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Quantum-mechanical phase locking in weak-link arrays

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Quantum-mechanical descriptions of arrays of Josephson weak links often invoke electric-field-energy storage in the weak-link capacitors. However, such quantum-electrodynamic mechanism is by no means a requirement for the notion of macroscopic quantum-mechanical wave functions for the array as a whole. These statements are illustrated for the phenomena of “quantum-mechanical phase locking” in one- and two-dimensional arrays in electromagnetic fields.

I. INTRODUCTION

Arrays of Josephson weak links are both of theoretical and experimental importance.¹ Apart from the tunneling of electron pairs through the weak links, the dynamics of arrays are usually treated from a classical electrodynamic viewpoint. For the case in which the weak link capacitance is sufficiently small, a quantum electrodynamic viewpoint is required since the electric field energy stored in the capacitors can be quite substantial.² However, such electromagnetic field energy storage is by no means a requirement for quantum mechanical effects to be made manifest in experimental systems.

Our purpose is to illustrate the above statements by discussing the phenomena of “quantum mechanical phase locking” in one- and two-dimensional arrays, here *neglecting capacitive electric field energy storage*. Nevertheless, it will be shown how the phenomena of “phase locking” can be understood on the basis of macroscopic quantum wave functions for the array as a whole. By “phase locking” we here mean that the phase of the superconducting order parameter maintains coherence across the whole array of grains and not merely coherence within each grain.

In previous treatments, the dynamics of “phase locking” has been theoretically computed from a classical circuit model,³ e.g. the resistively shunted junction model for each weak link. Using this classical view, considerable progress has been achieved in our understanding of the dynamics.⁴ On the other hand, it is also of importance to understand what is happening at the quantum mechanical level, and the work which follows may be regarded as a complimentary description of the classical “phase locking” process in macroscopic quantum mechanical terms.

In Sec. II we discuss the problem of the dynamics of a single electron pair injected into a one-dimensional array of weak links. If an electric field were applied in this model, then the well-known Stark ladder of energy levels is discussed and reviewed. In Sec. III the quantum mechanics of a single pair is generalized to a macroscopic

quantum mechanical wave function for a one-dimensional weak link array in a phase locked quantum state. What appeared as a Stark ladder for the single electron pair, here arises as a ladder of energy levels separated by the Josephson frequency

$$\omega_0 = qV/\hbar, \tag{1a}$$

$$q = 2e. \tag{1b}$$

In Sec. IV we consider a two-dimensional quantum mechanical weak link array in a magnetic field. The magnetic macroscopic Schrödinger equation for the phase locked states is discussed. In the concluding Sec. V, the Hall effect for the two-dimensional array is computed.

II. A SINGLE ELECTRON PAIR

Consider a single electron pair moving through a one-dimensional array as shown in Fig. 1. The electron pair (in this initial model) can be located on any one of the superconducting grains $n = 0, \pm 1, \pm 2, \dots$, in the state $|n\rangle$. It will be assumed that the electron pair can tunnel from one grain to a neighboring grain with nonzero matrix elements

$$\langle n \pm 1 | H | n \rangle = -\hbar v / 2. \tag{2}$$

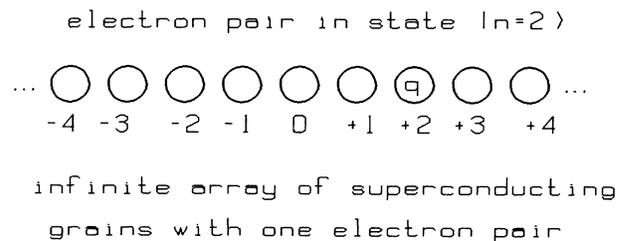


FIG. 1. A single electron pair of charge $q = 2e$ can be injected onto the n th grain in state $|n\rangle$. Shown in the state $|2\rangle$. The electron pair tunneling from grain to neighboring grain gives rise to a tight-binding energy band as in Eqs. (3).

In Eq. (2) ν turns out to be the maximum frequency with which the pair can tunnel from grain to grain. The tight-binding assumption is equivalent to that made by Josephson in his original work on the Josephson effect. If ν grows too large, then the current is still a periodic function to the Josephson phase, but higher harmonics must be used besides a single cosine term. As a result of Eq. (2), the electron pair moves in a tight-binding one-dimensional energy band:

$$h|k\rangle = \varepsilon(k)|k\rangle, \quad (3a)$$

$$|k\rangle = \sum_{n=-\infty}^{\infty} e^{-ikbn}|n\rangle, \quad (3b)$$

$$\varepsilon(k) = -\hbar\nu \cos(kb), \quad (3c)$$

where b is the distance between neighboring grains. The electron pair glides along the one-dimensional array with a group velocity (using the position operator $nb = i\partial/\partial k$) of

$$\begin{aligned} v(k) &= (i/\hbar)[\varepsilon(k), i(\partial/\partial k)] \\ &= \partial\varepsilon(k)/\hbar\partial k = \nu b \sin(kb). \end{aligned} \quad (4)$$

If an electric field F were applied to the system, then the Hamiltonian of the single electron pair would be represented by

$$h(F) = \varepsilon(k) - iqF(\partial/\partial k), \quad (5)$$

and the eigenstates determine the well-known Stark ladders of a one-dimensional tight-binding Hamiltonian.⁵ The Stark ladder states are

$$h(F)\phi_m(k) = \varepsilon_m \phi_m(k), \quad (6a)$$

$$\phi_m[k + (2\pi/b)] = \phi_m(k), \quad (6b)$$

$$\phi_m(k) = e^{i[kbm + (\hbar\nu/qFb)\sin(kb)]}, \quad (6c)$$

$$\varepsilon_m = m\hbar\omega_F = mqFB, \quad m = 0, \pm 1, \pm 2, \dots, \quad (6d)$$

where ω_F is the Stark ladder frequency of the energy levels in Eq. (6d).

There are two important points to be made about the properties of this one electron pair model: (i) The model assumption is that the electron pair as a quantum mechanical object. (ii) The Stark ladder frequency can (nevertheless) be described in quasiclassical terms. Suppose, in a classical fashion, we equate the rate of change of "crystal" momentum $\hbar k$ to the force qF on the pair due to the electric field:

$$\hbar dk/dt = qF. \quad (7)$$

Then, Eqs. (4) and (7) yield a classical oscillation frequency in the electron pair-group velocity, precisely equal to that obtained by the spacing between the quantum mechanical Stark ladder levels:

$$v(t) = \nu b \sin(\omega_F t), \quad (8a)$$

$$\omega_F = qFb/\hbar. \quad (8b)$$

While a quasiclassical viewpoint can indeed be invoked to explain Eq. (8b), the quantum mechanical viewpoint of

Eqs. (6) yields a better insight into the true physical picture.

III. ONE-DIMENSIONAL ARRAYS

In Fig. 2 is shown a one-dimensional array of weak links between bulk superconductors. A classical "phase-locked" state of N -weak links is present if there is a unique phase θ for the array as a whole. The classical energy W of a "phase-locked" state for N -identical weak links is given by

$$W(\theta) = -N\hbar\nu \cos(\theta), \quad (9)$$

where θ is the phase difference in the superconducting order parameter across each weak link. Comparing Eqs. (3) and (9) one notes the similarity between the one-electron pair problem and the problem in which there is a "Bose condensate of electron pairs" all in the same Bose state. The close analogy extends to the case of applying an electric field F , i.e. the analogue of Eq. (5) is given by

$$H(F) = -N[\hbar\nu \cos(\theta) + iqFb(\partial/\partial\theta)], \quad (10a)$$

$$F \equiv (i/qbN)[H(F), \theta]. \quad (10b)$$

We note that if capacitors and electric field energy were taken into account, the quantum electrodynamic commutators would involve different representations of the canonical commutation relations. As stated in the introductory sections these are here neglected. The phase here used is that of the Bloch states of the electron pairs in the condensate as a whole.

The energy levels which follow from Eq. (10a) may be found by solving for the "phase locked" macroscopic wave functions for Bose condensate as a whole, i.e.,

$$H(F)\psi_m(\theta) = E_m\psi_m(\theta), \quad (11a)$$

$$\psi_m(\theta + 2\pi) = \psi_m(\theta), \quad (11b)$$

yields

$$\psi_m(\theta) = \exp[i(m\theta + (\hbar\nu/qFb)\sin(\theta))], \quad (12a)$$

$$E_m = m\hbar\omega_0 - mqV, \quad m = 0, \pm 1, \pm 2, \dots, \quad (12b)$$

where the voltage V across the whole weak link array is given by



One-Dimensional Weak-Link Array
Weak-Link Spacing = b

FIG. 2. Shown is a one-dimensional array of weak links with a separation length b . The voltage across the whole array of N -weak links is given by $V = Nfb$, where F is the electric field.

$$V = NFb , \quad (13)$$

and $\omega_0 = qV/\hbar$ is the usual Josephson frequency corresponding to the total voltage across the one-dimensional array.

Thus, the spacing between phase-locked energy levels for a one-dimensional array of N -weak links $\hbar\omega_0$ is precisely N times the spacing between energy levels $\hbar\omega_F$ in a Stark ladder for the one Boson (single electron pair) model, i.e., $\omega_0 = N\omega_F$. The factor N is due to the Bose condensate acting collectively in the macroscopic quantum state describing phase locking. Equations (12) describe the macroscopic wave functions and energy levels corresponding to (classically) phase locked or (quantum mechanically) Bose (electron pair) condensed states.

IV. TWO-DIMENSIONAL ARRAYS

Here we consider a two-dimensional square array, as shown in Fig. 3, in a magnetic field B normal to the array plane. For the single Boson pair model, the two-dimensional tight-binding band now becomes

$$\varepsilon(\mathbf{k}) = -\hbar v [\cos(k_x b) + \cos(k_y b)] , \quad (14)$$

where b is the length of one square cell of the array. In order to include the magnetic field, one must replace the wave vector \mathbf{k} in Eq. (14) with a new magnetic wave vector \mathbf{K} whose commutation relations reflect the existence of \mathbf{B} .⁶ The magnetic field Hamiltonian for a single electron pair is then deduced from Eq. (14) by imposing the required magnetic commutation relations, i.e.

$$[K_x, K_y] = -iqB/\hbar c , \quad (15a)$$

and then writing

$$h = \varepsilon(\mathbf{K}) = -\hbar v [\cos(K_x b) + \cos(K_y b)] , \quad (15b)$$

$$K_y = (iqB/\hbar c) \partial/\partial K_x . \quad (15c)$$

The single-boson Hamiltonian of Eqs. (15) can be generalized to macroscopic phase-locked quantum states for the two-dimensional array using similar physical arguments which lead from Eq. (5) to Eq. (10a). If θ denotes the

phase for the columns of the array, and ϕ denotes the phase for the rows of the array, and one uses the two-dimensional array analogue of the single electron pair Eq. (15c), i.e.,

$$\phi = (iqBb^2/\hbar c) \partial/\partial \theta , \quad (16)$$

then an $M \times M$ square array ($N = M^2$) has the Hamiltonian

$$H(B) = -\hbar v N [\cos(\theta) + \cos(\phi)] . \quad (17)$$

Note that the quantum mechanical nature of θ and ϕ in Eq. (16), i.e., the uncertainty principle forbids the notion that the square array can exhibit phase locking both between rows and columns when a magnetic field B is applied. This magnetic field induced "quantum frustration" in locking both θ and ϕ is merely a consequence of the macroscopic Schrödinger equation

$$H(B)\psi(\theta) = E\psi(\theta) . \quad (18)$$

Using the expansion

$$\psi(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} , \quad (19)$$

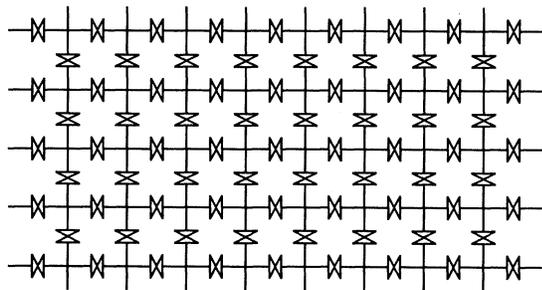
together with Eqs. (16), (17), and (18) we find that

$$-N\hbar v [\frac{1}{2}(a_{n+1} + a_{n-1}) + a_n \cos(nqBb^2/\hbar c)] = E a_n . \quad (20)$$

Our "quantum frustration" energy Eq. (20) is (of course) not new. The previously understood "classical frustration" for an "XY model" in mean-field theory gives precisely the same answer⁷ (which appears to us remarkable). Our electron pair Bose condensate macroscopic quantum picture, as stated in Sec. I, then appears to be an alternative view of the classical "frustrated" physical phase-locked states. However, our point here is that Eq. (18), or the equivalent Eq. (20), does not merely look like a macroscopic Schrödinger equation, but it really represents a macroscopic Schrödinger equation. No matter how well disguised in classical costumes, the actors (electron pairs in a magnetic field) move by quantum mechanical rules. This in no way detracts from the utility of classical circuit models in describing much of the experimental data in two-dimensional arrays, but it surely is a reminder to be aware of possible uniquely quantum macroscopic effects.

V. DISCUSSION

If two-dimensional arrays of weak links are indeed described by macroscopic quantum superposition of amplitudes, then it should be possible to obtain physical results which are not so easily described in classical circuit terms. We conclude our discussion with one such result, i.e., the Hall impedance R_H of a two-dimensional array in a magnetic field B . We consider the regime in which the array currents are well below the critical value so that the array does not dissipate. Nevertheless, a surface current per unit length (say J_x) will be associated with an electric field in a perpendicular direction (say F_y).



Two-Dimensional Array

FIG. 3. Shown is a two-dimensional weak link array.

To see what is involved, consider how our one-dimensional array Eq. (10b) is generalized using our two-dimensional array Hamiltonian $H(B)$ in Eq. (17), i.e.,

$$F_y = (i/qbN)[H(B), \phi] . \quad (21)$$

Equations (16), (17), and (21) imply

$$F_y = (vbB/c) \sin(\theta) . \quad (22)$$

On the other hand, the condensed electron pair (Josephson) surface current per unit length is given by

$$J_x = (qv/b) \sin(\theta) . \quad (23)$$

Equations (22) and (23) yield the superconducting array surface Hall impedance R_H . It is

$$F_y = R_H J_x , \quad (24a)$$

$$R_H = b^2 B / qc . \quad (24b)$$

Thus, Eqs. (24) for the array surface Hall impedance follow directly from the macroscopic quantum mechanical view of phase-locked states. To the authors knowledge there exists no present derivation of the Hall effect in superconducting weak link arrays from the viewpoint of classical circuit theory.

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