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A. Widom
Northeastern University

Y. N. Srivastava

C. Vittoria
Northeastern University

H. How

R. Karim

See next page for additional authors

Author(s)
A. Widom, Y. N. Srivastava, C. Vittoria, H. How, R. Karim, and H. Jiang

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A. Widom and Y. N. Srivastava
Physics Department, Northeastern University, Boston, Massachusetts 02115
and Physics Department, University of Perugia, Perugia, Italy

C. Vittoria, H. How, R. Karim, and H. Jiang
Department of Electrical Engineering, Northeastern University, Boston, Massachusetts 02115
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In the quantum-electrodynamic theory of vortex motions in type-II superconductors, the energy stored in the electric field must be added to the conventional energy in the magnetic field. The resulting quantum-electrodynamic Hamiltonian produces the appropriate quantum vortex dynamics. The implications of the theory for vortex-induced irreversibility in type-II superconductors are also explored.

I. INTRODUCTION

The nature of a vortex line in a type-II superconductor is of long-standing interest. To a first approximation, the vortex line may be viewed as a "string" with a tension (i.e., free energy per unit length) τ. In type-II superconductors, the tension τ can be measured in terms of the critical magnetic intensity H_{c1} for which it becomes favorable to create the vortex line inside the superconductor,[1] i.e.,

$$\tau=\phi_0 H_{c1}/4\pi \ , \tag{1a}$$

where

$$\phi_0=2\pi\hbar c/q \tag{1b}$$

is the magnetic-flux quantum and

$$q=2e \tag{1c}$$

is the electron-pair charge.

In Ginzburg-Landau theory, the tension τ can be calculated to be

$$\tau=(\phi_0/4\pi\lambda)^2[\ln(\kappa)+\eta] \ , \tag{2a}$$

where λ is the London penetration depth, ξ is the coherence length, and

$$\kappa=\lambda/\xi \ . \tag{2b}$$

The parameter η can be neglected in the extreme type-II limit ($\kappa \gg 1$) and has been calculated in other regimes[2] in which the details of how the order parameter goes to zero in the core of the vortex becomes important. If the London penetration depth λ is expressed in terms of the superfluid density n_s as

$$1/\lambda^2=4\pi q^2 n_s/Mc^2 \ , \tag{3a}$$

where

$$M=2m \tag{3b}$$

is the electron-pair mass, then the tension in Eqs. (2) can equally well be written as

$$\tau=(\pi\hbar^2 n_s/M)[\ln(\kappa)+\eta] \ . \tag{4}$$

(The superfluid density is related to the Ginzburg-Landau-model order parameter via $n_s=|\psi_{\rm GL}|^2$.)

The above well-known results concerning the vortex-string tension τ (in and by themselves) give no insights into the nature of vortex motions, e.g., string vibrations or vortex motions inducing dissipation in type-II superconductors. To describe such phenomena, previous work relied on "hydrodynamic models" coupled with the Lorentz force. In the work which follows, a quantum-electrodynamic approach to the problem will be employed. This implies that, in addition to the conventional estimates of the magnetic-field energies of vortices in superconductors, we will also include the electric-field energy,

$$K=(1/8\pi)\int d^3r\,|\mathbf{E}|^2 \ . \tag{5}$$

One effect (to be discussed in what follows) of the electric-field-energy storage is that the vortex string develops a mass per unit length,

$$\mu=(\phi_0/4\pi c\xi)^2 \ . \tag{6}$$

Since in quantum-electrodynamic theory

$$[E_i(\mathbf{r}),B_j(\mathbf{r}')]=4\pi i\epsilon_{ijk}\hbar c\,\partial_k\delta(\mathbf{r}-\mathbf{r}') \ , \tag{7}$$

the dynamics is fully determined by the Hamiltonian $H[\mathbf{E},\mathbf{B}]$ obtained from the electromagnetic-field energy. In what follows the effective Hamiltonian will be derived from the vortex-string Lagrangian.

In Sec. II the hydrodynamic theory[3] of vortex-string vibrations will be discussed in quantum-mechanical form. In Sec. III the fully quantum-electrodynamic theory will be presented, and the mass per unit length of the vortex string will be derived. In Sec. IV the Casimir effect for the vortex string will be presented, and the implied temperature dependence of the string tension will be discussed. In Sec. V the damping of vortex motions is computed. The theory is applied to the quantum-electrodynamic nucleation of vortices in the concluding Sec. VI.

II. HYDRODYNAMIC THEORY

Let us consider first a very thin film of superconductor of thickness b. As in Fig. 1, let us consider the action associated with moving a vortex with position coordinates (X,Y) around a closed path P in the plane of the film. Each of the electron pairs contained within the included area of the closed path picks up a phase $(\pm 2\pi)$ with a sign depending on the chirality of the vortex. In terms of the hydrodynamic action,

$$S_h = \pm 2\pi\hbar N_s = \pm \pi\hbar n_s b \int_P (Y\,dX - X\,dY) . \tag{8}$$

Equation (8) corresponds to a kinetic term in the Lagrangian

$$K_h = \pm \pi\hbar n_s b (Y\dot{X} - X\dot{Y}) . \tag{9}$$

For the case of a long vortex along the z axis, with a displacement $\mathbf{u}(z,t)$ from the equilibrium straight-line configuration, Eq. (9) becomes [with $\mathbf{u}=(u_x,u_y)$],

$$K_h = \pm \pi\hbar n_s \int dz (u_y \dot{u}_x - u_x \dot{u}_y) . \tag{10}$$

Defining the complex field

$$\Psi = \sqrt{\pi n_s}(u_x \mp i u_y) , \tag{11}$$

Eqs. (10) and (11) read (for a fixed chirality)

$$K_h = \frac{i\hbar}{2}\int dz \left[\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right] . \tag{12}$$

On the other hand, the energy required to displace a string with tension τ from a straight line along the z axis is given by

$$U = \frac{\tau}{2}\int dz \left| \frac{\partial \mathbf{u}}{\partial z} \right|^2 , \tag{13a}$$

$$U = \frac{\tau}{2\pi n_s}\int dz \left| \frac{\partial \Psi}{\partial z} \right|^2 . \tag{13b}$$

The hydrodynamic Lagrangian is thereby

$$L_h = K_h - U , \tag{14a}$$

$$L_h = \int dz \left[\frac{i\hbar}{2} \left[\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t} \right] - \frac{\hbar^2}{2M_0} \left| \frac{\partial \Psi}{\partial z} \right|^2 \right] , \tag{14b}$$

where

$$M_0 = (\pi\hbar^2 n_s/\tau) = \{M/[\ln(\kappa)+\eta]\} . \tag{15}$$

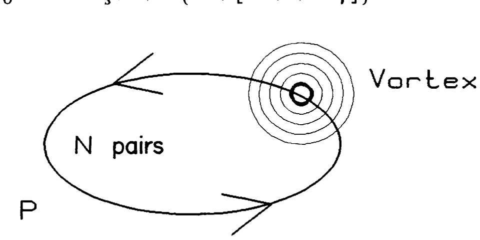

FIG. 1. When a vortex center moves around a closed path P, each of the N enclosed electron pairs in the condensate undergoes a 2π phase change leading to the action in Eq. (8).

The equation of motion for the complex vortex-string field in Eq. (11) follows from the Lagrangian in Eqs. (14) to be

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2M_0} \left[\frac{\partial}{\partial z} \right]^2 \Psi , \tag{16}$$

with a frequency spectrum of

$$\omega_Q = \hbar Q^2 / 2M_0 . \tag{17}$$

Equation (16) is the conventional hydrodynamic equation for vortex-string vibrations[3] when written in terms of $\mathbf{u}$ using Eq. (11). The advantage of the Lagrangian formalism here employed is that the quantization of the Ψ field becomes evident from the Lagrangian. For the quantized field

$$[\Psi(z), \Psi^\dagger(z')] = \delta(z-z') , \tag{18}$$

the hydrodynamic Hamiltonian is that of free bosons:

$$H_h = \frac{\hbar^2}{2M_0}\int dz \frac{d\Psi^\dagger(z)}{dz} \frac{d\Psi(z)}{dz} . \tag{19}$$

The energy spectrum of the hydrodynamic Hamiltonian is that of free bosons each with energy $\hbar\omega_Q$.

III. QUANTUM-ELECTRODYNAMIC THEORY

Let us here consider a straight-line vortex moving at velocity $\mathbf{V}$ normal to the lien of the vortex core. A moving vortex produces in the superconductor an electric field $\mathbf{E}$ as will now be shown. The electric field can be viewed as being a consequence of a changing (in time) of the superconducting phase θ,

$$\mathbf{E} = \frac{\hbar}{q} \text{grad} \left[\frac{\partial \theta}{\partial t} \right] \tag{20}$$

or, equivalently,

$$\mathbf{E} = -(\hbar/q)\text{grad}[\mathbf{V}\cdot\text{grad}(\theta)] , \tag{21}$$

where θ is in Eq. (21) the polar angle around the axis of the vortex. In magnitude,

$$|\mathbf{E}|^2 = \frac{(\hbar V/q)^2}{r^4} , \tag{22}$$

where r is the distance from a point in the superconductor to the axis of the vortex string.

The electric-field energy of a moving vortex is then given by

$$K = (1/8\pi)\int d^3r |\mathbf{E}|^2 = (b/8\pi)\int_\xi^\infty (2\pi r) dr |\mathbf{E}|^2 , \tag{23}$$

where b is the vortex length and the integral is cut off at the vortex-core radius ξ. In terms of the mass density (per unit length of vortex) μ,

$$K = \tfrac{1}{2} b\mu |\mathbf{V}|^2 , \tag{24a}$$

$$\mu = (\hbar/2q\xi)^2 . \tag{24b}$$

From Eqs. (24) it is seen that the electric-field-energy storage plays the role of a "mechanical" kinetic energy of the string.

For a displacement $\mathbf{u}(z,t)$ of the string from the straight-line position, as in Sec. II, the kinetic energy has the form

$$K=\frac{\mu}{2}\int dz\left|\frac{\partial \mathbf{u}}{\partial t}\right|^2 , \tag{25a}$$

$$K=\frac{\mu}{2\pi n_s}\int dz\left|\frac{\partial \Psi}{\partial t}\right|^2 . \tag{25b}$$

The total Lagrangian in the quantum-electrodynamic theory comes from adding the electric-field-energy storage to the hydrodynamic Lagrangian in Eqs. (14); i.e.,

$$L=\frac{1}{2\pi n_s}\int dz\left\{\mu\left|\frac{\partial \Psi}{\partial t}\right|^2\right\}+L_h . \tag{26}$$

With the transformation

$$\Psi=e^{i\pi\hbar n_s t/\mu}\psi , \tag{27}$$

the quantum-electrodynamic vortex-string Lagrangian becomes

$$L=\frac{1}{2\pi n_s}\int dz\left\{\mu\left|\frac{\partial \psi}{\partial t}\right|^2-\tau\left|\frac{\partial \psi}{\partial z}\right|^2-\frac{\pi^2\hbar^2 n_s^2}{\mu}|\psi|^2\right\} . \tag{28}$$

The Lagrangian in Eq. (28) is well known as describing a free-boson system in one dimension (along the string). The spectrum is given by

$$\Omega_Q=(v_0^2Q^2+\Omega_0^2)^{1/2} , \tag{29a}$$

where

$$v_0/c=(\tau/\mu c^2)^{1/2}=(1/\kappa)\sqrt{\ln(\kappa)+\eta} \tag{29b}$$

and

$$\Omega_0=\pi\hbar n_s/\mu=Mc^2/\hbar\kappa^2 . \tag{29c}$$

The hydrodynamic excitation spectrum is determined by

$$\omega_Q=\Omega_Q-\Omega_0=\hbar Q^2/2M_0+\cdots , \tag{30}$$

for $\Omega_0 \gg v_0 Q$, in agreement with Eq. (17).

The notion that the vortex string has a mass density μ per unit length has experimental consequences that have recently been discussed.[4] The quantum-electrodynamic approach discussed in this section is a natural way in which to understand this concept as arising directly out of the electric-field-energy storage.

IV. CASIMIR EFFECT

The entropy per unit length s of a vortex string is related to the way in which the tension (free energy per unit length) τ varies with temperature,

$$s=-\frac{d\tau}{dT} . \tag{31}$$

In laboratory experiments the entropy per unit length of a vortex line can be obtained from the way in which H_{c1} varies with temperature; i.e., Eqs. (1) and (31) imply

$$\frac{dH_{c1}(T)}{dT}=-\frac{4\pi}{\phi_0}s(T) . \tag{32}$$

The change in the free energy per unit length of the vortex string due to the oscillations can be computed using the excitation spectra

$$\tau(T)=\tau(0)+k_BT\int_{-\infty}^{\infty}\frac{dQ}{2\pi}\ln(1-e^{-\hbar\omega_Q/k_BT}) . \tag{33}$$

The entropy per unit length of the vortex string due to thermal excitation of vortex oscillations is then (as $T\to 0$) given by

$$s(T)/k_B=\tfrac{3}{2}\zeta(\tfrac{3}{2})(M_0k_BT/2\pi\hbar^2)^{1/2}+\cdots , \tag{34a}$$

where we are using the conventional definition of the ζ function,

$$\zeta(z)=\sum_{n=1}^{\infty}(1/n^z) . \tag{34b}$$

Apart from producing an added entropy term, the oscillations also produce an energy renormalization for vortex strings with finite length. From the Lagrangian Eq. (28) follows the zero-point fluctuation energy of a vortex string of length L,

$$E_0(\Omega_0)=2\sum_{n=1}^{\infty}(\hbar/2)[\Omega_0^2+(\pi v_0/L)^2n^2]^{1/2} . \tag{35}$$

Techniques for obtaining a finite renormalization from a divergent sum, as in Eq. (35), have been recently reviewed.[5] For example, for $\Omega_0=0$ one may employ ζ-function regulation to write Eqs. (34b) and (35) as

$$E_0(0)=(\pi\hbar v_0/L)\zeta(-1)=-(\pi/12)\hbar v_0/L . \tag{36}$$

For the problem at hand, we need a generalized ζ function defined by

$$Z(a,z)=\sum_{n=1}^{\infty}(a^2+n^2)^{-(z/2)} , \tag{37a}$$

$$Z(0,z)=\zeta(z) , \tag{37b}$$

so that Eqs. (35) and (37) read

$$E_0(\Omega_0)=(\hbar\Omega_0/a)Z(a,-1) , \tag{38a}$$

$$a=\Omega_0L/\pi v_0 . \tag{38b}$$

Using the Γ function

$$\Gamma(z)=\int_0^{\infty}\frac{dt}{t}t^ze^{-t} , \tag{39}$$

Eq. (37a) reads

$$Z(a,z)+1/2a^z$$
$$=[\tfrac{1}{2}\Gamma(z/2)]\sum_{n=-\infty}^{\infty}\int_0^{\infty}\frac{dt}{t}t^{(z/2)}\exp[-t(a^2+n^2)] . \tag{40}$$

The identity

$$\sum_{n=-\infty}^{\infty}\exp(-n^2t)=\sqrt{\pi/t}\sum_{m=-\infty}^{\infty}\exp(-\pi^2m^2/t) , \tag{41}$$

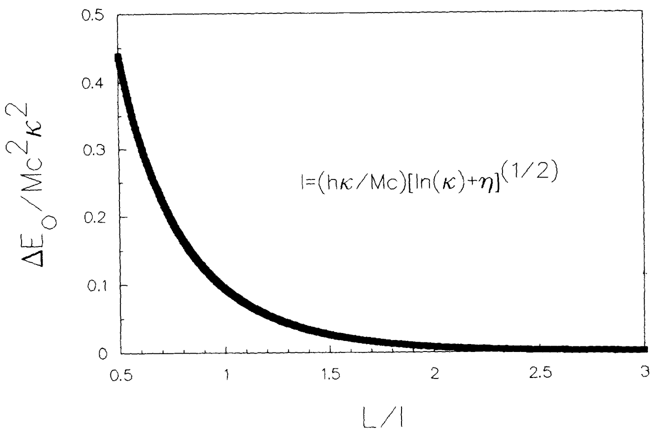

FIG. 2. Casimir energy ΔE_0 is shown as a function of the vortex length L in reduced units as in Eqs. (45) and (46).

combined with Eq. (40) yields

$$Z(a,z) = -1/2a^z + [\sqrt{\pi}\Gamma((z-1)/2)/2a^{(z-1)}\Gamma(z/2)] + W(a,z) , \tag{42}$$

where

$$W(a,z) = [\sqrt{\pi}/a^{(z-1)}\Gamma(z/2)] \times \sum_{m=1}^{\infty} \int_0^{\infty} \frac{dt}{t} t^{(z-1)/2} e^{-[(a^2 t)+(\pi^2 m^2/t)]} . \tag{43}$$

The finite size of a vortex string then has a Casimir-energy contribution determined by Eqs. (38), (42), and (43) to be

$$\Delta E_0 = (\hbar\Omega_0/a) W(a, -1) . \tag{44}$$

Equations (43) and (44) read, in a closed-form integral,

$$\Delta E_0 = \frac{\hbar v_0 a^2}{L} \int_0^{\infty} \frac{ds}{s^2} \frac{s^2-1}{e^{\pi a(s+s^{-1})}-1} . \tag{45}$$

The Casimir effect for finite length L is appreciable somewhat above the microscopic length scale

$$l = (\hbar\kappa/Mc)\sqrt{\ln(\kappa)+\eta} , \tag{46}$$

as shown in Fig. 2. The added energy for forming a vortex string of finite length is appreciable for very short strings in the $\kappa \gg 1$ limit.

V. VORTEX MOTION DAMPING

At nonzero temperatures the superconductor is expected to have a normal electron current which can give rise to Ohmic heating. The power generated by such currents is determined by

$$P = \sigma_n \int d^3r \, |\mathbf{E}|^2 . \tag{47}$$

From Eqs. (5) and (47) it follows that the heating rate is proportional to the vortex kinetic energy,

$$P = 8\pi\sigma_n K = 4\pi\sigma_n \mu b |\mathbf{V}|^2 , \tag{48}$$

where Eqs. (24) have been invoked. The drag force per unit vortex length of a vortex moving at velocity $\mathbf{V}$ follows from Eq. (48) to be

$$\mathbf{f} = -\eta_{\mathrm{drag}} \mathbf{V} , \tag{49}$$

where the drag coefficient

$$\eta_{\mathrm{drag}} = 4\pi\mu\sigma_n = \phi_0^2 \sigma_n / 4\pi c^2 \xi^2 = \pi\hbar^2 \sigma_n / q^2 \xi^2 . \tag{50}$$

Equation (50) is essentially the Stephen-Bardeen[6] result.

The quantum-electrodynamic picture of damped vortex motion can now be employed to discuss the motions of mesoscopic vortex excitations in type-II superconductors. For example, consider a vortex ring of radius R when there is a current density J normal to the area πR^2. The energy of such a configuration is given by

$$U(R,J) = 2\pi R \tau - (\phi_0 J/c)\pi R^2 . \tag{51}$$

The kinetic energy of the ring if it expands or contracts is given by

$$K(\dot{R},R) = \tfrac{1}{2}(2\pi R\mu)\dot{R}^2 . \tag{52}$$

The Lagrangian (for an expanding or contracting) vortex ring is then

$$L(\dot{R},R) = K(\dot{R},R) - U(R) . \tag{53}$$

The equation of motion (including damping) implied by Eqs. (49) and (53) is given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{R}} = \frac{\partial L}{\partial R} - 2\pi R \eta_{\mathrm{drag}} \dot{R} . \tag{54}$$

Explicitly,

$$\left[\frac{d}{dt}\right]^2 R + \frac{1}{2R}\left[\frac{dR}{dt}\right]^2 + 4\pi\sigma_n \frac{dR}{dt} + \frac{\tau}{\mu R} = \frac{\phi_0 J}{\mu c} . \tag{55}$$

From Eq. (55) it follows that the ring will expand from rest if $R > R_0$ and the ring will contract from rest if $R < R_0$, where the nucleation radius is given by

$$R_0 = c\tau/\phi_0 J . \tag{56}$$

From Eq. (51) it follows that the energy barrier to nucleating a vortex is given by

$$U_0(J) = U(R_0, J) = \pi c \tau^2/\phi_0 J . \tag{57}$$

For a current fed into a bulk type-II superconductor from a narrow channel, as in previous experiments,[7] the vortex-ring-nucleation mechanism gives reasonable values for observed critical currents. In narrow-channel systems the nucleation can take place near the end of the channel in the bulk of the superconductor when the channel has a length scale less than the penetration depth. The quantum-nucleation results will be discussed in the concluding Sec. VI.

VI. CONCLUSIONS

We have discussed the quantum-electrodynamic approach to the motions of vortices in type-II superconductors. In this conclusion let us consider the effects of such motions on the irreversible behavior of such superconductors using the vortex-ring example of Sec. V.

The nonlinear electric-field current-density relation implied by the vortex-ring-nucleation mechanism is given by

$$E=(J/\sigma_n)P(J)\ , \tag{58}$$

where $P(J)$ is the probability of nucleating a vortex ring. Since the nucleation potential in Eq. (51) is an inverted parabola and quantum tunneling with dissipation for such a model can be solved,[8] we merely quote the result; i.e., in the W.K.B.J. tunneling limit,

$$P(J)=e^{-U_0(J)/\hbar\Omega(J)}\ , \tag{59}$$

where $U_0(J)$ is defined in Eq. (57) and the "unstable inverted parabolic" frequency $\Omega(J)$ is given by

$$\Omega(J)=2\pi\sigma_n+[4\pi^2\sigma_n^2+(\phi_0^2J^2/c^2\tau\mu)]^{1/2}\ . \tag{60}$$

There are two regimes of interest: (i) In the low-damping limit in which $J \gg (c\sigma_n\sqrt{\tau\mu}/\phi_0)$, Eq. (58) reads

$$\sigma_n E=J\exp[-(J_{\text{undamped}}/J)^2]\ , \tag{61a}$$

$$J_{\text{undamped}}=\sqrt{\pi/15\hbar}(8c/\phi_0)\tau^{5/4}(2\mu)^{1/4}\ . \tag{61b}$$

(ii) In the high-damping limit $(c\sigma_n\sqrt{\tau\mu}/\phi_0) \gg J$, Eq. (58) reads

$$\sigma_n E=J\exp(-J_{\text{damped}}/J)\ , \tag{62a}$$

$$J_{\text{damped}}=(c\tau^2/4\hbar\phi_0\sigma_n)\ . \tag{62b}$$

The above calculations are illustrative of the manner in which a quantum-electrodynamic view of vortex motions allows for the computations of irreversible voltage-current characteristics by means of mesoscopic quantum nucleation of vortex lines. Quantum tunneling away from vortex-line pinning centers can proceed in an analogous fashion once the effective Lagrangian for such pinning sites is understood. The view is complementary to that employed in thermal-activation models for temperatures near critical. However, we expect vortex motions for temperatures $T \ll T_c$ to also be present.

ACKNOWLEDGMENTS

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