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Field theory of absorbing phase transitions with a nondiffusive conserved field

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We investigate the critical behavior of a reaction-diffusion system exhibiting a continuous absorbing-state phase transition. The reaction-diffusion system strictly conserves the total density of particles, represented as a nondiffusive conserved field, and allows an infinite number of absorbing configurations. Numerical results show that it belongs to a wide universality class that also includes stochastic sandpile models. We derive microscopically the field theory representing this universality class.

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The directed percolation (DP) [1] universality class is recognized as the canonical example of the critical behavior in the transition from an active to a single absorbing state. This universality class appears to be robust with respect to microscopic modifications, and non-DP behavior emerges only in the presence of additional symmetries, such as symmetric absorbing states [2], long-range interactions [3], or infinitely many absorbing states [4].

Recently, a new universality class of absorbing-state phase transitions (APT) [1] coupled to a nondiffusive conserved field has been identified [5]. This class characterizes the critical behavior of several models showing APT with a dynamics that strictly conserves the density of particles, that is represented by a conserved static (nondiffusive) field. The models are tuned to criticality by varying the particle density, and exhibit an infinite number of absorbing states. This universality class is particularly interesting because it embraces also the large group of stochastic sandpile models [6] (and in particular, the Manna model [7]) which are the prototypical examples that illustrate the ideas of self-organized criticality (SOC) [8]. These are driven dissipative models in which sand (or energy) is injected into the system and dissipated through the boundaries, leading eventually to a stationary state. In the limit of infinitesimally slow external driving, the systems approach a critical state characterized by an avalanchelike response. Recently, it has been pointed out that this critical state is equivalent to the APT present in the *fixed energy* case; that is, in automata with the same microscopic rules defining the sandpile, but without driving or dissipation [9–11].

The numerical evidence for the existence of such a general universality class [5] is corroborated by the observation that all the models analyzed share the same structure and basic symmetries; namely, a conserved and static noncritical field dynamically coupled to a nonconserved order parameter field, identified as the density of active particles. These observations have led to the conjecture that, in the absence of additional symmetries, *all stochastic models with an infinite number of absorbing states in which the order parameter evolution is coupled to a nondiffusive conserved field define a unique universality class* [5].

In this Rapid Communication, we study the nondiffusive field limit for the two species reaction-diffusion (RD) model introduced in Ref. [12] (see also Ref. [13]). In this limit the model has a phase transition with infinitely many absorbing states, and it conserves the total number of particles that is associated with a nondiffusive conserved field. We present extensive numerical simulations of the model in two and three dimensions, and determine the full set of critical exponents. The obtained values are compatible with the new universality class conjectured in Ref. [5]. This definitely shows the existence of a broad universality class that includes RD

processes, stochastic sandpile models, and lattice gases with the same symmetries. For the present RD model, it is possible to derive microscopically a field theory (FT) description. The resulting action and Langevin equations exhibit the basic symmetries that characterize this universality class, and represent a microscopic derivation of a FT for sandpile models. Notably, the resulting FT description recovers a phenomenological Langevin approach proposed for sandpiles [9,10]. The analysis provided here is a very promising path for a coherent description of several nonequilibrium critical phenomena now rationalized in a single universality class.

We consider the two-component RD process identified by the following set of reaction equations:

$$B \rightarrow A \quad \text{with rate } k_1, \tag{1}$$

$$B + A \rightarrow 2B \quad \text{with rate } k_2. \tag{2}$$

In this system, B particles diffuse with diffusion rate $D_B \equiv D$, and A particles *do not diffuse*; that is, $D_A = 0$. This corresponds to the limit $D_A \rightarrow 0$ of the model introduced in Ref. [12]. From the rate Eqs. (1) and (2), it is clear that the dynamics conserves the total density of particles $\rho = \rho_A + \rho_B$, where ρ_i is the density of component $i = A,B$. In this model, the only dynamics is due to B particles, which we identify as *active* particles. A particles do not diffuse and cannot generate spontaneously B particles. More specifically, A particles can only move via the motion of B particles that later on transform into A through Eq. (2). In the absence of B particles, ρ_A is thus a static field. This implies that any configuration devoid of B particles is an absorbing state in which the system is trapped forever.

It is easy to see [12] that the RD process defined by Eqs. (1) and (2) exhibits a phase transition from an active to an absorbing phase for a nontrivial value of the total particle density $\rho = \rho_c$. The critical value ρ_c depends upon the reaction rates k_1, k_2. The nature of this phase transition for $D_A \neq 0$ has been discussed in [12]; the static field case ($D_A = 0$), on the other hand, has never been explored to our knowledge. It is clear that the static field conserved RD (SFCRD) model allows, for any density ρ, an infinite number (in the thermodynamic limit) of absorbing configurations, in which there are no B particles. This is the key difference with respect to the case in which $D_A \neq 0$. In the latter case a configuration devoid of B particles consists of many diffusing A particles. In the long run, all particles can visit all sites in the lattice, and therefore, in a statistical sense, all configurations with a fixed number of A's are equivalent and the absorbing state can be considered unique [14].

The SFCRD model seems to possess all the required symmetries (stochastic dynamics, many absorbing states, static conserved field) for being part of the universality class conjectured in Ref. [5]. In order to test this possibility, we have performed numerical simulations of the model in a d-dimensional hypercubic lattice with $N = L^d$ sites. Each site can store any number of A and B particles; that is, our model can be represented by bosonic variables. Initial conditions are generated by randomly placing $N\rho_A^{(0)}$ particles A and $N\rho_B^{(0)}$ particles B, corresponding to a particle density $\rho = \rho_A^{(0)} + \rho_B^{(0)}$. The results are independent of the particular initial ratio $\rho_A^{(0)}/\rho_B^{(0)}$, apart from very early time transients.

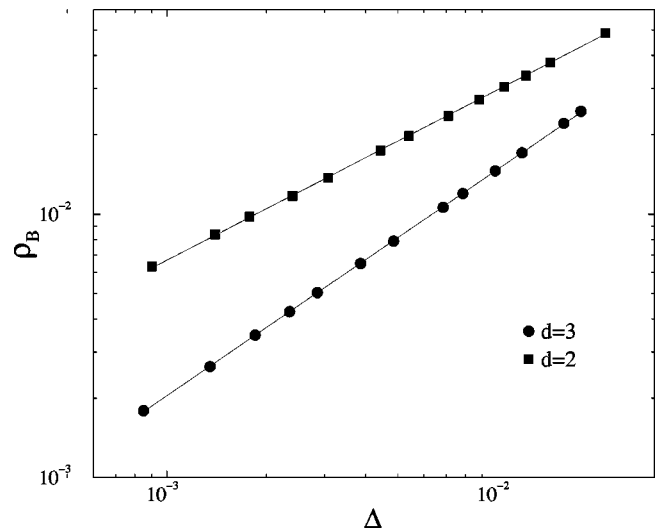

FIG. 1. Order parameter behavior (stationary density of B particles) as a function of $\Delta = \rho - \rho_c$ for the reaction-diffusion model in $d = 2$ and 3. The slope of the straight lines is $\beta = 0.65$ in $d = 2$ and $\beta = 0.86$ in $d = 3$.

The dynamics proceeds in parallel. Each time step, we update the lattice according to the following rules: (a) Diffusion: on each lattice site, each B particles moves into a randomly chosen nearest neighbor site. (b) After all sites have been updated for diffusion, we perform the reactions: (i) On each lattice site, each B particle is turned into an A particle with probability r_1. (ii) At the same time, each A particle becomes a B particle with probability $1 - (1 - r_2)^{n_B}$, where n_B is the total number of B particles in that site. This corresponds to the average probability for an A particle of being involved in the reaction of Eq. (2) with any of the B particles present on the same site. The probabilities r_1 and r_2 are proportional to the reaction rates k_1 and k_2 defined in Eqs. (1) and (2). The order parameter of the system is ρ_B, measuring the density of dynamical entities.

As we vary ρ, the system exhibits a continuous transition separating an absorbing phase ($\rho_B = 0$) from an active phase ($\rho_B \neq 0$) at a critical point ρ_c. The order parameter is null for $\rho < \rho_c$, and follows a power law $\rho_B \sim (\rho - \rho_c)^\beta$, for $\rho \geqslant \rho_c$. The system correlation length ξ and time τ, which define the exponential relaxation of space and time correlation functions, diverge as $\rho \rightarrow \rho_c$ [1]. In the critical region the system is characterized by a power law behavior, $\xi \sim |\rho - \rho_c|^{-\nu_\perp}$ and $\tau \sim |\rho - \rho_c|^{-\nu_\parallel}$. The dynamical critical exponent is defined as $\tau \sim \xi^z$, with $z = \nu_\parallel / \nu_\perp$. These exponents fully determine the critical behavior of the stationary state of the model [1].

We have studied the steady-state properties of the model in $d = 2$ and 3, by performing numerical simulations for systems with size ranging up to $L = 512$ and $L = 125$, respectively. Averages were performed over $10^4 - 10^5$ independent initial configurations. The values considered for the rates r_i are $r_1 = 0.1$ and $r_2 = 0.5$ in $d = 2$, and $r_1 = 0.4$ and $r_2 = 0.5$ in $d = 3$. From the finite-size scaling analysis for APT [1], we obtain the critical point ($\rho_c = 0.3226(1)$ in $d = 2$ and $\rho_c = 0.95215(15)$ in $d = 3$) and the complete set of critical exponents. A detailed presentation of these results will be reported elsewhere. In Fig. 1 we show as an example the order

TABLE I. Critical exponents for spreading and steady-state experiments in $d=2$. Figures in parenthesis indicate the statistical uncertainty in the last digit. Manna exponents from Refs. [5,10,18,19].

	Steady-state exponents				
	β	$\beta/\nu_\perp$	$\nu_\perp$	z	$\nu_\parallel$
SFCRD	0.65(1)	0.78(2)	0.83(3)	1.55(5)	1.29(8)
Manna	0.64(1)	0.78(2)	0.82(3)	1.57(4)	1.29(8)
	Spreading exponents				
	τ_s	D	z	η	δ
SFCRD	1.27(1)	2.75(1)	1.54(2)	0.29(2)	0.50(2)
Manna	1.28(1)	2.76(1)	1.55(1)	0.30(3)	0.48(2)

parameter behavior with respect to the control parameter $\Delta=\rho-\rho_c$, from which it is possible to calculate directly the β exponent. The results obtained in $d=2$ and 3 are reported in Tables I and II and compared with the Manna sandpile model in the respective dimension.

In APT it is possible to obtain more information on the critical state by studying the evolution (spread) of activity in systems that start close to an absorbing configuration [15]. In each *spreading* simulation, a small perturbation is added to an absorbing configuration. It is then possible to measure the spatially integrated activity $N(t)$, averaged over all runs, and the survival probability $P(t)$ of the activity after t time steps. Only at the critical point do we have power law behavior for these magnitudes. In the case of many absorbing states, the choice of the initial absorbing state is not unique [16]. There are several methods to perform spreading exponents in this case, and we have followed the technique outlined in Ref. [5], which amounts to the study of critical spreading with the so-called ''natural initial conditions'' at $\rho=\rho_c$ [16]. The probability distribution $P_s(s)$ of having a spreading event involving s sites, as well as the the quantities $N(t)$ and $P(t)$, can thus be measured. At criticality, the only characteristic length is the system size L, and we can write the scaling forms $P_s(s)=s^{-\tau_s}\mathcal{F}_1(s/L^D)$, $N(t)=t^{\eta}\mathcal{F}_2(t/L^z)$, and $P(t)=t^{-\delta}\mathcal{F}_3(t/L^z)$ [15]. The scaling functions $\mathcal{F}_i(x)$ are decreasing exponentially for $x\gg 1$, and we have considered that the spreading characteristic time and size are scaling as L^z and L^D, respectively. In this case simulations were performed for systems of size up to $L=1024$ in $d=2$ and $L=200$ in $d=3$, averaging over at least 5×10^6 spreading experiments. The new scaling exponents τ_s, D, δ, and η are measured using the now standard moment analysis technique [17,18]. The resulting exponents are summarized in Tables I and II, and can be compared with the avalanche exponents usually measured in stochastic sandpile models. As a further consistency check of our results, we have checked that our exponents fulfill all scaling and hyperscaling relations in standard APT. Despite the apparent diversity in the dynamical rules, we can safely include that the SFCRD and the Manna models are in the same universality class.

TABLE II. Critical exponents for spreading and steady-state experiments in $d=3$. Figures in parenthesis indicate the statistical uncertainty in the last digit. Manna exponents from Refs. [5,10,18,19].

	Steady-state exponents				
	β	$\beta/\nu_\perp$	$\nu_\perp$	z	$\nu_\parallel$
SFCRD	0.86(2)	1.39(4)	0.62(3)	1.80(5)	1.12(8)
Manna	0.84(2)	1.40(2)	0.60(3)	1.80(5)	1.08(8)
	Spreading exponents				
	τ_s	D	z	η	δ
SFCRD	1.41(2)	3.32(2)	1.74(2)	0.16(2)	0.76(3)
Manna	1.43(2)	3.31(2)	1.75(2)	0.16(2)	0.75(3)

From a theoretical point of view, the SFCRD allows the construction of a field theory description that also will represent the critical behavior of all models belonging to the same universality class. The construction of the FT follows standard steps [20], and it consists of recasting the master equation implicit in Eqs. (1) and (2) into a ''second quantized form'' via a set of creation and annihilation bosonic operators for particles A and B on each site. It is then possible to map the solution of the master equation into a path integral over the density fields, weighted by the exponential of a functional action S [20]. In our case, we can quote the elegant results of Ref. [12], just considering that we have $D_A=0$. The action of the FT is thus

$$\begin{aligned} S=\int dxdt\{&\overline{\psi}[\partial_t+(r-D\nabla^2)]\psi+\overline{\phi}[\partial_t\phi-\lambda\nabla^2\psi]\\ &+u_1\overline{\psi}\psi(\psi-\overline{\psi})+u_2\overline{\psi}\psi(\phi+\overline{\phi})+v_1\overline{\psi}^2\psi^2\\ &+v_2\overline{\psi}\psi(\psi\overline{\phi}-\overline{\psi}\phi)+v_3\overline{\psi}\psi\overline{\phi}\phi\}, \end{aligned} \tag{3}$$

where ψ and ϕ are auxiliary fields, defined such that their average values coincide with the average density of B particles and the total density of particles, respectively, $\overline{\psi}$ and $\overline{\phi}$ are response fields, and the coupling constants are related to the reaction rates k_i. Namely, D represents the diffusion coefficient of B particles, λ is initially also proportional to D, and r is the critical parameter that is related to the difference of the total density with respect to the critical density ρ_c. By standard power-counting analysis, one realizes that the reduced couplings u_i/D have critical dimension $d_c^{(1)}=4$, while the couplings v_i/D have on their part $d_c^{(2)}=2$. This means that when applying the renormalization group (RG) and performing a perturbative expansion around the critical dimension 4, one could in principle drop all the couplings v_i [21]. The critical parameter of this theory is the density of active sites ψ, while ϕ serves just to propagate interactions. We can exploit some symmetry considerations of the FT to relate the physics of the system to the corresponding analytical description. By neglecting irrelevant terms in the power-counting analysis, action (3) is invariant under the shift transformation

$$\phi\rightarrow\phi'=\phi+\delta,\quad r\rightarrow r'=r-u_2\delta, \tag{4}$$

where δ is any constant. This symmetry has a very intuitive meaning: If we increase everywhere the density of particles by an amount δ, we must be closer to the critical point by an amount proportional to δ. In other words, this symmetry represents the conserved nature of the system. It is also interesting to write the set of corresponding Langevin equations (up to the irrelevant terms v_i) by integrating out the response fields $\overline{\psi},\overline{\phi}$ in the action S,

$$\partial_t \psi = D\nabla^2 \psi - r\psi - u_1 \psi^2 - u_2 \psi\phi + \eta_\psi, \tag{5}$$

$$\partial_t \phi = \lambda \nabla^2 \psi + \eta_\phi. \tag{6}$$

Here, η_ψ and η_ϕ are noise terms with zero mean and correlations $\langle \eta_\psi(x,t)\eta_\psi(x',t')\rangle = 2u_1\psi(x,t)\delta(x-x')\delta(t-t')$, $\langle \eta_\psi(x,t)\eta_\phi(x',t')\rangle = -u_2\psi(x,t)\delta(x-x')\delta(t-t')$ and $\langle \eta_\phi(x,t)\eta_\phi(x',t')\rangle = 0$. The noise terms have a multiplicative nature [22], that is the standard form in APT. Note that v_i couplings of Eq. (3) contribute to noises correlations with higher order terms. These equations have a very clear physical interpretation. The field ϕ is conserved [23] and static, i.e., it only diffuses via the activity of B particles, represented by the field ψ. On its turn, the field ψ is locally coupled to the field ϕ, but is nonconserved. Noticeably, this set of equations recovers (up to the discarded couplings v_i) the Langevin description proposed on a phenomenological level for the sandpiles in Refs. [9,10], with the extra information of the cross-correlation term $\langle \eta_\psi \eta_\phi \rangle$. Indeed, the stochastic sandpile model has the same basic symmetries of the present RD model, once the local density field ρ is replaced by the local sand-grain (energy) density and the order parameter is identified with the density of toppling sites field [9,10]. It is then natural to expect that the very same basic structure is reflected in a unique theoretical description [24].

The complete RG analysis of the field theory would allow us to extract estimates for the critical exponents to compare with simulations in $d=2$ and 3. Unfortunately, some severe technical problems are encountered in this case. In general, as pointed out in Ref. [12], the couplings v_i become relevant and should be taken into account in the RG analysis. The importance of the couplings v_i can be argued by the change of the energy shift symmetry, Eq. (4), in the case of the full action Eq. (3). Second, and more important, is the presence of the singular bare propagator for the field ϕ, which cannot be regularized by adding a mass term $m^2\phi\bar{\phi}$, since it will obviously break the symmetry (4). This singular propagator gives rise to divergences in the RG perturbative expansions, and the results of Ref. [12] cannot be extended ''*tout-court*'' to the limit $D_A \to 0$. In particular, some Feynman diagrams in the ϵ-expansion presented in Refs. [12,13] are proportional to $1/D_A$. Hence, the limit $D_A \to 0$ in the theory with $D_A \neq 0$ is nonanalytic; any infinitesimal amount of diffusion in the energy field renormalizes to a finite value, and definitely changes the universality class of the model. Work is in progress to provide a suitable regularization that will allow an ϵ-expansion calculation of the critical exponents.

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