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## Isotopic spin formulation of the Josephson effect

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A model for a quantum-mechanical weak link between two superconductors is discussed in which the tunneling of electron pairs is described by “isotopic spin” operators  $(T_1, T_2, T_3)$  obeying  $[T_a, T_b] = i\epsilon_{abc} T_c$ . In the thermodynamic limit of infinite size for the superconductors, the model reduces to the quantum “Josephson pendulum.” However, the model also holds true for mesoscopic grains, and thus allows for the quantum description of lowest-order finite-size effects.

### I. INTRODUCTION

The central result of Josephson on the nature of a weak link separating two bulk superconductors is that the weak link behaves as a “pendulum”.<sup>1</sup> For the case of a weak link biased at voltage  $V$ ,<sup>2</sup> the “Josephson-pendulum” Hamiltonian reads

$$H_\infty = -\frac{q^2}{2C} \left[ \frac{\partial}{\partial \phi} \right]^2 + iqV \frac{\partial}{\partial \phi} - qv \cos(\phi), \quad (1)$$

where  $q = 2e$  is the charge of a tunneling pair,  $C$  is the weak-link capacitance, and  $qv$  is the weak-link critical current. Implicit in the derivation of Eq. (1) is the notion of going to the thermodynamic limit of infinite size for the bulk superconductors on each side of the weak link. In what follows we shall develop the model in a manner which takes into account (to lowest order) the mesoscopic finite size of the superconductors. Other studies<sup>3</sup> have considered in detail the manner in which a BCS state is influenced by finite-size effects.

The operator representation here of interest involves “isotopic spins” describing electron-pair charges. The general notion of using isotopic-spin representations of BCS theory is well known,<sup>4</sup> and the further notion that the Josephson effect is implicit in such a representation has been previously discussed.<sup>5</sup> For the problem at hand, i.e., a weak link between two superconductors, the isotopic spin representation of the Hamiltonian is given by

$$H_T = (q^2/2C)T_3^2 - qVT_3 - \lambda T_1, \quad (2)$$

where the isotopic spin algebra is determined by

$$[T_a, T_b] = i\epsilon_{abc} T_c, \quad (3a)$$

$$T_1^2 + T_2^2 + T_3^2 = T(T+1). \quad (3b)$$

Our purpose is to discuss the physical basis for employing Eqs. (2) and (3), and the physical consequences of the isotopic-spin representation for quantum circuits.

In Sec. II we discuss the Feynman electron-pair boson model for the Josephson effect in quantum-mechanical

form and (employing a boson-spin representation) arrive at Eqs. (2) and (3). We also show that in the thermodynamic limit of infinite size ( $T \rightarrow \infty$ ), Eq. (2) is equivalent to Eq (1). In Sec. III we discuss the ground-state energy and charge-voltage characteristics of the model which are computed numerically. In Sec. IV the lifetimes of the energy levels due to interactions with the environmental electromagnetic fields are calculated in the large capacitance limit ( $C \rightarrow \infty$ ). Finally, in the concluding Sec. V, the notion of mesoscopic quantum-circuit effects without direct capacitive-charging energies are explored.

### II. ISOTOPIC SPIN

Let us begin with the original picture of the Josephson effect in terms of the tunneling of electron pairs. We introduce the boson operators  $\psi_1$  and  $\psi_2$  for the electron-pair condensates in superconductors 1 and 2 (respectively):

$$[\psi_i, \psi_j] = 0 = [\psi_i^\dagger, \psi_j^\dagger], \quad (4a)$$

$$[\psi_i, \psi_j^\dagger] = \delta_{ij}. \quad (4b)$$

Our isotopic-spin operators are then defined as

$$2T_1 = (\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1), \quad (5a)$$

$$2iT_2 = (\psi_1^\dagger \psi_2 - \psi_2^\dagger \psi_1), \quad (5b)$$

$$2T_3 = (\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2), \quad (5c)$$

$$2T = (\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2). \quad (5d)$$

The isotopic-spin algebra of Eqs. (3) then follows directly from Eqs. (4) and (5).

Feynman has given a very simple formulation<sup>6</sup> of the Josephson effect, treating  $\psi_1$  and  $\psi_2$  as  $c$ -numbers. Feynman’s equations are

$$i\hbar \frac{\partial \psi_1}{\partial t} = -\frac{qV}{2} \psi_1 - \frac{\lambda}{2} \psi_2, \quad (6a)$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = \frac{qV}{2} \psi_2 - \frac{\lambda}{2} \psi_1. \quad (6b)$$

In quantum-mechanical terms, Eqs. (6) are equivalent to the Hamiltonian

$$H_F = -(qV/2)(\psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2) - (\lambda/2)(\psi_1^\dagger\psi_2 + \psi_2^\dagger\psi_1). \quad (7a)$$

In the isotopic-spin representation,

$$H_F = -qVT_3 - \lambda T_1. \quad (7b)$$

Our model Hamiltonian Eq. (2) follows from Eqs. (7) by adding a term corresponding to the energy stored in the electric field, i.e., the charging energy stored in a finite capacitance,

$$H_T = (q^2/2C)T_3^2 - qVT_3 - [\hbar v/\sqrt{T(T+1)}]T_1, \quad (8)$$

for fixed total isotopic spin  $T$ . In Eq. (8) we have defined

$$\hbar v = \lambda\sqrt{T(T+1)}. \quad (9)$$

Let us now demonstrate that Eq. (8) is equivalent to the quantum Josephson pendulum Eq. (1) in the limit  $T \rightarrow \infty$ .

The Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H_T|\Psi\rangle, \quad (10)$$

in the representation

$$T_3|Tm\rangle = m|Tm\rangle, \quad (11a)$$

$$|\Psi\rangle = \sum_{m=-T}^T c_{Tm}|Tm\rangle, \quad (11b)$$

reads

$$i\hbar\dot{c}_{Tm} = [(q^2m^2/2C) - (qVm)]c_{Tm} - \hbar v \sum_{m'=-T}^T R_T(m, m')c_{Tm'}, \quad (11c)$$

$$R_T(m, m') = \langle Tm|T_1|Tm'\rangle/\sqrt{T(T+1)}. \quad (11d)$$

Since in the limit  $T \rightarrow \infty$  we have

$$R_\infty(m, m') = \frac{1}{2}[\delta_{m+1, m'} + \delta_{m-1, m'}], \quad (12)$$

it follows that

$$i\hbar\dot{c}_{\infty(m)} = [(q^2m^2/2C) - (qVm)]c_{\infty(m)} - (\hbar v/2)[c_{\infty(m+1)} + c_{\infty(m-1)}]. \quad (13)$$

Defining the Josephson-pendulum wave function as

$$\psi(\phi, t) = \sum_{m=-\infty}^{\infty} c_{\infty(m)}(t)e^{im\phi}, \quad (14)$$

yields

$$i\hbar\partial\psi(\phi, t)/\partial t = H_\infty\psi(\phi, t), \quad (15)$$

in virtue of Eqs. (1), (13), and (14). The proof of the equivalence of our model [Eqs. (2) and (9)] and the Josephson-pendulum model (in the  $T \rightarrow \infty$  limit) is now completed.

### Voltage Biased Weak Link

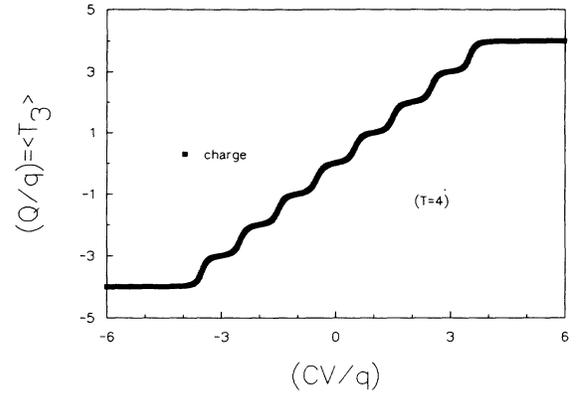


FIG. 1. Ground-state charge as a function of the voltage bias is shown for  $T=4$ . The value  $(\hbar Cv/q^2)=0.2$  is chosen for the purpose of illustration.

### III. WEAK-LINK GROUND STATE

The ground state  $|0\rangle$  of the model Eq. (8) obeys

$$H_T(V)|0\rangle = E(V)|0\rangle, \quad (16)$$

from which it is possible to compute the mean charge stored on the weak link when viewed as a nonlinear capacitor:

$$Q(V) = q\langle 0|T_3|0\rangle = -\frac{dE(V)}{dV}. \quad (17)$$

In Fig. 1 the charge on the weak link is plotted as a function of the external voltage for the case  $T=4$ . There is a “staircase” of  $(2T+1)$  steps with both the lowest and highest steps acting as a breakpoint beyond which no more charge can be added. In the  $T \rightarrow \infty$  limit, there is simply an “infinite staircase” representing the ground state of the quantum Josephson pendulum. Experimental tests<sup>7</sup> of the Josephson (voltage-biased) ground state have been discussed elsewhere.

### IV. INFINITE-CAPACITANCE LIMIT

In the limit of infinite capacitance ( $C \rightarrow \infty$ ), quantum-circuit behavior for the weak link is still present in the model. If we choose the original Feynman model in the quantum-mechanical form of Eqs. (7), then we arrive at the energy spectrum

$$H_F|M\rangle = \epsilon_{TM}|M\rangle, \quad (18a)$$

$$\epsilon_{TM} = M\{q^2V^2 + [\hbar^2v^2/T(T+1)]\}^{1/2}, \quad (18b)$$

$$M = 0, \pm 1, \pm 2, \dots, \pm T. \quad (18c)$$

In the thermodynamic limit of infinite size ( $T \rightarrow \infty$ ), the spectrum is an infinite ladder of energy levels spaced at the Josephson voltage-biased frequency

$$\omega_V = (qV/\hbar), \quad (19a)$$

$$\epsilon_{\infty M} = M\hbar\omega_V, \quad (19b)$$

$$M = 0, \pm 1, \pm 2, \pm 3, \dots. \quad (19c)$$

The lifetimes of the states can be computed on the basis of methods in common use in spin-resonance theory. For example, suppose that there is an interaction of the weak link with the electromagnetic field from environmental circuits. This may be described by an additional circuit Hamiltonian

$$\Delta H = -q\Delta VT_3, \quad (20)$$

where  $\Delta V$  is the voltage contribution of the fluctuating electromagnetic fields. For the case of weak coupling, we may employ Fermi's golden rule in the form

$$W(i+I \rightarrow f+F) = (2\pi q^2/\hbar^2) |(T_3)_{fi}|^2 |\Delta V_{FI}|^2 \delta(\omega_{if} - \Omega_{FI}), \quad (21)$$

where  $i$  ( $f$ ) are the initial (final) isotopic-spin states,  $I$  ( $F$ ) are the initial (final) electromagnetic-field states, and

$$\hbar\omega_{if} = \epsilon_i - \epsilon_f, \quad (22a)$$

$$\hbar\Omega_{FI} = E_F - E_I. \quad (22b)$$

Averaging over initial electromagnetic states and summing over final electromagnetic-field states in Eq. (21) yields (for the rate of isospin transitions per unit time)

$$\bar{W}(i \rightarrow f) = (2\pi q^2/\hbar^2) |(T_3)_{fi}|^2 S_V(\omega_{if}). \quad (23)$$

In Eq. (23),  $S_V(\omega)$  is the spectral function for voltage noise across the weak link:

$$S_V(\omega) = \sum_{IF} p_I |\Delta V_{FI}|^2 \delta(\omega - \Omega_{FI}). \quad (24)$$

From an electrical engineering viewpoint, the voltage noise can be computed from the radiative impedance across the weak link. In virtue of the Nyquist theorem,

$$S_V(\omega) = (\hbar\omega/\pi) \text{Re}Z(\omega + i0^+) / (1 - e^{-\hbar\omega/kT}). \quad (25)$$

The mean radiated power corresponding to an initial isotopic-spin state  $i$  is given by

$$P_i = \sum_f \hbar\omega_{if} \bar{W}(i \rightarrow f). \quad (26)$$

In the thermodynamic limit  $T \rightarrow \infty$ , the power is given by

$$P = (q^2 v^2 / 2) \text{Re}Z(\omega_V + i0^+). \quad (27)$$

The above result is central for computing current-voltage characteristics in the  $T \rightarrow \infty$  case ( $P = \bar{I}V$ ),

$$\bar{I}(V) = (qv/2)(v/\omega_V)(q^2/\hbar) \text{Re}Z(\omega_V + i0^+), \quad (28)$$

as well as the lifetime  $\tau$  of the quantum states in the ladder,

$$(1/\tau) = (v^2/2\omega_V)(q^2/\hbar) \text{Re}Z(\omega_V + i0^+). \quad (29)$$

## V. CONCLUSIONS

The quantum Josephson-pendulum model [Eq. (1)] has long served as a canonical model for describing quantum-mechanical weak-link circuits. While a finite-capacitance  $C$  in the Josephson model is sufficient for the existence of quantum-electrodynamic effects, it is by no means a requirement for quantum-mechanical behavior. In particular, even in the limit  $C \rightarrow \infty$ , mesoscopic quantum mechanics is present. In the context of arrays of weak links, this has been previously discussed.<sup>8</sup>

The isotopic spin formalism, here discussed, is particularly suited to the discussion of quantum-circuit behavior in mesoscopic weak-link systems. The isotopic-spin commutation relations dictate the quantum kinematics of the weak-link circuits and allow for a rigorous treatment of quantum phenomena in circumstances in which capacitive electric-field-energy storage can safely be neglected.

## ACKNOWLEDGMENTS

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