects that the Yukawa couplings generated by such interactions will be suppressed by a factor of M_G/M_{st} and thus would naturally explain the smallness of the first and second generation Yukawa couplings. Regarding the Yukawa coupling hierarchy between the first generation and the second generation it will have to be explained by a hierarchy in the couplings associated with the Planck suppressed operators. Thus, for example, such a hierarchy could arise from a flavor symmetry that distinguishes generations. On the other hand the couplings of the third generation are much larger and it is difficult to understand the largeness of these Yukawa couplings from the quartic

[1]The choice of the basis where only one generation couples to the 45-plet and the 10-plet can be seen more easily as follows: Thus suppose all the three generations couple to the 45-plet and the 10-plet. In this case the mass matrices for the up quark, the down quark, and the charged lepton, will be 5×5. Each of these will have two superheavy modes and two massless modes and one normally massive mode. The superheavy modes can be discarded at low energy. Of the remaining three, the two modes which are massless may be labeled first generation and second generation, while the remaining massive mode will be the third generation fermion. Thus, regardless of how the 45-plet and the 10-plet couple to the 16_i, only the top, the bottom, and the tau will acquire a mass from the cubic interactions.

interactions. [Of course, Planck suppressed operators can also contribute to the third generation masses, but presumably their contributions are smaller than those arising from the cubic couplings.] We have proposed in this paper a new scenario where we have introduced a 10-plet and a 45-plet of matter fields which mix with the third generation 16-plet of matter and lead to cubic couplings in the superpotential of type 16.144.10 and $16.\overline{144}.45$. The third generation quarks and leptons are then an admixture of the components from the 16, 10, and 45-plets, and their Yukawa couplings are not suppressed by large factors and thus third generation masses which are much larger than the masses for the first two generations naturally arise. Further, under reasonable constraints one finds $b-\tau-t$ unification. Quite remarkably such a unification occurs at low values of $\tan\beta$ in contrast to the large $\tan\beta$ scenarios needed for $b-\tau-t$ unification in some $SO(10)$ scenarios. The small values of $\tan\beta$ will have important implications for $SO(10)$ phenomenology. One obvious implication is in the supersymmetric proton decay where the lifetime is inversely proportional to the square of $\tan\beta$. [For recent works on proton decay in $SO(10)$ see Refs. [7] and for a recent review see Ref. [8]]. Thus a smaller value of $\tan\beta$ can help stabilize the proton. We note that the fermion mass generation mechanism discussed here would work equally well for the nonsupersymmetric case as well.

Finally, we note that the explicit analysis of the $16.\overline{144}.45$ and of 16.144.10 vertices given here has another use. Thus the analysis of these cubic couplings can be used to compute the quartic couplings of the type $16_{\acute{a}}.16_{\acute{b}}.\overline{144}.\overline{144}$ and $16_{\acute{a}}.16_{\acute{b}}.144.144$ along the 45 and 10 contractions, respectively. This can be done by integrating out the matter 45-plet and 10-plet, i.e., by setting the F-terms associated with these fields from Eqs. (21) and (22) to zero. The structure of these quartic couplings is important in understanding the higher generation masses and couplings.

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APPENDIX A

In this appendix we display the symbolic forms of the Higgs sector quartic couplings for GUT and electroweak symmetry breaking. The explicit forms of the couplings that appear in the text are

$$(\overline{144}_H \times 144_H)_{45_1}(\overline{144}_H \times 144_H)_{45_1} = \frac{\lambda_{45_1}}{M'}\langle \hat{\Upsilon}^*_{(-)\mu}|B\Sigma_{\rho\lambda}|\hat{\Upsilon}_{(+)\mu}\rangle \times \langle \hat{\Upsilon}^*_{(-)\nu}|B\Sigma_{\rho\lambda}|\hat{\Upsilon}_{(+)\nu}\rangle, \tag{A1}$$

$$(\overline{144}_H \times 144_H)_{45_2}(\overline{144}_H \times 144_H)_{45_2} = \frac{\lambda_{45_2}}{M'}\langle \hat{\Upsilon}^*_{(-)[\mu}|B|\hat{\Upsilon}_{(+)\nu]}\rangle \times \langle \hat{\Upsilon}^*_{(-)\acute{c}[\mu}|B|\hat{\Upsilon}_{(+)\nu]}\rangle, \tag{A2}$$

$$\begin{aligned}(\overline{144}_H \times 144_H)_{210}(\overline{144}_H \times 144_H)_{210} &= \frac{\lambda_{210}}{M'}\langle \hat{\Upsilon}^*_{(-)\mu}|B\Gamma_{[\rho}\Gamma_\sigma\Gamma_\lambda\Gamma_{\xi]}|\hat{\Upsilon}_{(+)\mu}\rangle \\ &\quad\times \langle \hat{\Upsilon}^*_{(-)\nu}|B\Gamma_{[\rho}\Gamma_\sigma\Gamma_\lambda\Gamma_{\xi]}|\hat{\Upsilon}_{(+)\nu}\rangle,\end{aligned} \tag{A3}$$

where B is an $SO(10)$ charge conjugation operator and Γ_μ is the Clifford element given by

$$\begin{aligned} B &= -i\prod_{k=1}^{5}(b_k - b_k^\dagger), \qquad \Gamma_\mu = (\Gamma_{2i}, \Gamma_{2i-1}); \\ \Gamma_{2i} &= (b_i + b_i^\dagger), \qquad \Gamma_{2i-1} = -i(b_i - b_i^\dagger),\end{aligned} \tag{A4}$$

and where $i = 1, \ldots, 5$ is a $SU(5)$ index.

The $SU(5)$ field content of the $SO(10)$ multiplets 16, $\overline{144}$ + 144, 45, and 10 is as follows

$$\begin{aligned} 16(\hat{\Psi}_{(+)}) &= 1(\hat{\mathbf{M}}) + \bar{5}(\hat{\mathbf{M}}_i) + 10(\hat{\mathbf{M}}^{ij}), \\ \overline{144}(\hat{\Upsilon}_{(+)\mu}) &= \bar{5}(\hat{\mathbf{P}}_i) + 5(\hat{\mathbf{P}}^i) + \overline{10}(\hat{\mathbf{P}}_{ij}) + \overline{15}(\hat{\mathbf{P}}^{(S)}_{ij}) \\ &\quad + 24(\hat{\mathbf{P}}^i_j) + \overline{40}(\mathbf{P}^i_{jkl}) + 45(\hat{\mathbf{P}}^{ij}_k), \\ 144(\hat{\Upsilon}_{(-)\mu}) &= \bar{5}(\hat{\mathbf{Q}}_i) + 5(\hat{\mathbf{Q}}^i) + 10(\hat{\mathbf{Q}}^{ij}) + 15(\hat{\mathbf{Q}}^{ij}_{(S)}) \\ &\quad + 24(\hat{\mathbf{Q}}^i_j) + 40(\mathbf{Q}^{ijk}_l) + \overline{45}(\hat{\mathbf{Q}}^i_{jk}), \\ 45(\hat{\mathbf{F}}^{(45)}_{\mu\nu}) &= 1(\hat{\mathbf{F}}^{(45)}) + 10(\hat{\mathbf{F}}^{(45)ij}) + \overline{10}(\hat{\mathbf{F}}^{(45)}_{ij}) + 24(\hat{\mathbf{F}}^{(45)i}_j), \\ 10(\hat{\mathbf{F}}^{(10)}_\mu) &= \bar{5}(\hat{\mathbf{F}}^{(10)}_i) + 5(\hat{\mathbf{F}}^{(10)i}). \end{aligned} \tag{A5}$$

Similarly the expansions of the 16-component spinor and of the 144-component vector spinor [3,6] expressed in their oscillator modes are

$$\begin{aligned}|\hat{\Psi}_{(+)}\rangle &= |0\rangle\hat{\mathbf{M}} + \tfrac{1}{2}b_i^\dagger b_j^\dagger|0\rangle\hat{\mathbf{M}}^{ij} \\ &\quad + \tfrac{1}{24}\epsilon^{ijklm}b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger|0\rangle\hat{\mathbf{M}}_i,\end{aligned} \tag{A6}$$

$$
\begin{aligned}
|\hat{\Upsilon}_{(\pm)\mu}\rangle &= (|\hat{\Upsilon}_{(\pm)c_n}\rangle, |\hat{\Upsilon}_{(\pm)\bar{c}_n}\rangle),\\
|\hat{\Upsilon}_{(+)c_n}\rangle &= |0\rangle\hat{\mathbf{P}}^n + \tfrac{1}{2}b_i^\dagger b_j^\dagger|0\rangle[\epsilon^{ijklm}\hat{\mathbf{P}}^n_{klm} - \tfrac{1}{6}\epsilon^{ijnlm}\hat{\mathbf{P}}_{lm}]\\
&\quad + \tfrac{1}{24}\epsilon^{ijklm}b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger|0\rangle\hat{\mathbf{P}}^n_i,\\
|\hat{\Upsilon}_{(+)\bar{c}_n}\rangle &= |0\rangle\hat{\mathbf{P}}_n + \tfrac{1}{2}b_i^\dagger b_j^\dagger|0\rangle[\hat{\mathbf{P}}^{ij}_n + \tfrac{1}{4}(\delta^i_n\hat{\mathbf{P}}^j - \delta^j_n\hat{\mathbf{P}}^i)]\\
&\quad + \tfrac{1}{24}\epsilon^{ijklm}b_j^\dagger b_k^\dagger b_l^\dagger b_m^\dagger|0\rangle[\tfrac{1}{2}\hat{\mathbf{P}}_{in} + \tfrac{1}{2}\hat{\mathbf{P}}^{(S)}_{in}],\\
|\hat{\Upsilon}_{(-)c_n}\rangle &= b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger|0\rangle\hat{\mathbf{Q}}^n + \tfrac{1}{12}\epsilon^{ijklm}b_k^\dagger b_l^\dagger b_m^\dagger|0\rangle\\
&\quad \times[\hat{\mathbf{Q}}^n_{ij} + \tfrac{1}{4}(\delta^n_i\hat{\mathbf{Q}}_j - \delta^n_j\hat{\mathbf{Q}}_i)] + b_i^\dagger|0\rangle[\tfrac{1}{2}\hat{\mathbf{Q}}^{in} + \tfrac{1}{2}\hat{\mathbf{Q}}^{in}_{(S)}],\\
|\hat{\Upsilon}_{(-)\bar{c}_n}\rangle &= b_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger b_5^\dagger|0\rangle\hat{\mathbf{Q}}_n + \tfrac{1}{12}b_k^\dagger b_l^\dagger b_m^\dagger|0\rangle\\
&\quad \times[\epsilon_{ijklm}\hat{\mathbf{Q}}^{klm}_n - \tfrac{1}{6}\epsilon_{ijnlm}\hat{\mathbf{Q}}^{lm}] + b_i^\dagger|0\rangle\hat{\mathbf{Q}}^i_n. \qquad \text{(A7)}
\end{aligned}
$$

APPENDIX B

Below we list the normalization of the relevant fields computations for which can be found in previous analyses [4]:

$$
\begin{aligned}
&(\hat{\mathbf{Q}}^i, \hat{\mathbf{P}}_i) \to (\hat{\mathbf{Q}}^i, \hat{\mathbf{P}}_i); \qquad (\hat{\mathbf{Q}}_i, \hat{\mathbf{P}}^i) \to \tfrac{2}{\sqrt{5}}(\hat{\mathbf{Q}}_i, \hat{\mathbf{P}}^i),\\
&(\hat{\mathbf{Q}}^i_j, \hat{\mathbf{P}}^i_j) \to (\hat{\mathbf{Q}}^i_j, \hat{\mathbf{P}}^j_i); \qquad (\hat{\mathbf{Q}}^k_{ij}, \hat{\mathbf{P}}^{ij}_k) \to (\hat{\mathbf{Q}}^k_{ij}, \hat{\mathbf{P}}^{ij}_k),\\
&(\hat{\mathbf{F}}^{(10)}_i, \hat{\mathbf{F}}^{(10)i}) \to \tfrac{1}{\sqrt{2}}(\hat{\mathbf{F}}^{(10)}_i, \hat{\mathbf{F}}^{(10)i});\\
&(\hat{\mathbf{F}}^{(45)}_{ij}, \hat{\mathbf{F}}^{(45)ij}) \to \sqrt{2}(\hat{\mathbf{F}}^{(45)}_{ij}, \hat{\mathbf{F}}^{(45)ij}),
\end{aligned} \qquad \text{(B1)}
$$

where the arrow indicates the replacement needed to achieve a normalized kinetic energy for the fields. Additionally as mentioned earlier there are also pairs of $SU(2)$ doublets: $(\tilde{\mathbf{Q}}_a, \tilde{\mathbf{P}}^a)$ that are contained in $\mathbf{Q}^k_{ij}$ and $\mathbf{P}^{ij}_k$. We project out these doublets as follows ($\alpha, \beta = 1, 2, 3$ and $a, b, c = 4, 5$):

$$
\begin{aligned}
&\mathbf{Q}^b_{ba} = -\mathbf{Q}^\beta_{\beta a} = \tilde{\mathbf{Q}}_a, \qquad \mathbf{P}^{ba}_b = -\mathbf{P}^{\beta a}_\beta = \tilde{\mathbf{P}}^a,\\
&\mathbf{Q}^\alpha_{\beta a} = \tilde{\mathbf{Q}}^\alpha_{\beta a} + \tfrac{1}{3}\delta^\alpha_\beta\tilde{\mathbf{Q}}_a, \qquad \mathbf{P}^{\alpha a}_\beta = \tilde{\mathbf{P}}^{\alpha a}_\beta + \frac{1}{3}\delta^\alpha_\beta\tilde{\mathbf{P}}^a,\\
&\tilde{\mathbf{Q}}^\alpha_{\alpha b} = 0 = \tilde{\mathbf{P}}^{\alpha b}_\alpha, \qquad \mathbf{Q}^a_{bc} = \delta^a_c\tilde{\mathbf{Q}}_b - \delta^a_b\tilde{\mathbf{Q}}_c,\\
&\mathbf{P}^{ab}_c = \delta^b_c\tilde{\mathbf{P}}^a - \delta^a_c\tilde{\mathbf{P}}^b, \qquad \tilde{\mathbf{Q}}^a_{ab} = 0 = \tilde{\mathbf{P}}^{ab}_a.
\end{aligned} \qquad \text{(B2)}
$$

The kinetic energy of the 45 and $\overline{45}$ fields of $SU(5)$ is given by

$$
\begin{aligned}
&-\tfrac{1}{2}\partial_A\mathbf{Q}^k_{ij}\partial^A\mathbf{Q}^{k\dagger}_{ij} - \tfrac{1}{2}\partial_A\mathbf{P}^{ij}_k\partial^A\mathbf{P}^{ij\dagger}_k\\
&\quad = -\tfrac{1}{2}[\tfrac{8}{3}\partial_A\tilde{\mathbf{Q}}_\alpha\partial^A\tilde{\mathbf{Q}}^\dagger_\alpha + \tfrac{8}{3}\partial_A\tilde{\mathbf{P}}^\alpha\partial^A\tilde{\mathbf{P}}^{\alpha\dagger} + \ldots]
\end{aligned} \qquad \text{(B3)}
$$

so the doublet fields are normalized according to

$$
(\tilde{\mathbf{Q}}_a, \tilde{\mathbf{P}}^a) \to \tfrac{\sqrt{3}}{2}(\tilde{\mathbf{Q}}_a, \tilde{\mathbf{P}}^a). \qquad \text{(B4)}
$$

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