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Persistence of Screening and Self-Criticality in the Scale Invariant Dynamics of Diffusion Limited Aggregation

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The origin of fractal properties in diffusion limited aggregation is related to the persistence of screening in the scale invariant growth regime. This effect is described by the effective noise reduction parameter S spontaneously generated by the scale invariant dynamics. The renormalization of this parameter under scale transformation shows the following: (i) The fixed point is attractive, implying the self-critical nature of the process. (ii) The fixed point value S^* is of the order of unity, showing that the small scale growth rules are already close to the scale invariant ones and that screening effects persist in the asymptotic regime.

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Diffusion limited aggregation (DLA) and the dielectric breakdown model (DBM) are considered as prototype fractal growth models for several physical phenomena [1] ranging from electric discharge to cluster growth by accretion of diffusing particles. These models are rather simple and lead to very complex fractal patterns, which are the result of a self-organizing stochastic process governed by the Laplace equation. The DLA and DBM have been objects of intense studies in the past years and, from a theoretical point of view, one can formulate the following hierarchy of problems: (a) Why DLA gives rise to fractal structures. (b) How to compute the fractal dimension D analytically. (c) Understanding also properties that go beyond the simple fractal dimension, like morphology, self-affinity and anisotropy, topological properties, eventual multifractality of the growth probability, and several others [1].

A number of authors have proposed several theoretical approaches [1] to DLA and DBM; however, the extension of standard theoretical methods to these problems turned out to be rather problematic [2]. It is for this reason that some years ago we introduced the new approach of the fixed scale transformation (FST) [3]. In this method one studies the properties of the system under dynamical evolution at a given scale. If the fixed point corresponding to this dynamical evolution is the same at all scales, then it is possible to derive the fractal dimension. In principle this would imply the use of asymptotic scale invariant growth rules [4–6]; however, consistent results for the fractal dimension could be obtained already with the use of the small scale growth rules. In this sense the original FST method allowed one to make some progress on point (b) but the lack of control on the scale invariant dynamics did not allow a proper treatment of point (a). In this Letter we address this question by characterizing some essential properties of the scale invariant dynamics.

These models are intrinsically critical in the sense that their dynamics evolves in the scale invariant one without tuning any parameter. However, the lack of a characteristic length in the original (small scale) growth rules does not guarantee that these hold at a generic scale. Therefore, the question of the universality and scale invariant properties [point (a)] of DLA and DBM are related to their effective asymptotic dynamics. In addition the knowledge of this effective dynamics is one of the key points in the understanding of why these models give rise to fractal structures. This problem is in general very complex due to the large number (in principle infinite) of parameters that would appear in the renormalized dynamics. In practice, however, one can fix a subset of parameters and study their evolution under scale change in the highly complex space of growth rules. This procedure is somewhat arbitrary because these restrictions are selected a priori and it is hard to make the approach systematic from this point of view. Some approaches along this line are nevertheless instructive [4,7–10].

In order to answer point (a) the FST framework points, however, to the essential concept: Fractals can be originated only if *screening* persists in the scale invariant regime [3]. Note that the presence of screening due to the Laplace equation in the original growth rules does not guarantee that a similar effect persists for coarse grained variables. In fact, if one studies the growth rules for a coarse grained cell, a problem of noise reduction [11,12] naturally appears. The larger the cell, the larger will be the number of particles necessary to span it. Naively therefore one could expect that, asymptotically, the effective noise reduction parameter (S) could diverge. This would then eliminate screening effects and lead to a compact structure. Therefore the key feature of the asymptotic growth rules with respect to point (a) is the identification of the *fixed point noise reduction parameter.*

The behavior of the noise reduction parameter has been up to now studied with respect to the overall shape of the cluster or with respect to the anisotropy problem in a square lattice [13]. It should be noted that the overall shape morphology and the internal fractal properties of the clusters are two different problems that have sometimes been confused, especially in relation to simulations

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in radial geometry. The overall morphology is linked to the velocity of propagation of the growing interface while the fractal properties should refer to the nature of the structure that is left behind asymptotically. The internal fractal properties are in fact similar for both lattice and off-lattice DLA and this becomes particularly clear in the cylinder geometry. Our present studies as well as the FST approach refer to the intrinsic fractal properties (and not to the overall shape of the radial DLA) that are similar for lattice and off-lattice growth. A discussion of the difference between the intrinsic fractal properties and the overall shape, that is instead lattice dependent, can be found in Ref. [2].

The problem of the asymptotic behavior of the noise reduction parameter S will be addressed by constructing a scheme of renormalization in real space. The main results are that the fixed point noise reduction parameter S^* turns out to be *of the order of unity* with an *attractive* fixed point. These results clarify therefore the self-critical nature of DLA patterns, opposite, for example, to percolation in which the fixed point p_c is repulsive [5,6]. This is due to the fact that, under scale change, noise is automatically generated by the dynamics of the system and it implies that a noise reduced DLA model flows essentially into the standard DLA process with respect to fractal properties. This is in contrast to the usual belief (that refers, however, only to the overall shape) that noise reduction accelerates the approach to the asymptotic behavior [11,13], but it is in agreement with recent studies that refer instead to the fractal properties [12].

In addition, the fact that the value of S^* is close to 1 shows that screening is asymptotically preserved and that the minimal scale growth rules are already rather close to the asymptotic ones. This allows one therefore to understand why the initial FST studies of DLA, in which the minimal scale growth rules were used, give indeed reasonable results for the fractal dimension [3]. Further details in the scale invariant growth rules would affect the precise value of D but not the fact that the resulting structure is fractal.

In the noise reduction generalization of the DLA and DBM growth rules a bond is grown only after having been hit by S particles [11]. A counter is raised by one each time a particle hits the respective bond. When a counter reaches the value S the corresponding bond is occupied and the new bonds near this one have their counters set to zero. The effect of this procedure is a systematic reduction of the noise. In fact the introduction of the parameter S corresponds to averaging over several realizations of the same stochastic process. This reduces the fluctuations and introduces, through the counters, a memory effect. For a finite value of S the branches acquire a finite thickness, while for $S \rightarrow \infty$ screening is suppressed and the structure is compact [11,12].

If one considers the original DLA process with $S=1$ at the minimal scale, it is clear that a nontrivial noise reduction parameter appears for a coarse grained cell because many bonds are needed to span the large cell. The problem is therefore to study the evolution of the effective noise reduction parameter as a function of scale transformation. The arrival of each particle corresponds actually to a Poisson process. If S particles have to arrive in order to lead to growth within a time t, one can show that [7]

$$1/S = \langle \delta t^2 \rangle / \langle t \rangle^2 . \tag{1}$$

For a coarse grained bond one also has to consider its internal structure. If the noise reduction parameter at the fine scale is S, the growth of the coarse grained bond corresponds to the superposition of different Poisson processes, each with a certain number of particles $N(S)$ and with the associated probability. One can then define the effective noise reduction parameter S' at the larger scale by using Eq. (1) to define the effective noise reduction at a generic scale. This leads to [7]

$$\frac{1}{S'} = \frac{1}{\langle N(S) \rangle} + \frac{\langle [\delta N(S)]^2 \rangle}{\langle N(S) \rangle^2} . \tag{2}$$

The key point is therefore to define the appropriate renormalization scheme in order to compute the averages that appear in Eq. (2). In order to do this analytically it is convenient to use the usual cell-to-bond transformation of Fig. 1. We define the following renormalization group (RG) transformation rules on the cell of Fig. 1: (i) The cell not spanned vertically by grown (thick) bonds is renormalized to a vertical perimeter (thin) bond. (ii) The cell spanned vertically by grown bonds is renormalized to a vertical grown (thick) bond.

The simplest way to define the RG transformation in practice is to consider only the renormalization along the vertical direction. In this case the renormalization transformation is exactly that of Eq. (2). In practice we have to consider the configurations renormalized in a perimeter bond and compute the quantities $\langle N(S) \rangle$ and $\langle N^2(S) \rangle$ corresponding to spanning the cell for each configuration. One therefore has to consider all the paths that span the cell, weighting them with the probabilities of the corresponding growth processes. In addition, all the possible configurations of the small scale counters have to be considered. Finally this process has to be repeated for all the possible starting configurations of the considered cell. These starting configurations are shown in Fig. 2 and are related to each other by the growth process [8]. The

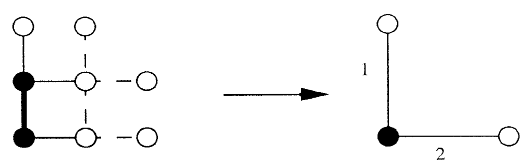

FIG. 1. Cell-bond renormalization scheme. The thick lines denote the growing bonds.

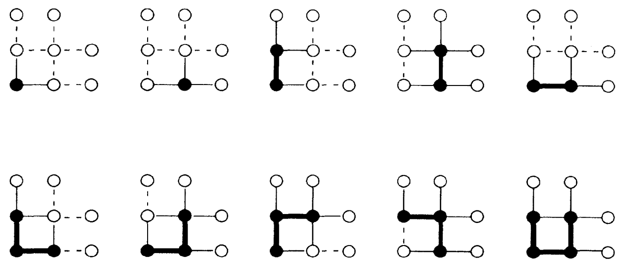

FIG. 2. There are ten different occupation configurations for a 2×2 cell. They are related to each other by the growth process. The thick black lines correspond to occupied bonds while the thin lines represent the bonds on which growth can occur.

weight $W(a)$ of a given starting configuration (a) can therefore be related to the probabilities of the elementary growth processes.

We can write therefore

$$\langle N(S)\rangle = \sum_a W^{(a)}(S) \sum_i P_i^{(a)}(S) N_i^{(a)}(S) \tag{3}$$

in which the index i refers to a particular path whose total probability is given by $P_i^{(a)}$ and $N_i^{(a)}$ is the corresponding total number of particles. An analogous equation can be written for $\langle N^2(S)\rangle$.

Each path leading to $P_i^{(a)}(S)$ is made by a sequence of elementary bond probabilities $p_j(S)$ that depend on the particular configuration (j). For example if we have a competition between two bonds, 1 and 2 as, for example, in the right part of Fig. 1, these will be characterized by the corresponding Laplacian potentials ϕ_1 and ϕ_2, which give the probabilities that a random walk reaches the corresponding bond. The probability $p_1(S)$ that, with a noise reduction parameter s, the bond 1 will grow before bond 2 can be written as

$$p_1(S) = \frac{1}{N} \sum_{k=0}^{S-1} \binom{S+k-1}{k} \left(\frac{\phi_1}{\phi_1+\phi_2}\right)^S \left(\frac{\phi_2}{\phi_1+\phi_2}\right)^k, \tag{4}$$

where N is the normalization factor. This expression corresponds to the probability that $S-1$ particles arrive at bond 1, while any number between 0 and $S-1$ arrives at bond 2, and finally the last particle reaches the bond 1.

This formalism can be naturally generalized to more complex structures that involve more than two growing bonds by using the multinomial generalization. It is also possible to extend the results to noninteger values of S by analytical continuation [14].

In this way it is possible to compute analytically the renormalization transformation given by Eq. (2). In fact, solving the Laplace equation for each configuration of the starting cell and spanning path, it is possible, through Eq. (4), to calculate the probabilities $W^{(a)}(S)$ and $P_i^{(a)}(S)$. Finally, $\langle N^2(S)\rangle$ and $\langle N(S)\rangle$ are obtained by Eq. (3). The details of the calculation will be reported elsewhere [14] while here we only discuss the results. These are essentially contained in Fig. 3, which shows the group equation $S'=R(S)$ that describes the change of the noise reduction parameter under scale transformation.

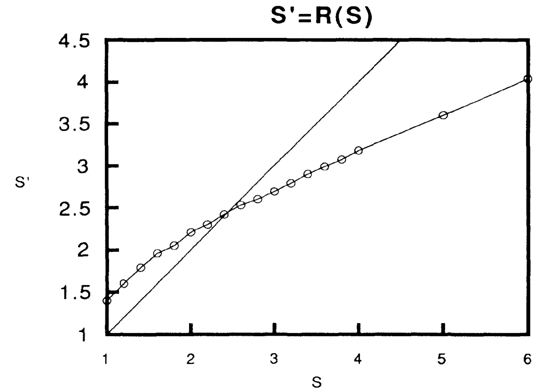

FIG. 3. Renormalization equation for the noise reduction parameter S' at the coarse grained scale as a function of the noise reduction parameter S of the finer scale. One can see that the fixed point is *attractive* and that the value of S^* is of the order of unity, implying that screening is preserved to the asymptotic scale.

The main results are the following:

(a) The fixed point for the noise reduction parameter S is *attractive*. This allows us to understand the self-critical nature of the dynamics of DLA because a large degree of noise is spontaneously generated even if one would start from a quasideterministic process (large S) at small scale.

(b) The fixed point value is $S^* \simeq 2.4$, therefore of the order of unity. This guarantees that screening is preserved to all scales in the asymptotic regime. The persistence of screening in the scale invariant dynamics of the growth process allows us to understand *why* the DLA process actually leads to a fractal structure. In addition the small value of S^* implies that the scale invariant growth rules are actually rather close to the small scale ones (in the bond or DBM version [4]). This allows us now to clarify why the FST method gives rather good results even with the standard small scale growth rules [3]. In principle this is not obvious because the scale invariant growth rules represent the nonuniversal critical parameters that should actually be used in the FST in order to derive the fractal dimension in a consistent way [5]. Preliminary calculations show that the use of the scale invariant dynamics in the FST approach leads to a value of the fractal dimension $D\simeq 1.70$ [14]. This should be compared with the value $D=1.64$ previously obtained using the small scale growth rules [3].

It should be noted that the results of Fig. 3 are very stable with respect to improved schemes to compute the renormalization transformation. For example we have considered larger cell schemes and the possibility of different noise reduction parameters for the growth direction (S_1) or the lateral direction (S_2). The results of this more complex scheme are essentially similar to those of Fig. 3 [14].

In order to understand the self-organized nature of the DLA process and the fact that S^* is of order of unity it is useful to consider the following reasoning. Suppose that we start at a small scale with a large value of S (small noise) and we study how this value will change under scale transformation. Given that at the small scale the probability distribution that it takes exactly a time t to occupy a bond is

$$W(S,t)=e^{-\lambda t}\frac{(\lambda t)^{(S-1)}}{(S-1)!}\lambda\,dt\,, \qquad (5)$$

we intend to estimate the probability distribution to span a cell and occupy a coarse grained bond.

The different occupation configurations of a cell (Fig. 2) will lead to various possible paths of different lengths appearing in Eq. (3). Typically for a 2×2 cell there will be paths of lengths of 1, 2, 3, and 4 steps, depending on the path and on the starting configuration. For simplicity we consider only the possibility of lengths of 2 and 3 with probability, respectively, of p and $1-p$. This would lead to a time distribution for the occupation of the coarse grained bond given by

$$\tilde{W}(S',t)=pW(2S,t)+(1-p)W(3S,t)\,. \qquad (6)$$

For large values of S this distribution will consist of two highly peaked functions at positions $2S$ and $3S$. Just the fact of having two peaks introduces a fluctuation in $\tilde{W}(S',t)$ that is larger than the intrinsic fluctuations of $W(2S,t)$ and $W(3S,t)$. Therefore if p is appreciably different from zero (this is actually unavoidable if different configurations are involved) we can treat these distributions as δ function. An upper limit for S can therefore be easily computed as

$$S'=\langle t\rangle^2/\langle\delta t^2\rangle\le(3-p)^2/p(1-p)\,. \qquad (7)$$

This shows that the value of S' becomes immediately of the order of unity even if the starting value of S at the lower scale is very large. The key point is that as soon as there are different paths the fluctuations between the lengths of these paths are necessarily of the order of the path lengths. This implies that $\langle\delta t^2\rangle$ is of the order $\langle t^2\rangle$ independently of the value of S at the smaller scale. In this way it is possible to understand how the DLA dynamics intrinsically generates a large noise asymptotically even if the small scale dynamics is characterized by a small noise. The fact that the results of our renormalization schemes can be understood in this simple way provides strong support for their general validity.

The main conclusions of this paper are therefore the following: From the FST approach to fractal growth one can see, in a mathematically controllable way, that the necessary condition for a growth model in order to generate fractal structures (as opposite to compact ones) is the persistence of screening in the asymptotic dynamics for coarse grained cells. For DLA the identification of the asymptotic dynamics is in general quite complex. In this respect we identify the critical parameter that governs the screening as the effective noise reduction. This is spontaneously generated in the coarse graining process for the growth rules. However, we show here by a renormalization analysis that not only does this parameter not diverge (this would eliminate screening), but its fixed point is attractive at a value of the order of unity. This demonstrates the self-critical nature of the process and it shows that the small scale growth rules are already rather close to the asymptotic ones.

[1] T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1992).

[2] M. Ben Amar, P. Pelcé, and P. Tabeling, *Growth and Form, Nonlinear Aspects,* NATO ASI Ser. B, Vol. 276 (Plenum, New York, 1991), see L. Pietronero, p. 345.

[3] L. Pietronero, A. Erzan, and C. Evertsz, Phys. Rev. Lett. **61**, 861 (1988); Physica (Amsterdam) **151A**, 207 (1988).

[4] R. De Angelis, M. Marsili, L. Pietronero, A. Vespignani, and H. J. Wiesmann, Europhys. Lett. **16**, 417 (1991).

[5] L. Pietronero, Physica (Amsterdam) **191A**, 85 (1992).

[6] L. Pietronero and A. Stella, Physica (Amsterdam) **170A**, 64 (1990).

[7] P. W. Barker and R. C. Ball, Phys. Rev. A **42**, 6289 (1990).

[8] X. R. Wang, Y. Shapir, and M. Rubinstein, J. Phys. A **22**, L507 (1989); Phys. Rev. A **39**, 5974 (1989).

[9] T. Nagatani, J. Phys. A **20**, L381 (1987); Phys. Rev. A **36**, 5812 (1987).

[10] X. Wang and Y. Huang, Phys. Rev. A **46**, 5038 (1992).

[11] J. Kertesz and T. Vicsek, J. Phys. A **19**, L257 (1986); J. Nittmann and H. E. Stanley, Nature (London) **321**, 663 (1986).

[12] C. Moukarzel, Physica (Amsterdam) **188A**, 469 (1992).

[13] J. P. Eckmann, P. Meakin, I. Procaccia, and R. Zeitak, Phys. Rev. Lett. **65**, 52 (1990); Phys. Rev. A **39**, 3185 (1989).

[14] R. Cafiero, R. De Angelis, L. Pietronero, A. Vespignani, and S. Zapperi (to be published).