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Duality of N=2 Heterotic – Type I Compactifications in Four Dimensions*

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Abstract

We discuss type I – heterotic duality in four-dimensional models obtained as a Coulomb phase of the six-dimensional $U(16)$ orientifold model compactified on T^2 with arbitrary $SU(16)$ Wilson lines. We show that Kähler potentials, gauge threshold corrections and the infinite tower of higher derivative F-terms agree in the limit that corresponds to weak coupling, large T^2 heterotic compactifications. On the type I side, all these quantities are completely determined by the spectrum of N=2 BPS states that originate from D=6 massless superstring modes.

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1. Introduction

In a previous paper [1] we presented a general computation of the perturbative prepotential in four-dimensional type I string vacua with $N = 2$ space-time supersymmetry. As we have shown, its one-loop contribution is determined entirely by the corresponding correction to the Planck mass. Finally, we applied the result to study duality in the so-called STU-model with three vector multiplets, which admits simultaneous type II and heterotic descriptions. This model, on the type I and heterotic side, corresponds to the Higgs phase of a six-dimensional vacuum with gauge group $U(16)$ [2, 3], compactified to four dimensions on T^2.

In this work we study heterotic – type I duality [4] in the context of the original $U(16) \times U(1)^3$ model. More precisely, we consider the Coulomb phase of the above model whose massless spectrum consists of $n_V = 19$ Abelian vector multiplets and $n_H = 4$ neutral hypermultiplets. Two vector multiplets, S and S', play a special role in type I theory: their scalar vacuum expectation values determine the tree-level gauge couplings of the vector bosons associated to $U(16)$. In this class of models, the perturbative prepotential has the form [1]:

$$\mathcal{F}^I = SS'U - \frac{1}{2}\sum_i (v_i S + v_i' S') A_i^2 + f^I(U, A) \ , \tag{1.1}$$

where v_i and v_i' are constants and $f^I(U, A)$ is the one-loop correction. Type I – heterotic duality maps S, S' and U into the "universal" S, T and U heterotic multiplets, respectively.

On the heterotic side, the perturbative prepotential has the form

$$\mathcal{F}^H = STU - \frac{1}{2}\sum_i v_i \, S A_i^2 + f^H(T, U, A) \ , \tag{1.2}$$

where the constants v_i are given by the Kač-Moody levels of the corresponding gauge groups and $f^H(T, U, A)$ is the one-loop correction. The limits $\mathrm{Im}T \to \infty$ and the respective $\mathrm{Im}S > \mathrm{Im}S' \to \infty$ take both theories into perturbative regimes, where it is possible to test duality by comparing perturbative prepotentials and/or the related Kähler metrics and

gauge threshold corrections [1]. In particular, on the heterotic side, such a limit corresponds to a weakly coupled theory compactified on large T^2. For a dual pair of models

$$f^H(T,U,A) \stackrel{T\to\infty}{\longrightarrow} -\frac{1}{2}\sum_i v_i' T A_i^2 + f^I(U,A) \ . \tag{1.3}$$

Note that a part of the one-loop heterotic prepotential is mapped into the tree-level type I couplings proportional to v_i' while the remaining part corresponds to type I one-loop corrections. This is of course also true for the related Kähler metrics and gauge threshold corrections, which can be extracted directly from the appropriate superstring scattering amplitudes. In the models under consideration, $v_i' = 0$ for all gauge fields associated to the $SU(16)$ subgroup of $U(16)$.

Prepotential is only the first representative of a large class of analytic quantities characterizing N=2 supersymmetric effective field theories. Effective Lagrangians contain also higher derivative F-terms of the form $I_g = \mathcal{F}_g W^{2g}$, where W is the Weyl superfield. The prepotential corresponds to $\mathcal{F}_0$ while higher $\mathcal{F}_g$'s correspond to analytic couplings associated to higher derivative interactions [5]. In ref. [6], type II – heterotic duality was tested by comparing $\mathcal{F}_g$'s of dual models, while in ref. [7] a similar analysis was done to test type I – heterotic duality in the STU-model. In this work, we apply similar tests to type I – heterotic duality, in the Coulomb phase of the $U(16)\times U(1)^3$ model *i.e.* in the presence of non-vanishing Wilson lines. On the type II side, $\mathcal{F}_g$'s are subject to an exact non-renormalization theorem, which implies that they are purely genus-g quantities. On the other hand, for type I and heterotic, all $\mathcal{F}_g$'s with $g \geq 2$ start appearing at the one-loop level. Duality can then be tested by comparing them in the weak coupling limits discussed before. An important difference between the type II – heterotic and type I – heterotic comparisons is that in the first case the tests are restricted to the holomorphic anomaly equation and to the behaviour of $\mathcal{F}_g$'s near singular points of the moduli space (such as conifolds), while in the latter the comparison is extended to their full expression in the appropriate limits.

The case of $\mathcal{F}_1$, which is related to the gravitational R^2 threshold corrections [8], is a little more subtle. Due to non-localities of the effective action produced by one-loop anomalies, $\mathcal{F}_1$ cannot be defined in a straightforward manner; this is exactly the same problem as was first encountered in the analysis of $N = 1$ threshold corrections [9] where it was shown that, in a strict sense, the gauge function does not exist. In this paper, we adopt the same convention as in ref. [6], with $\mathcal{F}_1$ defined as the coefficient of the R^2 term. For a heterotic model,

$$\mathcal{F}_1^H = 4\pi \mathrm{Im} S + f_1^H(T, U, A), \tag{1.4}$$

while for a type I model

$$\mathcal{F}_1^I = 4\pi \mathrm{Im} S + 4\pi v'_{\mathrm{grav}} \mathrm{Im} S' + f_1^I(U, A) \ , \tag{1.5}$$

where v'_{grav} is a constant. For a dual pair,

$$f_1^H(T, U, A) \stackrel{T\to\infty}{\longrightarrow} 4\pi v'_{\mathrm{grav}} \mathrm{Im} T + f_1^I(U, A) \ , \tag{1.6}$$

so similarly to the prepotential, a part of one-loop heterotic corrections is mapped into the tree-level type I coupling. In the model under consideration, $v'_{\mathrm{grav}} = 1$ consistently with T-duality of the type I vacuum, which inverts the volume of K_3 [1].

The paper is organized as follows. In section 2, we determine the moduli Kähler metrics for the $U(16) \times U(1)^3$ type I model with $SU(16)$ Wilson lines. In section 3 we give the free-fermionic heterotic description of this model and compute its Kähler metrics. We show that the heterotic and type I results agree in appropriate limits. We also make a comparison of the $SU(16)$ gauge group threshold corrections, which were computed on the type I side in ref. [10]. In section 4 we discuss $\mathcal{F}_g$ couplings, showing that the whole infinite towers of higher derivative interactions are identical for the dual pairs of models; this provides an overwhelming evidence for type I – heterotic duality.

2. Type I model

The model we consider on the type I side is constructed as an orientifold of type IIB [2] compactified on $T^4/\mathbf{Z}_2 \times T^2$. Its massless spectrum from closed strings consists of the $N = 2$ gravity multiplet, three Abelian vector multiplets and four neutral hypermultiplets in the untwisted sector, as well as 16 additional singlet twisted hypermultiplets. On the other hand, one has open strings with ends moving on 2×16 nine-branes and/or 2×16 five-branes, giving rise to a $U(16)$ gauge group together with two hypermultiplets in the $\mathbf{120}_{1/2}$ representation (from the 99-sector) and sixteen in the $\mathbf{16}_{1/4}$ of $U(16)$ (from the 95-sector). Moreover, in the case when the five-branes are located at different fixed points of the orbifold, there is a $U(1)^{16}$ gauge group from the 55-sector (which eventually become massive by coupling to the sixteen twisted hypermultiplets).

In the following, we consider the Coulomb phase of the 99-gauge group by turning on the sixteen Wilson lines on T^2. One thus obtains 3+16 Abelian vector multiplets whose moduli space in $N = 2$ special coordinates is parametrized by the complex fields S, S', U and $A_i = a_4^i - a_5^i U$ [1]. As usual, U parametrizes the complex structure of T^2 while $a_{4,5}^i$ denote the sixteen (real) Wilson lines along the two compact directions (4 and 5) of T^2. The fields S and S' are appropriate combinations involving the dilaton and the volumes of T^2 and T^4. Their real parts are associated to perturbative continuous Peccei-Quinn (PQ) symmetries while their imaginary parts contain a factor of the inverse string coupling and go to infinity in the weak coupling. Under type I – heterotic duality, they are mapped into the heterotic dilaton and Kähler modulus of the T^2, S and T respectively.

Due to analyticity and PQ symmetries, the perturbative prepotential receives one-loop corrections only, that depend on U and A_i. Using standard $N = 2$ formulae, these can be extracted from the one-loop Kähler metric $K^{I(1)}_{U\bar{U}}$, which in type I strings is determined uniquely by the one-loop corrections to the Einstein kinetic term. Applying the general

results of ref. [1], we find[1]

$$K^{I(1)}_{U\bar{U}} = \frac{\sqrt{G}}{16\pi^2 \mathrm{Im}S'} \partial_U \partial_{\bar{U}} \int_0^\infty \frac{dt}{t^2} \left\{ \sum_{a^I+a^J+\Gamma_2} s_{IJ} + \sum_{2a^I+\Gamma_2} -16 \sum_{a^I+\Gamma_2} \right\} e^{-\pi t|p|^2/2\sqrt{G}} \ , \tag{2.1}$$

where $\sqrt{G}$ is the volume of T^2, and Γ_2 is the two-dimensional lattice of Kaluza-Klein momenta $G^{-1/4}p$, with

$$p = \frac{m_4 - m_5 U}{\sqrt{2\mathrm{Im}U}} \ , \tag{2.2}$$

and integer m_4, m_5. The lattice is shifted by the Wilson lines as indicated in eq. (2.1): for instance, the shift $a^I + \Gamma_2$ implies that $m_{4,5} \to m_{4,5} + a^I_{4,5}$. For convenience, we have introduced the index $I \equiv i$ or $\bar{\imath}$, with i and $\bar{\imath}$ running over the **16** and $\overline{\mathbf{16}}$ of $SU(16)$, respectively, and $a^{\bar{\imath}} \equiv -a^i$. Also, $s_{IJ} = -1$ or 1, depending on whether I and J belong to the same or conjugate representations. Finally, the partial derivatives with respect to U and $\bar{U}$ are taken by keeping the Wilson lines $a^I_{4,5}$ fixed. Note that the volume $\sqrt{G}$ of T^2 drops out after a rescaling of t.

The three terms in the r.h.s. of eq. (2.1) correspond to the contributions of the annulus in the 99-sector, Möbius strip in the 99-sector and annulus in the 95-sector, respectively, while the numerical coefficient of the latter comes from the multiplicity of 5-branes. The torus contribution vanishes because of extended supersymmetry, whereas the Klein bottle and annulus in the 55-sector contributions cancel one another. Note the similarity of the above expression with the one giving the threshold corrections to gauge couplings [10], where the eigenvalues of the charge-squared operator $(q_I + q_J)^2$, $4q_I^2$ and q_I^2 are inserted in the three terms of eq. (2.1) correspondingly.

Using the identity

$$\partial_U \partial_{\bar{U}} e^{-\pi t|p|^2/2} = -\frac{1}{(U-\bar{U})^2} t\partial_t^2 t e^{-\pi t|p|^2/2} \ , \tag{2.3}$$

[1]Here, the Wilson lines correspond to the $SU(16)$ generators and are subject to the condition $\sum_i a^i_{4,5} = 0$. The $U(1)$ factor, which is anomalous in six dimensions, requires special treatment [11, 3] and will not be discussed in the present work.

which follows from eq. (2.2), and performing an integration by parts, we find that the boundary term vanishes and

$$K^{I(1)}_{U\bar{U}} = -\frac{1}{(U-\bar{U})^2}\frac{1}{16\pi^2 \mathrm{Im}S'}\int_0^\infty \frac{dt}{t^2}\partial_t \left\{ \sum_{a^I+a^J+\Gamma_2} s_{IJ} + \sum_{2a^I+\Gamma_2} - 16 \sum_{a^I+\Gamma_2} \right\} t\ e^{-\pi t|p|^2/2}\ . \quad (2.4)$$

3. Heterotic realization and threshold corrections

The above model also has a perturbative heterotic description as $SO(32)$ or $E_8 \times E_8$ compactified on $T^4/\mathbf{Z}_2 \times T^2$ with instanton numbers 24 or $(12, 12)$, respectively. Note that the T^4 moduli belong to neutral hypermultiplets, so they do not appear in the Kähler metric of vector moduli. Hence, without loss of generality, we can consider the fermionic point that corresponds to particular values of T^4 radii. In six dimensions, the model is then generated by three vectors of boundary conditions for the world-sheet fermions: the identity $\mathbf{1}$, the supersymmetry vector $\mathbf{S}$ and a vector $\mathbf{b}$ that breaks half of the supersymmetries, as well as the gauge group $SO(32)$ down to $U(16)$. However, at the fermionic point, one also obtains an additional $SO(4)^2$ gauge group factor.

Following refs. [12], we denote by $(\partial X^\mu, \psi^\mu)$ the left-moving supercoordinates and $y^{6,\dots,9}$, $\omega^{6,\dots,9}$, $\chi^{6,\dots,9}$ the remaining left-moving real fermions. The right-movers are $\bar{\partial} X^\mu$ together with the real fermions $\bar{y}^{6,\dots,9}, \bar{\omega}^{6,\dots,9}$ and the complex fermions $\bar{\eta}^i$, $(i = 1, ..., 16)$. In this notation,

$$\begin{aligned} \mathbf{S} &= \{\psi^\mu, \chi^{6,\dots,9}\} \\ \mathbf{b} &= \{\chi^{6,\dots,9}, y^{6,\dots,9}; \bar{y}^{6,\dots,9}, \underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{\bar{\eta}^i}\}\ , \end{aligned} \quad (3.1)$$

where the fermions present in the sets are periodic with the exception of $\bar{\eta}^i \to -i\bar{\eta}^i$, while the remaining are antiperiodic under parallel transport along the string. Note that the vectors $\mathbf{1}$, $\mathbf{S}$ and $2\mathbf{b}$ generate the untwisted sector of the $T^4/\mathbf{Z}_2$ orbifold at the fermionic point, giving rise to $SO(32) \times SO(8)$ with $N = 2$ supersymmetry in $D = 6$. The matter

multiplets in the massless spectrum consist of two $\mathbf{120}_{1/2}$'s of $U(16)$ from the vectors $\mathbf{0}$, $\mathbf{S}$ of the untwisted orbifold sector, sixteen $\mathbf{16}_{1/4}$'s from the vectors $\pm\mathbf{b}$, $\mathbf{S}\pm\mathbf{b}$, $\mathbf{1}+\mathbf{S}\pm\mathbf{b}$, $\mathbf{1}\pm\mathbf{b}$ of the twisted orbifold sector, as well as an additional untwisted hypermultiplet in the $(\mathbf{4,4})$ representation of $SO(4)^2$. The latter can be higgsed away, giving rise to four singlet hypermultiplets, while $SU(16)$ can be broken to $U(1)^{15}$ by turning on the Wilson lines upon toroidal compactification to four dimensions.

The one-loop corrections to the Kähler metric take the general form [13]:

$$K^{H(1)}_{U\bar{U}} = \frac{i}{32\pi^2(U-\bar{U})^2}\sum_s \sum_{P\in\Gamma^{(2,18)}_s} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2}\bar{F}^{(s)}_P(\bar{\tau})\partial_{\bar{\tau}}\left(\tau_2 e^{i\pi\tau|p_L|^2}e^{-i\pi\bar{\tau}|p_R|^2}\right) , \tag{3.2}$$

where $\tau=\tau_1+i\tau_2$ is the modular parameter of the world-sheet torus which is integrated inside its fundamental domain $\mathcal{F}$, and the sum is extended over the momenta P of three Lorenzian lattices $\Gamma^{(2,18)}_s$ associated to the three possible $\mathbf{Z}_2$ orbifold boundary conditions, $s\equiv(1,0),(0,1),(0,0)$ (1 and 0 stand correspondingly for periodic and antiperiodic coordinates, or equivalently for fermion bilinears $y\omega\equiv\partial X$). $\bar{F}^{(s)}_P(\bar{\tau})$ is the contribution to the partition function of the right-moving (bosonic) sector of the heterotic superstring, with the exception of the T^2 zero modes, which are the left and right momenta p_L, p_R. Note that the untwisted $N=4$ sector $s=(1,1)$ gives vanishing contribution, so that the result is independent of the T^4 moduli as mentioned before.

For vanishing Wilson-lines, the right-moving partition functions are factorized

$$\sum_{P\in\Gamma^{(2,18)}_s} \bar{F}^{(s)}_P(\bar{\tau})e^{i\pi\tau|p_L|^2}e^{-i\pi\bar{\tau}|p_R|^2}\Big|_{a^i_{4,5}=0} = \bar{F}^{(s)}Z^{(2,2)} , \tag{3.3}$$

where

$$Z^{(2,2)} = \sum_{p_L,p_R\in\Gamma^{(2,2)}} e^{i\pi\tau|p_L|^2}e^{-i\pi\bar{\tau}|p_R|^2} , \tag{3.4}$$

and

$$\begin{aligned} p_L &= \frac{1}{\sqrt{2\mathrm{Im}T\mathrm{Im}U}}(m_4-m_5U-n_5T-n_4TU) \\ p_R &= \frac{1}{\sqrt{2\mathrm{Im}T\mathrm{Im}U}}(m_4-m_5\bar{U}-n_5T-n_4T\bar{U}) . \end{aligned} \tag{3.5}$$

$\bar{F}^{(s)}$ can be obtained in a straightforward way by using the defining basis vectors (3.1):

$$\begin{aligned}
\bar{F}^{(1,0)} &= \frac{1}{\bar{\eta}^4}\frac{4\bar{\eta}^2}{\bar{\Theta}^2\binom{1}{0}}\frac{\bar{\Theta}^{16}\binom{0}{1/2}+\bar{\Theta}^{16}\binom{0}{-1/2}-\bar{\Theta}^{16}\binom{1}{1/2}-\bar{\Theta}^{16}\binom{1}{-1/2}}{\bar{\eta}^{16}} \\
\bar{F}^{(0,1)} &= -\frac{1}{\bar{\eta}^4}\frac{4\bar{\eta}^2}{\bar{\Theta}^2\binom{0}{1}}\frac{\bar{\Theta}^{16}\binom{1/2}{0}-\bar{\Theta}^{16}\binom{1/2}{1}+\bar{\Theta}^{16}\binom{-1/2}{0}-\bar{\Theta}^{16}\binom{-1/2}{1}}{\bar{\eta}^{16}} \\
\bar{F}^{(0,0)} &= i\frac{1}{\bar{\eta}^4}\frac{4\bar{\eta}^2}{\bar{\Theta}^2\binom{0}{0}}\frac{\bar{\Theta}^{16}\binom{1/2}{1/2}-\bar{\Theta}^{16}\binom{1/2}{-1/2}-\bar{\Theta}^{16}\binom{-1/2}{1/2}+\bar{\Theta}^{16}\binom{-1/2}{-1/2}}{\bar{\eta}^{16}} \ ,
\end{aligned} \tag{3.6}$$

where

$$\Theta\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix} = \sum_{k\in\mathbf{Z}} e^{i\pi(k+\frac{\varepsilon}{2})^2\tau+i\pi k\varepsilon'} \ , \tag{3.7}$$

and η is the Dedekind eta function. The three factors in the r.h.s. of each $\bar{F}^{(s)}$ in eq. (3.6) correspond to the contribution of the (right-moving) space-time and T^2 oscillator modes, the T^4 twisted coordinates, and the sixteen complex fermions $\bar{\eta}^i$ generating the $U(16)$, respectively.

The Wilson lines can now be introduced as boosting parameters of the three (2,2+16) Lorentzian lattices, in a way that when they vanish we recover the above factorized form (3.3): $\Gamma_s^{(2,18)}|_{a^i=0} = \Gamma_s^{(16)} \oplus \Gamma^{(2,2)}$. This can be implemented by going from 6 to 4 dimensions using coordinate dependent compactification, which amounts to changing the boundary conditions in a way that respects modular invariance [14]. In ref. [14], this procedure was presented for a compactification on S^1. It is not difficult to extend these results in the case of T^2 and find that in the presence of the 16 Wilson lines $a^i_{4,5}$, each term of eq. (3.3) that is of the form $Z^{(2,2)}\bar{\Theta}\binom{u}{v}^{16}$ (for $u, v = 0, 1, \pm 1/2$) is modified:

$$\begin{aligned}
Z^{(2,2)}\bar{\Theta}\begin{pmatrix}u\\ v\end{pmatrix}^{16} \to \frac{\mathrm{Im}T}{\tau_2}\sum_{A=\begin{pmatrix}-n_5 & l_5\\ -n_4 & l_4\end{pmatrix}} e^{2i\pi\bar{T}\det A} e^{-\frac{\pi \mathrm{Im}T}{\tau_2 \mathrm{Im}U}\left|(1,U)A\binom{\tau}{1}\right|^2} \ \times \\
\times \prod_{i=1}^{16} e^{-i\pi u(l_4a_4^i+l_5a_5^i)} e^{i\frac{\pi}{2}(n_4a_4^i+n_5a_5^i)(l_4a_4^i+l_5a_5^i)} \quad \bar{\Theta}\begin{pmatrix}u-2(n_4a_4^i+n_5a_5^i)\\ v+2(l_4a_4^i+l_5a_5^i)\end{pmatrix}(\bar{\tau}) \ ,
\end{aligned} \tag{3.8}$$

where the integers $l_{4,5}$ correspond to $m_{4,5}$ of eq. (3.5) after Poisson resummation. Performing a Poisson resummation back to $m_{4,5}$, one finds that the T^2 momenta p_L, p_R for non-vanishing

$a^i_{4,5}$ have the same form as in eq. (3.5) with the replacement:

$$m_{4,5} \to m_{4,5} - \sum_{i=1}^{16} a^i_{4,5}\left(k_i + \frac{\alpha+\epsilon}{4}\right) + \frac{1}{2}\sum_{i=1}^{16} a^i_{4,5}(n_4 a^i_4 + n_5 a^i_5) \ , \tag{3.9}$$

where we defined $\alpha = 2u - \epsilon = \pm 1$. Furthermore, the partition functions $\bar{F}^{(s)}_P$ become:

$$\bar{F}^{(\epsilon,\epsilon')}_P = \sum_{\alpha=\pm} \alpha^{1+\epsilon'} \zeta_{\epsilon,\epsilon'} \frac{8}{\bar{\eta}^{18}\bar{\Theta}^2\binom{\epsilon}{\epsilon'}} e^{-i\pi\bar{\tau}\sum_{i=1}^{16}(k_i+\frac{\alpha+\epsilon}{4}-n_4 a^i_4 - n_5 a^i_5)^2} \ , \tag{3.10}$$

where $(\epsilon,\epsilon') = (1,0)$, $(0,1)$ or $(0,0)$ and the coefficients $\zeta_{\epsilon,\epsilon'}$ are given by:

$$\zeta_{1,0} = -\cos\left(\frac{\pi}{2}\sum_{i=1}^{16} k_i\right) \qquad \zeta_{0,1} = -\frac{1-(-)^{\sum_{i=1}^{16} k_i}}{2} \qquad \zeta_{0,0} = \sin\left(\frac{\pi}{2}\sum_{i=1}^{16} k_i\right) \ . \tag{3.11}$$

As a result, the momenta P of the $\Gamma^{(2,18)}_s$ lattices are characterized by $4+16$ integers $m_{4,5}$, $n_{4,5}$ and k_i associated to the T^2 coordinates and $U(16)$ right-moving fermions, respectively, as well as by the parameter $\alpha = \pm 1$ related to $u = 0, 1, \pm 1/2$ labelling the four conjugacy classes of $SU(16)$ level-one.

Since under heterotic – type I duality the T-modulus is mapped into S', in order to compare the above result with the type I expression (2.4), one has to take the limit $\mathrm{Im}T \to \infty$ [1]. From eq. (3.2) and the expression of the T^2 momenta (3.5), it follows that only the Kaluza-Klein momenta corresponding to $n_4 = n_5 = 0$ survive in this limit. Moreover, by making the change of variable

$$\tau_2 = \frac{t}{4}\mathrm{Im}T \ , \tag{3.12}$$

one can easily show that the integration domain becomes the strip $t \geq 0$, $-1/2 \leq \tau_1 \leq 1/2$, up to exponentially small corrections in $\mathrm{Im}T$. We now perform the τ_1 integration, which takes the form:

$$\int_{-1/2}^{1/2} d\tau_1 e^{-2i\pi\tau_1\left\{\sum_{i=1}^{16}(k_i+\frac{\alpha+\epsilon}{4})^2/2 - \frac{3+\epsilon}{4} + \frac{1+\epsilon}{2}N\right\}} \ , \tag{3.13}$$

where N is a non-negative integer coming from the oscillator expansion of the partition function (3.10). In the untwisted sector $\epsilon = 1$ the frequencies are integers and the intercept is at -1, while in the twisted sector $\epsilon = 0$ the frequencies are half-integers and the intercept

is at $-3/4$. An inspection of the exponent in eq. (3.13) using the expressions (3.11) shows that the coefficient of $2i\pi\tau_1$ is always an integer. In fact, this follows immediately in the sector $(1,0)$, where $\epsilon = 1$ and $\sum_i k_i$ is even, while in the remaining two sectors where $\epsilon = 0$ and $\sum_i k_i$ is odd it is half-integer. However, since the sum of these sectors is proportional to $(1-\alpha(-)^{N+(1-\sum_i k_i)/2})$, one can show that only integer values survive. As a result, the τ_1 integration (3.13) implies that the exponent vanishes:

$$\frac{1}{2}\sum_{i=1}^{16}\left(k_i+\frac{\alpha+\epsilon}{4}\right)^2-\frac{3+\epsilon}{4}+\frac{1+\epsilon}{2}N=0\ . \tag{3.14}$$

Equation (3.14) in the $(1,0)$ sector can be satisfied only for $\alpha=-1$; the solutions are $N=1$ and $k_i=0$ or $N=0$ and $\sum_i k_i^2=2$. In the sum of the other two sectors, it can be satisfied only for $N=0$; the solutions in this case are $k_i=(0,\cdots,0,\alpha,0,\cdots,0)$. Using eqs. (3.2), (3.5) and (3.9-3.11), we find:

$$K^{H(1)}_{U\bar{U}}\to-\frac{1}{(U-\bar{U})^2}\frac{1}{8\pi^2\mathrm{Im}T}\int_0^\infty\frac{dt}{t^2}\partial_t\left\{16\sum_{\Gamma_2}-\sum_{\substack{i<j\\ \varepsilon_1,\varepsilon_2=\pm1\\ \varepsilon_1 a^i+\varepsilon_2 a^j+\Gamma_2}}\varepsilon_1\varepsilon_2-8\sum_{\substack{i\\ \varepsilon=\pm1\\ \varepsilon\left(a^i-\sum_j\frac{a^j}{4}\right)+\Gamma_2}}\right\}t\ e^{-\pi t|p|^2/2}\ , \tag{3.15}$$

where p's are given in eq. (2.2). By introducing again the index $I=i,\bar{\imath}$ defined in section 2 and using the traceless condition of $SU(16)$ generators $\sum_j a^j=0$, it is easy to show that the third term inside the bracket of eq. (3.15) can be written as $-8\sum_{a^I+\Gamma_2}$, while the first two become $\frac{1}{2}\sum^{I\neq J}_{a^I+a^J+\Gamma_2}$. Comparing with the type I expression (2.4), we conclude that the $(1,0)$ sector reproduces the contribution of the 99-open strings on the annulus and Möbius strip, while the twisted sector $(0,1)+(0,0)$ reproduces the 95-contribution.

A similar analysis can be done for the threshold corrections to the $SU(16)$ gauge couplings Δ, which are given by:

$$\begin{aligned}\Delta^H\ =\ &-\frac{1}{8}\sum_s\sum_{P\in\Gamma_s^{(2,18)}}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\bar{F}_P^{(\epsilon,\epsilon')}\left\{\left[\sum_{i=1}^{16}\left(k_i+\frac{\alpha+\epsilon}{4}-n_4a_4^i-n_5a_5^i\right)q_i\right]^2-\frac{1}{2\pi\tau_2}\right\}\\ &\times e^{i\pi\tau|p_L|^2}e^{-i\pi\bar{\tau}|p_R|^2}\ ,\end{aligned} \tag{3.16}$$

in the notation of eqs. (3.2), (3.5) and (3.9-3.11). Taking the limit $\mathrm{Im}T \to \infty$ as before, one finds

$$\Delta^H(T,U,A_i) \to c_0\,\mathrm{Im}T + \Delta^I(U,A_i) + 4\pi^2 K^{I(1)} \tag{3.17}$$

up to exponentially suppressed corrections. The last term, which originates from the $1/2\pi\tau_2$ "universal" part of the threshold corrections (3.16), is proportional to $1/\mathrm{Im}T$ and reproduces the type I Kähler metric, eq. (2.4), upon the replacement $T \to S'$.

The T-independent term Δ^I comes from the "group-dependent" part of threshold corrections (3.16) and is given by

$$\begin{aligned}\Delta^I &= \frac{1}{8}\int_0^{\prime\infty}\frac{dt}{t}\left\{\sum_{\substack{i<j\\ \varepsilon_1,\varepsilon_2=\pm1\\ \varepsilon_1 a^i+\varepsilon_2 a^j+\Gamma_2}}\varepsilon_1\varepsilon_2(\varepsilon_1 q_i+\varepsilon_2 q_j)^2 + 16\sum_{\substack{i\\ \varepsilon=\pm1\\ \varepsilon\left(a^i-\sum_{j=1}^{16}\frac{a^j}{4}\right)+\Gamma_2}}\left(q_i-\sum_{j=1}^{16}\frac{q_j}{4}\right)^2\right\}e^{-\pi t|p|^2/2}\\ &= -\frac{1}{8}\int_0^{\prime\infty}\frac{dt}{t}\left\{\sum_{a^I+a^J+\Gamma_2}s_{IJ}(q_I+q_J)^2 + \sum_{2a^I+\Gamma_2}(2q_I)^2 - 16\sum_{a^I+\Gamma_2}q_I^2\right\}e^{-\pi t|p|^2/2}\,,\end{aligned} \tag{3.18}$$

where in the second line we used again the index $I = i, \bar{\imath}$ (such that $q_{\bar{\imath}} = -q_i$) and the traceless condition of the $SU(16)$ generators $\sum_j q_j = 0$. The prime in the integral indicates that the apparent quadratic divergence in the ultraviolet limit $t \to 0$ has been subtracted. Equation (3.18) then coincides with the threshold corrections to the corresponding type I model [10]. There, the ultraviolet divergence is removed by a regularization prescription, which amounts to introducing a uniform cutoff in the transverse closed string channel as dictated by the tadpole cancellation. Note that the integrals (3.16) and (3.18) are infrared finite due to non-vanishing Wilson lines.

The first term of eq. (3.17) is linear in $\mathrm{Im}T$ with a constant coefficient c_0 that controls the quadratic ultraviolet divergence appearing in the limit $\mathrm{Im}T \to \infty$. In fact, $\mathrm{Im}T$ acts as a regulator in the heterotic integral (3.16) after the change of variable (3.12). In order to derive eq. (3.17), one goes back to the Poisson-resummed expression (3.8) with $n_4 = n_5 = 0$.

Then, the term linear in ImT arises from the $l_4 = l_5 = 0$ contribution with

$$c_0 = -\frac{1}{8}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_s \sum_{P\in\Gamma_s^{(16)}} \bar{F}_P^{(s)} \left\{ \left[\sum_{i=1}^{16} \left(k_i + \frac{\alpha+\epsilon}{4} \right) q_i \right]^2 - \frac{1}{2\pi\tau_2} \right\} . \tag{3.19}$$

This term is mapped under duality to the part of the type I tree-level gauge coupling proportional to ImS', cf. eqs. (1.1-1.3) for prepotentials. In the type I model under consideration, such a term is absent, which implies that the integral (3.19) should vanish. This is indeed the case as will be shown in the next section when considering similar contributions to the gravitational couplings.

4. Higher derivative F-terms

We now consider the class of higher derivative F–terms of the form

$$I_g = \mathcal{F}_g W^{2g} , \tag{4.1}$$

for integer $g \geq 0$ and W being the Weyl superfield

$$W_{\mu\nu} = F_{\mu\nu} - R_{\mu\nu\lambda\rho}\theta^1\sigma_{\lambda\rho}\theta^2 + \cdots , \tag{4.2}$$

which is anti-self-dual in the Lorentz indices. $F_{\mu\nu}$ and $R_{\mu\nu\lambda\rho}$ are the (anti-self-dual) graviphoton field strength and Riemann tensor, respectively. The couplings $\mathcal{F}_g$ are holomorphic sections of degree $2-2g$ of the vector moduli, up to a holomorphic anomaly for $g \geq 1$ [15]. They generalize the well-known prepotential $\mathcal{F} \equiv \mathcal{F}_0$, so that $\mathcal{F}_1$ determines the gravitational R^2 couplings etc. On the type II side, $\mathcal{F}_g$ is determined entirely from genus g, while on the heterotic and type I, these couplings arise already at one loop (with additional tree-level contibutions for $\mathcal{F}_0$ and $\mathcal{F}_1$).

On the heterotic side, the torus amplitude involving two gravitons and $(2g-2)$ graviphotons was used in ref. [6] to extract the $\mathcal{F}_g^H$ functions. The result involves a universal $2g$-point bosonic correlator $G_g^H(\tau,\bar{\tau})$ derived from the generating function

$$G^H(\lambda,\tau,\bar{\tau}) = \sum_{g=1}^{\infty} \lambda^{2g} G_g^H(\tau,\bar{\tau}) = \left(\frac{2\pi i\lambda\bar{\eta}^3}{\bar{\theta}_1(\lambda,\bar{\tau})} \right)^2 e^{-\pi\frac{\lambda^2}{\tau_2}} + 1 , \tag{4.3}$$

which is modular invariant under $\tau \to \dfrac{a\tau+b}{c\tau+d}$, $\lambda \to \pm\dfrac{\lambda}{c\tau+d}$. Here, θ_1 is the odd theta function. Then, the generating function

$$\mathcal{F}^H(\lambda, T, U, A_i) = \sum_{g=1}^{\infty} g^2 \lambda^{2g} \mathcal{F}_g^H(T, U, A_i)\,, \tag{4.4}$$

is given by [6, 16]:

$$\mathcal{F}^H(\lambda, T, U, A_i) = \frac{\lambda^2}{32\pi^2} \sum_s \sum_{P\in\Gamma_s^{(2,18)}} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \bar{F}_P^{(s)}(\bar{\tau}) e^{i\pi\tau|p_L|^2} e^{-i\pi\bar{\tau}|p_R|^2} \frac{d^2}{d\tilde{\lambda}^2} G^H(\tilde{\lambda}, \tau, \bar{\tau})\ , \tag{4.5}$$

where we used the notation of eqs. (3.2), (3.5), (3.9-3.11), (4.3) and $\tilde{\lambda} = \dfrac{\bar{p}_L \tau_2 \lambda}{\sqrt{2\,\mathrm{Im}T\mathrm{Im}U}}$.

Now, following the steps of section 3, we can take the limit $\mathrm{Im}T \to \infty$:

$$\begin{aligned}\mathcal{F}^H \to\ & c_1\lambda^2 \mathrm{Im}T - \frac{\lambda^2}{32}\int_0^{\prime\infty} \frac{dt}{t} \left\{ 8\sum_{\Gamma_2} e^{-\pi t|p|^2/2} \frac{d^2}{d\lambda^2}\left[\left(\frac{\lambda}{\sin\pi\tilde{\lambda}}\right)^2 \left(2-\sin^2\pi\tilde{\lambda}\right)\right] \right. \\ & \left. - \sum_{\substack{i<j\\ \varepsilon_1,\varepsilon_2=\pm1\\ \varepsilon_1 a^i+\varepsilon_2 a^j+\Gamma_2}} \varepsilon_1\varepsilon_2 e^{-\pi t|p|^2/2} \frac{d^2}{d\lambda^2}\left(\frac{\lambda}{\sin\pi\tilde{\lambda}}\right)^2 - 8\sum_{\substack{i\\ \varepsilon=\pm1\\ \varepsilon\left(a^i-\sum_j \frac{a^j}{4}\right)+\Gamma_2}} e^{-\pi t|p|^2/2} \frac{d^2}{d\lambda^2}\left(\frac{\lambda}{\sin\pi\tilde{\lambda}}\right)^2 \right\},\end{aligned} \tag{4.6}$$

where p is given in eq. (2.2) and after the change of variables (3.12) $\tilde{\lambda}$ becomes $\tilde{\lambda} = \dfrac{\bar{p}t\lambda}{4\sqrt{2\mathrm{Im}U}}$. The first two sums on the r.h.s. arise from the untwisted $(1,0)$ sector, whereas the third one is due to the twisted $(0,1)+(0,0)$ sectors. Equation (4.6) can be rewritten as:

$$\begin{aligned}\mathcal{F}^H \ \to\ & c_1\lambda^2 \mathrm{Im}T - \frac{\lambda^2}{32\pi^2}\int_0^{\prime\infty} \frac{dt}{t} \left\{ \sum_{a^I+a^J+\Gamma_2} s_{IJ} \frac{d^2}{d\lambda^2}\left(\frac{\pi\lambda}{\sin\pi\tilde{\lambda}}\right)^2 \right. \\ & \left. + \sum_{2a^I+\Gamma_2} \frac{d^2}{d\lambda^2}\left(\frac{\pi\lambda}{\sin\pi\tilde{\lambda}}\right)^2 - 32\pi^2 \sum_{\Gamma_2} - 16 \sum_{a^I+\Gamma_2} \frac{d^2}{d\lambda^2}\left(\frac{\pi\lambda}{\sin\pi\tilde{\lambda}}\right)^2 \right\} e^{-\pi t|p|^2/2}\ .\end{aligned} \tag{4.7}$$

The above manipulations in taking the limit $\mathrm{Im}T \to \infty$ are valid for all $\mathcal{F}_g^H$ with $g \geq 2$ for which the t-integration converges. For $\mathcal{F}_1^H$ the integral diverges in the infrared limit $t \to +\infty$, while it has also an apparent quadratic ultraviolet divergence as $t \to 0$, which has been subtracted as indicated by the prime. The infrared divergence is already present in the original expression (4.5) before taking the limit and reproduces the trace anomaly of the effective field theory. Note that in contrast to the case of threshold corrections to

gauge couplings, the infrared divergence in the gravitational couplings cannot be regulated by non-vanishing Wilson lines since there are still massless states that contribute to the trace anomaly. In fact the gravitational beta-function [8] is $b_{\rm grav} = (23+n_H-n_V)/12 = 2/3$, which is reproduced by the infrared divergence arising from the third and first terms (with $a^I + a^J = 0$) in the integrand (4.7). On the other hand, the ultraviolet divergence is regulated by ImT as discussed in the previous section, and gives rise to a term linear in ImT, which is mapped under duality to a tree-level term proportional to ImS'. Following the same steps as in the derivation of c_0 in eq. (3.19), one finds

$$c_1 = -\frac{1}{8}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_s \bar{F}^{(s)} \left(\frac{2i}{\pi}\partial_{\bar\tau} \ln \bar\eta - \frac{1}{2\pi\tau_2} \right) , \tag{4.8}$$

where we used eq. (4.5) and

$$\frac{d^2}{d\lambda^2} G^H \bigg|_{\lambda=0} = -8i\pi\partial_{\bar\tau} \ln \bar\eta + \frac{2\pi}{\tau_2} \equiv -\bar{G}_2 \tag{4.9}$$

from eq. (4.3).

The integral (4.8) can be evaluated using the method of ref. [17]. After replacing $\frac{1}{\tau_2} = \frac{i}{\pi}\partial_{\bar\tau}\bar{G}_2$, the integral becomes a total derivative and one is left with an integration along the boundary of the fundamental domain $\mathcal{F}$. We can then easily show that for arbitrary integer power n,

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \bar{G}_2^n F(\bar{q}) = \frac{1}{(n+1)\pi} \left[\bar{G}_2^{n+1} F(\bar{q}) \right]_{\bar{q}^0} , \tag{4.10}$$

where the subscript $\bar{q}^0$ denotes the constant term in the $\bar{q} = e^{-2i\pi\bar\tau}$ expansion. Using the above formula for $n = 1$ and the expansions

$$\bar{G}_2 = \frac{2\pi^2}{3}(1 - 24\bar{q} + \cdots) , \quad \sum_s \bar{F}^{(s)} = \frac{2}{\bar{q}}(1 - 240\bar{q} + \cdots) \tag{4.11}$$

following from eqs. (3.6), we obtain

$$c_1 = 4\pi . \tag{4.12}$$

After combining the one-loop contribution (4.12) with the tree-level term, we obtain (in our conventions):

$$\mathcal{F}_1^H = 4\pi(\mathrm{Im}S + \mathrm{Im}T) + f_1^H , \tag{4.13}$$

where f_1^H represents the contribution contained in the regularized integral of eq. (4.7). After the replacement $T \to S'$, the first two terms reproduce the tree-level type I expression for $\mathcal{F}_1^I$, cf. eqs. (1.4-1.6). Note the symmetry of this result under exchange of S and S', which is a consequence of T-duality in the type I theory (inverting the volume of K_3) but a non-perturbative symmetry on the heterotic side. Finally f_1^H, as well as all higher $\mathcal{F}_g^H$'s encoded in the integral (4.7), should be compared with the corresponding one-loop contribution in type I theory.

Before discussing the type I contribution, we can compute at this point the constant c_0 entering in the expression of threshold corrections (3.17) and given by the integral (3.19). Using the expressions (3.19) and (4.8), it is easy to check that the difference $c_1 - c_0$ is given by an integral of the form (4.10) with $n = 0$:

$$\begin{aligned} c_1 - c_0 &= \frac{1}{8}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_s \sum_{P\in\Gamma_s^{(16)}} \bar{F}_P^{(s)} \left\{ \left[\sum_{i=1}^{16} \left(k_i + \frac{\alpha+\epsilon}{4} \right) q_i \right]^2 - \frac{2i}{\pi}\partial_{\bar\tau} \ln \bar\eta \right\} \\ &= \frac{1}{8}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} (164 - \frac{1}{6} j(\bar q)) \,, \end{aligned} \tag{4.14}$$

where j is the modular invariant function with a simple pole at infinity with residue 1, $j(\bar q) = 1/\bar q + 744 + \ldots$ We now use eq. (4.10) and the expansions (3.6), (4.11) to obtain $c_1 - c_0 = 4\pi$ and from the result (4.12) we deduce

$$c_0 = 0 \,, \tag{4.15}$$

as was already announced in the previous section.

Let us now turn to the type I calculation of the $\mathcal{F}_g^I$'s. They can be determined by the one-loop type I amplitude involving two gravitons and $(2g-2)$ graviphotons [18, 7]. Following the steps of ref. [7] and using the results of ref. [1], we obtain the following general expression for the generating function (4.4):

$$\mathcal{F}^I(\lambda) = \frac{\lambda^2}{64\pi^2} \sum_{\sigma=\mathcal{A},\mathcal{M},\mathcal{K}} \sum_{s,\Gamma_2^{(s)}} \int_0^\infty \frac{dt}{t} \mathcal{I}_\sigma^{(s)} e^{-\pi t|p|^2/2} \frac{d^2}{d\tilde\lambda^2} G_\sigma^I(\tilde\lambda, t) + \mathcal{F}^I_{\text{torus}} \,, \tag{4.16}$$

where the first sum is extended over the three different open string surfaces, annulus ($\mathcal{A}$), Möbius strip ($\mathcal{M}$) and Klein bottle ($\mathcal{K}$); s denotes the various sectors of the theory (orbifold as well as open string boundary conditions), and $\mathcal{I}_\sigma^{(s)}$ is an index associated to K_3 which counts the open string spinors and closed string Ramond-Ramond bosons weighted with the fermion-parity operator $(-)^{F_{\rm int}}$. The generating partition functions G_σ^I were computed in refs. [18, 7] and they have no explicit t-dependence:

$$\begin{aligned} G_{\mathcal{A}}^I &= G_{\mathcal{M}}^I = \left(\frac{\pi\tilde{\lambda}}{\sin\pi\tilde{\lambda}}\right)^2 , \\ G_{\mathcal{K}}^I &= \left(\frac{2\pi\tilde{\lambda}}{\tan\pi\tilde{\lambda}}\right)^2 = \left(\frac{2\pi\tilde{\lambda}}{\sin\pi\tilde{\lambda}}\right)^2 - 4\tilde{\lambda}^2\pi^2 . \end{aligned} \tag{4.17}$$

It is easy to see that as in the heterotic case, the t-integration converges for all $\mathcal{F}_g^I$ with $g \geq 2$. Moreover, $\mathcal{F}_1^I$ has an infrared divergence as $t \to \infty$, which reproduces the same trace anomaly $b_{\rm grav} = 2/3$ (after taking into account the torus contribution $\mathcal{F}_{\rm torus}^I$ given below). On the other hand, the apparent ultraviolet divergence at $t = 0$ is cancelled among the contributions of different surfaces when the integral is appropriately regularized as discussed in section 3.

The last term in eq. (4.16) stands for the contribution of the world-sheet torus, which is non-vanishing only for $\mathcal{F}_1^I$ due to the odd-odd and even-even spin structures. The odd-odd contribution is [7]:

$$\mathcal{F}_{\rm torus}^I\Big|_{\text{odd-odd}} = \frac{\lambda^2}{16}\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{s,\Gamma^{(2,2)}} \mathcal{I}_{\mathcal{T}}^{(s)} e^{i\pi\tau|p_L^{\rm II}|^2} e^{-i\pi\bar{\tau}|p_R^{\rm II}|^2} , \tag{4.18}$$

where $\mathcal{I}_{\mathcal{T}}^{(s)} = {\rm Tr}_R^{(s)}(-)^{F_{\rm int}}$ is the Witten index in the sector s and $p_L^{\rm II}$, $p_R^{\rm II}$ are given by eq. (3.5) with the replacement $T \to i\sqrt{G}$. The τ-integration can be performed explicitly with the result [9]:

$$\mathcal{F}_{\rm torus}^I\Big|_{\text{odd-odd}} = \frac{\lambda^2}{16}\mathcal{I}_{\mathcal{T}}\left[\ln\left({\rm Im}U|\eta(U)|^4\right) + \ln\left(\sqrt{G}|\eta(i\sqrt{G})|^4\right) + c'\right] , \tag{4.19}$$

where $\mathcal{I}_{\mathcal{T}} = \sum_s \mathcal{I}_{\mathcal{T}}^{(s)}$ and c' is an infinite constant due to the infrared divergence. The contribution of even-even spin structures yields a similar expression with a relative minus

sign between the two terms in the r.h.s. We thus find

$$\mathcal{F}^I_{\rm torus} = \frac{\lambda^2}{8}\mathcal{I}_{\mathcal{T}} \ln\left({\rm Im}U|\eta(U)|^4\right) \ , \tag{4.20}$$

up to a moduli independent additive constant. This result is independent of $\sqrt{G}$ and can be reproduced by considering the integral (4.18) in the limit $\sqrt{G} \to \infty$. The complex integration variable τ is then replaced by the real t and the momenta become those of eq. (2.2):

$$\mathcal{F}^I_{\rm torus} = \frac{\lambda^2}{8}\sum_{\Gamma_2}\mathcal{I}_{\mathcal{T}}\int_0^\infty \frac{dt}{t}e^{-\pi t|p|^2/2} \ , \tag{4.21}$$

where the ultraviolet divergence at $t=0$ is regularized according to the type I prescription.

The second term in the r.h.s. of eq. (4.19) depends, in the context of type IIB, on the Kähler modulus of T^2 with its real part projected away by the type I reduction. As was shown in ref. [15] in general, this term does not couple to the gravitational R^2 couplings, which depend only on the complex structure moduli. On the contrary, in type IIA the situation is reversed because the even-even contribution comes with a minus sign, and it is the second term that survives. Moreover, under duality these corrections are mapped into gravitational instantons on the heterotic side [19]. It is interesting to note that if such corrections survive for some quantities in the context of type I strings, since under duality $\sqrt{G} \to ({\rm Im}T{\rm Im}S/V_{K_3})^{1/2}$ (with V_{K_3} the volume of K_3), they are mapped on the heterotic side into stringy-like non-perturbative effects of the type $e^{-1/g}$.

Using the above results in eqs. (4.16), (4.17) and (4.21), we can write the following general expression for the generating function

$$\begin{aligned}\mathcal{F}^I(\lambda) = \quad & \frac{\lambda^2}{64\pi^2}\sum_{s,\Gamma_2^{(s)}}\int_0^\infty \frac{dt}{t}(\mathcal{I}^{(s)}_{\mathcal{A}}+\mathcal{I}^{(s)}_{\mathcal{M}}+4\mathcal{I}^{(s)}_{\mathcal{K}})e^{-\pi t|p|^2/2}\frac{d^2}{d\lambda^2}\left(\frac{\lambda\pi}{\sin\pi\tilde{\lambda}}\right)^2 \\ & +\frac{\lambda^2}{8}\int_0^\infty\frac{dt}{t}\sum_{s,\Gamma_2}(\mathcal{I}^{(s)}_{\mathcal{T}}-\mathcal{I}^{(s)}_{\mathcal{K}})e^{-\pi t|p|^2/2} \ .\end{aligned} \tag{4.22}$$

The sum $(\mathcal{I}^{(s)}_{\mathcal{A}}+\mathcal{I}^{(s)}_{\mathcal{M}}+4\mathcal{I}^{(s)}_{\mathcal{K}})/4$ is reduced to a sum over $N=2$ BPS hypermultiplets minus vector multiplets (which come only from massless states in six dimensions), hence for $g \geq 2$,

$\mathcal{F}_g^I$'s are given by expressions that are very similar to the prepotential [1].[2] However, $\mathcal{F}_1^I$ has a different form due to the presence of the second term in eq. (4.22). In fact, the difference $(\mathcal{I}_{\mathcal{T}}^{(s)} - \mathcal{I}_{\mathcal{K}}^{(s)})$ receives contributions only from the (untwisted) $N = 4$ sector of the theory, which are already present at the level of the corresponding $N = 4$ supersymmetric type IIB theory [19].

Specializing to the model under consideration, we obtain the following contributions for the various open string boundary conditions:

$$\begin{aligned}
&\text{99-sector} \quad : \quad \frac{1}{4}\sum_{s,\Gamma_2^{(s)}} \mathcal{I}_{\mathcal{A}}^{(s)} = -\frac{1}{2}\sum_{a^I+a^J+\Gamma_2} s_{IJ} \quad ; \quad \frac{1}{4}\sum_{s,\Gamma_2^{(s)}} \mathcal{I}_{\mathcal{M}}^{(s)} = -\frac{1}{2}\sum_{2a^I+\Gamma_2} \\
&\text{95-sector} \quad : \quad \frac{1}{4}\sum_{s,\Gamma_2^{(s)}} \mathcal{I}_{\mathcal{A}}^{(s)} = 8\sum_{a^I+\Gamma_2} \quad ; \quad \mathcal{I}_{\mathcal{M}}^{(s)} = 0 \\
&\text{55-sector} \quad : \quad \frac{1}{4}\sum_{s,\Gamma_2^{(s)}} (\mathcal{I}_{\mathcal{A}}^{(s)} + \mathcal{I}_{\mathcal{M}}^{(s)}) = -16\sum_{\Gamma_2} ,
\end{aligned} \tag{4.23}$$

where the sum over s refers to the orbifold sectors. On the other hand, in the closed string sector we have $\mathcal{I}_{\mathcal{K}} = 16$ and $\mathcal{I}_{\mathcal{T}} = 24$ (the Euler number of K_3). Using these results we find that the generating function (4.22) yields the heterotic result for f_1^H [defined in eq. (4.13)], as well as for all higher $\mathcal{F}_g^H$'s given in eq. (4.7). The first two terms in the integral of the r.h.s. of eq. (4.7) now correspond to the annulus and Möbius strip contribution in the 99-sector, the last term to the contribution of the 95-sector and the third term to the contribution of the torus.

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[2]See also the recent preprint quoted in ref. [20].

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