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# Permeability tensor of cubic arrays of magnetic spherical particles

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We have calculated the Polder permeability tensor of a composite consisting of cubic arrays of magnetic spherical particles. Magnetic potentials associated with particle-particle interactions have been expressed in terms of multipole fields. The permeability tensor can be then expanded in power series of the particle volume loading. These results are in agreement with other method of calculation involving the free energy of a particle composite.

## INTRODUCTION

The study of composite materials has been the subject of considerable interest in recent years because of possible technological applications of these materials. In the past we have formulated the free energy of a magnetic composite upon which the Polder permeability tensor was calculated.<sup>1</sup> In that approach (static) particle-particle interactions are included in the free energy of a particle and the permeability tensor was derived in a straightforward manner. The classical Clausius-Mossotti problem as related to the average conductivity of a metal-dielectric composite has been considered by Rayleigh<sup>2</sup> and by Doyle.<sup>3</sup> In this paper we apply a multipole expansion method to calculate the Polder tensor of a composite consisting of magnetic spherical particles. We have generated the permeability of the composite analytically as a power series of the loading volume of the particles. The value of single-particle polarizability is then treated as a parameter. Permeability values derived in this manner prove to be more convenient in applications regarding magnetic composites.

## CALCULATIONS

Consider a composite consisting of magnetic spheres dispersed in cubic lattices. Assume the matrix binder is nonmagnetic. Let the sphere radii be  $a$  and the lattice constant be  $\alpha$ . A dc field  $H_0$  is applied in the  $z$  direction which saturates the magnetization of the particles. We consider first the case of simple cubic lattice. Results for other cubic lattices (body centered and face centered) can be directly generalized. Under magnetostatic approximation the magnetostatic potential  $\psi$  satisfies the Walker equation within the particles and the Laplace equation when outside the particles.<sup>4</sup>  $\psi$  can be written as follows:

$$\begin{aligned} \psi(\mathbf{r}) &= \sum_{n,m} C_{nm} R_n^m(\mathbf{r}; \kappa), \quad \text{for } r < a, \\ &= \sum_{n,m} (A_{nm} r^n + B_{nm} r^{-n-1}) P_n^m(\cos \theta) e^{im\phi}, \\ &\quad \text{for } r > a, \end{aligned}$$

where  $P_n^m$  is the associated Legendre function and

$$P_n^{-m} \equiv (-1)^m P_n^m.$$

$R_n^m$  denotes the Walker modes within the spheres which may be shown to have the following form:

$$R_n^m(\mathbf{r}; \kappa) = \frac{(-i)^m}{2\pi} e^{im\phi} \int_{-\pi}^{\pi} P_n \left[ \frac{r}{a} \left( \beta \cos \theta - \frac{i \sin \theta}{\sqrt{\kappa}} \cos \eta \right) \right] \cos m\eta d\eta,$$

where  $m$  and  $n$  are integers with  $n \geq 0$  and  $|m| \leq n$ ,  $P_n$  is the Legendre polynomial of order  $n$ , and  $\beta$  is defined as

$$\beta \equiv (1 + 1/\kappa)^{1/2}.$$

The Polder tensor elements  $\kappa$  and  $\nu$  are given by

$$\kappa = \Omega_H / (\Omega_H^2 - \Omega^2), \quad \nu = \Omega / (\Omega_H^2 - \Omega^2),$$

and the normalized frequencies  $\Omega$  and  $\Omega_H$  are defined as

$$\Omega = \omega / 4\pi\gamma M_s, \quad \Omega_H = H_i / 4\pi M_s.$$

$\omega$  is the applied frequency,  $\gamma$  the gyromagnetic ratio,  $4\pi M_s$  the saturation magnetization,  $H_i$  the internal dc field given by

$$H_i = H_o - 4\pi M_s / 3,$$

and  $H_o$  is the external field. The boundary conditions require  $\psi$  and  $\hat{\mathbf{r}} \cdot \mathbf{b}$  to be continuous across the particle surface. This implies

$$B_{nm} = D_{nm} A_{nm},$$

with

$$D_{nm} = \frac{n - mv - [\beta P_n^{|m|}(\beta) / P_n^{|m|}(\beta)]}{n + 1 + mv + [\beta P_n^{|m|}(\beta) / P_n^{|m|}(\beta)]}.$$

Here  $P_n^{|m|}(\beta)$  denotes derivative of  $P_n^{|m|}(\beta)$  with respect to  $\beta$ . Relationship between  $C_{nm}$  and  $A_{nm}$  can also be deduced.

Let circularly polarized external rf field be applied. Denote  $+$  ( $-$ ) as the right-(left-) hand polarization. The static potential outside the particle can be then written as

$$\begin{aligned} \psi^{(\pm)}(\mathbf{r}) &= (x \pm iy) + \sum_{n,m} [(A_{nm}^{(\pm)} r^n \\ &\quad + B_{nm}^{(\pm)} r^{-n-1}) P_n^m(\cos \theta) e^{im\phi}], \end{aligned}$$

where  $B_{nm}^{(\pm)} = D_{nm} A_{nm}^{(\pm)}$ . In view of the Green's theorem developed by Rayleigh discussing the average conductivity in a metal-insulator composite,<sup>2</sup> one may deduce directly the Polder polarization tensor in the present case of circular polarization as

$$\chi^{(\pm)} = -B_{1\pm 1}^{(\pm)}/\alpha^3.$$

The cross polarization for the left- and right-hand polarizations is zero, since the two polarizations are the diagonalized states in the present case. The permeability tensor is

$$\vec{\mu} = \vec{1} + 4\pi \vec{\chi}.$$

Note that  $\vec{\mu}$  is Hermitian for loss-less materials.

We require  $\partial\psi/\partial z = 0$  such as to have transverse  $\mathbf{m}$ . This implies  $A_{nm}^{(\pm)}$  and  $B_{nm}^{(\pm)}$  vanish unless  $m = \pm n$ . Furthermore, it will become clear later that the fourfold symmetry of the lattice structure restricts  $n$  to be of the form  $n = 4p + 1$  for integer values of  $p$ . According to Rayleigh<sup>2</sup> the field components having positive exponents arise from those terms having negative exponents associated with other particles. Therefore one may write

$$\begin{aligned} \mp (x \pm iy) \pm A_1^{(\pm)}(x \pm iy) \pm A_5^{(\pm)}(x \pm iy)^5 + \dots \\ = \sum_i \left( \pm B_1^{(\pm)} \frac{(x' \pm iy')}{r'^3} \pm B_5^{(\pm)} \frac{(x' \pm iy')^5}{r'^{11}} + \dots \right), \end{aligned} \quad (1)$$

where  $A_n^{(\pm)}$  and  $B_n^{(\pm)}$  are defined as

$$A_n^{(\pm)} \equiv (2n - 1)!! A_{n\pm n}^{(\pm)} \quad B_n^{(\pm)} \equiv (2n - 1)!! B_{n\pm n}^{(\pm)}$$

and  $x', y', z'$ , and  $r'$  are given by

$$x' = x - \xi_b \quad y' = y - \eta_b \quad z' = z - \zeta_b$$

$$r' = (x'^2 + y'^2 + z'^2)^{1/2}$$

with  $(\xi_b, \eta_b, \zeta_b)$  being the cubic lattice point. The summation in Eq. (1) is over all the lattice points  $i$  except the one located at the origin,  $i = 0$ . Coefficients  $A_1^{(\pm)}, A_5^{(\pm)}, \dots$ , can be therefore found by applying repetitive differentiation  $\partial/\partial(x \pm iy)$ ,  $\partial^5/\partial(x \pm iy)^5, \dots$ , respectively, on both sides of Eq. (1) followed by setting  $x$  and  $y$  equal to zero. We define the following lattice harmonics

$$\begin{aligned} F_{nm}^{(\pm)} &\equiv \sum_i \left[ \left( \frac{\partial}{\partial(x \pm iy)} \right)^m \frac{(x' \pm iy')^n}{r'^{2n+1}} \right]_{x=0, y=0} \\ &= \sum_i \left( \frac{(-1)^m (2m - 1) 2^{-m}}{P_{n+m}^{n-m}(1)} \right) \\ &\quad \times \frac{P_{n+m}^{n-m}(\cos \theta_i) \exp[\pm i(n - m)\phi_i]}{r_i^{n+m+1}}, \end{aligned} \quad (2)$$

where  $(r_b, \theta_b, \phi_b)$  is the polar coordinate of lattice point  $(\xi_b, \eta_b, \zeta_b)$ . The symmetry of the problem will reduce the imaginary part of Eq. (2) to zero and hence

$$F_{nm}^{(+)} = F_{nm}^{(-)} \equiv F_{nm}$$

$F_{nm}$  vanishes unless  $(n - m)$  is a integer multiple of 4. Therefore, one obtains

$$\begin{aligned} -1 + A_1^{(\pm)} &= F_{11} B_1^{(\pm)} + F_{51} B_5^{(\pm)} + F_{91} B_9^{(\pm)} + \dots \\ 5! A_5^{(\pm)} &= F_{15} B_1^{(\pm)} + F_{55} B_5^{(\pm)} + F_{95} B_9^{(\pm)} + \dots \end{aligned}$$

$$9! A_9^{(\pm)} = F_{19} B_1^{(\pm)} + F_{59} B_5^{(\pm)} + F_{99} B_9^{(\pm)} + \dots$$

and the coefficients  $A$ 's and  $B$ 's can be solved as accurate as desired by involving many multipoles in the above expansion. We normalized all the distances with respect to the lattice constant  $\alpha$  and define

$$d_n^{(\pm)} \equiv -D_{n\pm n} \cdot a^{2n+1} = 1 + (2 + 1/n)(\Omega_H \mp \Omega).$$

$\chi^{(\pm)}$  can be then solved as

$$\begin{aligned} (\chi^{(\pm)})^{-1} &= d_1^{(\pm)} \cdot a^{-3} + F_{11} - \frac{F_{51} F_{15}}{120 d_5^{(\pm)} + F_{55} \cdot a^{11}} \cdot a^{11} \\ &\quad - \frac{F_{91} F_{19}}{362880 d_9^{(\pm)}} \cdot a^{19} + O(a^{27}). \end{aligned} \quad (3)$$

As noted by Rayleigh,<sup>2</sup> value of  $F_{11}$  is not unique, which depends on the manner that the summation is evaluated.  $F_{11}$  is expressed as

$$F_{11} = \sum_i (3\xi_i^2 - r_i^2)/2r_i^5,$$

which incidentally has the same expression as the parameter  $S_2$  defined in Rayleigh's paper. If  $F_{11}$  is calculated for a space bounded by a cube, it vanishes. If one takes the  $z$  axis as the elongated direction, as in the case considered for the calculation of the average conductivity in a metal-dielectric composite,<sup>2</sup> the summation should be designated as over  $-u < \xi_i < u$ ,  $-u < \eta_i < u$ , and  $-\infty < \zeta_i < \infty$ , and  $u$  is a very large positive number. In this case Rayleigh derived  $S_2$  to be  $2\pi/3$ . However, in the present case we should have the transverse direction ( $x$  or  $y$ ) as the elongated direction. After using similar method one may deduce  $F_{11} = -\pi/3$ . Note that  $S_2 + 2F_{11} = 0$  as expected. Eq. (3) may be generalized to include all the cubic lattices as

$$\begin{aligned} (4\pi\chi^{(\pm)})^{-1} &= (\chi_p^{(\pm)} f_v)^{-1} - 1/12 \\ &\quad - \{A + B[33(\chi_p^{(\pm)})^{-1} + 4] \\ &\quad \times f_v^{-11/3}\}^{-1} - C[57(\chi_p^{(\pm)})^{-1} \\ &\quad + 8]^{-1} f_v^{19/3} + \dots, \end{aligned} \quad (4)$$

where coefficients  $A$ ,  $B$ , and  $C$  are defined as

$$\begin{aligned} A &= \frac{16}{945} \frac{F_{55}}{F_{51}^2} 4\pi n, \quad B = \frac{128}{315} \frac{1}{F_{51}^2} \left( \frac{4\pi n}{3} \right)^{14/3}, \\ C &= \frac{765765}{229376} F_{91}^2 \left( \frac{4\pi n}{3} \right)^{-22/3}, \end{aligned}$$

$\chi_p^{(\pm)}$  is the polarizability of a single particle defined by<sup>5</sup>

$$\chi_p^{(\pm)} = [1/3 + (\Omega_H \mp \Omega)]^{-1} = 4\pi M_s / (H_o \mp \omega/\gamma),$$

$f_v$  is the volume fraction of the particles, and  $n$  is the number of particles per unit cell ( $= 1, 2, 4$ , for sc, bcc, and fcc lattices, respectively). The first term in Eq. (4) represents the dilute (noninteracting) particle contribution. The second term corresponds to the similar Maxwell-Garnett interaction which presumes a value different from its scalar

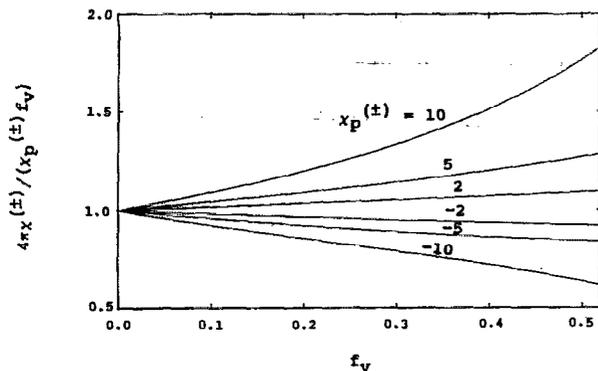


FIG. 1. Normalized composite polarizability as a function of particle volume loading for simple-cubic particle coordination.

case (1/3 compared to 1/12). Higher-order terms in Eq. (4), though small, must be included in order to distinguish interactions arising from different particle coordinations.

## RESULTS

Figure 1 shows the normalized polarizability of the particle composite,  $4\pi\chi^{(\pm)}/(\chi_p^{(\pm)}f_v)$ , as a function of particle loading volume,  $f_v$ , with  $\chi_p^{(\pm)}$ , the single particle polarizability, treated as a parameter,  $\chi_p^{(\pm)} = \pm 2, \pm 5$ , and  $\pm 10$ . Note that  $\chi_p^{(-)}$  can take only non-negative values. Figure 1 shows results only for simple cubic lattices. Results for other cubic lattices can be similarly obtained. Figure 1 shows that a magnetic composite can be treated as in the dilute particle-loading limit if and only if  $f_v$  and/or  $\chi_p^{(\pm)}$  values are small. In these limits particle-particle interactions are negligible and the composite can be approximated as composing only a dipole field. Figure 2 shows

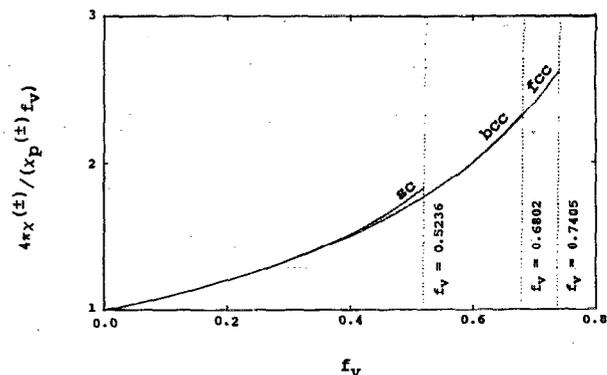


FIG. 2. Normalized composite polarizability as a function of particle volume loading for single-particle polarizability equal to 10.

$4\pi\chi^{(\pm)}/(\chi_p^{(\pm)}f_v)$  as a function of  $f_v$  for sc, bcc, and fcc particle coordinations. In Fig. 2  $\chi_p^{(\pm)}$  was taken to be 10. It is seen in Fig. 2 that the composite presumes only slightly different magnetic permeability values for different lattice structures.  $4\pi\chi^{(\pm)}$  becomes more and more distinguished from each other only when  $f_v$  approaches the percolation limits (0.5236, 0.6802, 0.7405 for sc, bcc, fcc coordinations, respectively). Similar results have also been reported in Ref. 1. Note that non-singular behavior of the permeability values are found for magnetic composites at the percolation limits of particle loading.

<sup>1</sup>H. How and C. Vittoria, Phys. Rev. B **43**, 8094 (1991).

<sup>2</sup>J. W. S. Rayleigh, Philos. Mag. **34**, 481 (1892).

<sup>3</sup>W. T. Doyle, J. Appl. Phys. **49**, 795 (1978).

<sup>4</sup>L. R. Walker, Phys. Rev. **105**, 390 (1957).

<sup>5</sup>B. Lax and K. J. Button, *Microwave Ferrites and Ferrimagnetics* (McGraw-Hill, New York, 1962), p. 159.