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Hierarchical organization in complex networks

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Many real networks in nature and society share two generic properties: they are scale-free and they display a high degree of clustering. We show that these two features are the consequence of a hierarchical organization, implying that small groups of nodes organize in a hierarchical manner into increasingly large groups, while maintaining a scale-free topology. In hierarchical networks, the degree of clustering characterizing the different groups follows a strict scaling law, which can be used to identify the presence of a hierarchical organization in real networks. We find that several real networks, such as the Worldwideweb, actor network, the Internet at the domain level, and the semantic web obey this scaling law, indicating that hierarchy is a fundamental characteristic of many complex systems.

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I. INTRODUCTION

In the past few years, an array of discoveries have redefined our understanding of complex networks (for reviews, see Refs. [1,2]). The availability of detailed maps, capturing the topology of such diverse systems as the cell [3–6], the Worldwideweb [7], or the sexual network [8] have offered scientists for the first time the chance to address in quantitative terms the generic features of real networks. As a result, we learned that networks are far from being random, but are governed by strict organizing principles that generate systematic and measurable deviations from the topology predicted by the random graph theory of Erdős and Rényi [9,10], the basic model used to describe complex webs in the past four decades.

Two properties of real networks have generated considerable attention. First, measurements indicate that most networks display a high degree of clustering. Defining the clustering coefficient for node i with k_i links as $C_i = 2n_i / k_i(k_i - 1)$, where n_i is the number of links between the k_i neighbors of i , empirical results indicate that C_i averaged over all nodes is significantly higher for most real networks than for a random network of similar size [1,2,11]. Furthermore, the clustering coefficient of real networks is to a high degree independent of the number of nodes in the network (see Fig. 9 in Ref. [1]). At the same time, many networks of scientific or technological interest, ranging from the Worldwideweb [7] to biological networks [3–6] have been found to be scale-free [12,13], which means that the probability that a randomly selected node has k links (i.e., degree k) follows $P(k) \sim k^{-\gamma}$, where γ is the degree exponent.

The scale-free property and clustering are not exclusive: for a large number of real networks, including metabolic networks [3,4], the protein interaction network [5,6], the World Wide Web [7], and even some social networks [14–16], the scale-free topology and high clustering coexist. Yet, most models that proposed to describe the topology of complex networks have difficulty capturing simultaneously these two features. For example, the random network model [9,10] can account neither for the scale-free nor for the clustered nature of real networks, as it predicts an exponential degree

distribution, and the average clustering coefficient $C(N)$ decreases as N^{-1} with the number of nodes in the network. Scale-free networks, capturing the power-law degree distribution, predict a much larger clustering coefficient than a random network. Indeed, numerical simulations indicate that for one of the simplest models [12,13], the average clustering coefficient depends on the system size as $C(N) \sim N^{-0.75}$ [1,2], significantly larger for large N than the random network prediction $C(N) \sim N^{-1}$. Yet, this prediction still disagrees with the finding that for several real systems, C is independent of N [1].

Here, we show that the fundamental discrepancy between models and empirical measurements is rooted in a previously disregarded, yet generic feature of many real networks: their hierarchical topology. Indeed, many networks are fundamentally modular: one can easily identify groups of nodes that are highly interconnected with each other, but have only a few or no links to nodes outside of the group to which they belong to. In society, such modules represent groups of friends or co-workers [17]; in the WWW, they denote communities with shared interests [18,19]; in the actor network, they characterize specific genres or simply individual movies. Some groups are small and tightly linked, others are larger and somewhat less interconnected. This clearly identifiable modular organization is at the origin of the high clustering coefficient seen in many real networks. Yet, models reproducing the scale-free property of real networks [1,2] distinguish nodes based only on their degree, and are blind to node characteristics that could lead to a modular topology.

In order to bring modularity, the high degree of clustering, and the scale-free topology under a single roof, we need to assume that modules combine into each other in a hierarchical manner, generating what we call a *hierarchical network*. The presence of a hierarchy and the scale-free property impose strict restrictions on the number and the degree of cohesiveness of the different groups present in a network, which can be captured in a quantitative manner using a scaling law, describing the dependence of the clustering coefficient on the node degree. We use this scaling law to identify the presence of a hierarchical architecture in several real networks, and the absence of such hierarchy in geographically organized webs.

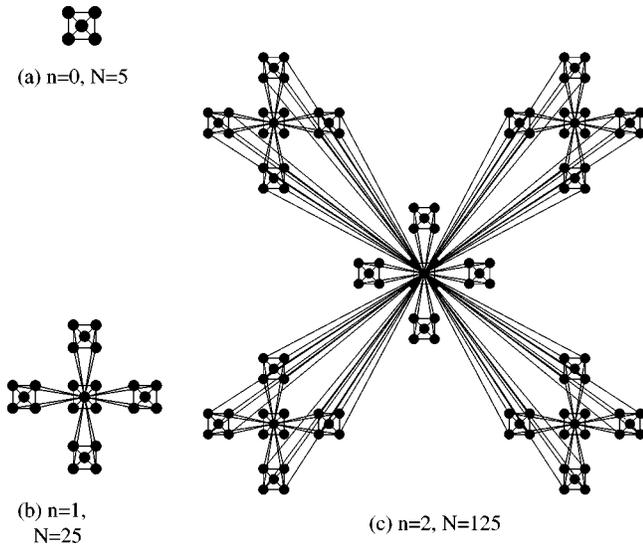


FIG. 1. The iterative construction leading to a hierarchical network. Starting from a fully connected cluster of five nodes shown in (a) (note that the diagonal nodes are also connected—links not visible), we create four identical replicas, connecting the peripheral nodes of each cluster to the central node of the original cluster, obtaining a network of $N=25$ nodes (b). In the next step, we create four replicas of the obtained cluster, and connect the peripheral nodes again, as shown in (c), to the central node of the original module, obtaining a $N=125$ -node network. This process can be continued indefinitely.

II. HIERARCHICAL NETWORK MODEL

We start by constructing a hierarchical network model that combines the scale-free property with a high degree of clustering. Our starting point is a small cluster of five densely linked nodes [Fig. 1(a)]. Next, we generate four replicas of this hypothetical module and connect the four external nodes of the replicated clusters to the central node of the old cluster, obtaining a large 25-node module [Fig. 1(b)]. Subsequently, we again generate four replicas of this 25-node module, and connect the 16 peripheral nodes to the

central node of the old module [Fig. 1(c)], obtaining a new module of 125 nodes. These replication and connection steps can be repeated indefinitely, in each step, increasing the number of nodes in the system by a factor 5.

Precursors to the model described in Fig. 1 have been proposed in Ref. [20] and extended and discussed in Ref. [21,22] as a method of generating deterministic scale-free networks. Yet, it was believed that aside from their deterministic structure, their statistical properties are equivalent with the stochastic models that are often used to generate scale-free networks. In the following, we argue that such hierarchical construction generates an architecture that is significantly different from the networks generated by traditional scale-free models. Most important, we show that this new feature of the model, its hierarchical character, are shared by a significant number of real networks.

First, we note that the hierarchical network model seamlessly integrates a scale-free topology with an inherent modular structure. Indeed, the generated network has a power-law degree distribution with degree exponent $\gamma = 1 + \ln 5 / \ln 4 = 2.161$ [Fig. 2(a)]. Furthermore, numerical simulations indicate that the clustering coefficient $C = 0.743$ is independent of the size of the network [Fig. 2(c)]. Therefore, the high degree of clustering and the scale-free property are simultaneously present in this network.

The most important feature of the network model of Fig. 1, not shared by either the scale-free [12,13] or random network models [9,10], is its hierarchical architecture. The network is made of numerous small, highly integrated five-node modules [Fig. 1(a)], which are assembled into larger 25-node modules [Fig. 1(b)]. These 25-node modules are less integrated but each of them is clearly separated from the other 25-node modules when we combine them into the even larger 125-node modules [Fig. 1(c)]. These 125-node modules are even less cohesive, but again will appear separable from their replicas if the network expands further.

This intrinsic hierarchy can be characterized in a quantitative manner using the recent finding of Dorogovtsev, Goltsev, and Mendes [21] that in the deterministic scale-free net-

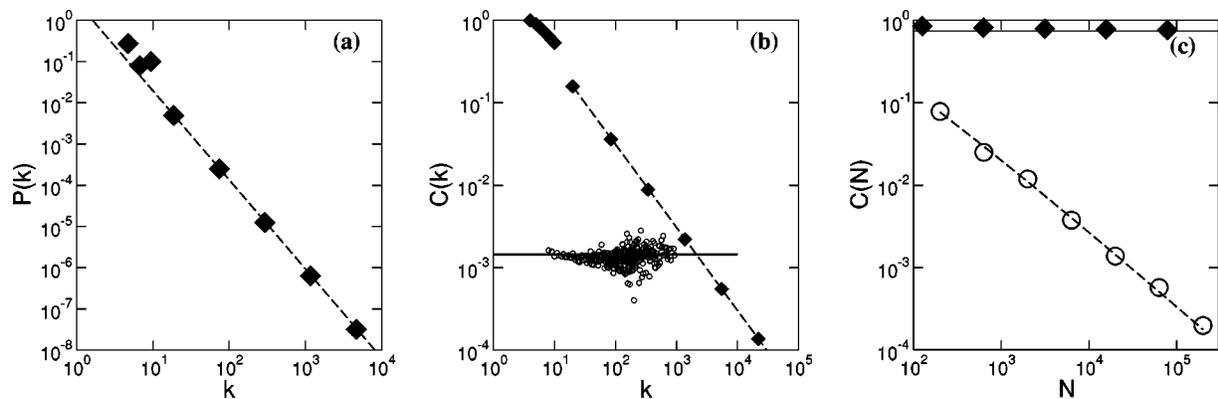


FIG. 2. Scaling properties of the hierarchical model shown in Fig. 1 ($N=5^7$). (a) The numerically determined degree distribution. The asymptotic scaling, with slope $\gamma = 1 + \ln 5 / \ln 4$, is shown as a dashed line. (b) The $C(k)$ curve for the model, demonstrating that it follows Eq. (1). The open circles show $C(k)$ for a scale-free model [12] of the same size, illustrating that it does not have a hierarchical architecture. (c) The dependence of the clustering coefficient C on the size of the network N . While for the hierarchical model C is independent of N (\blacklozenge), for the scale-free model $C(N)$ decreases rapidly (\circ).

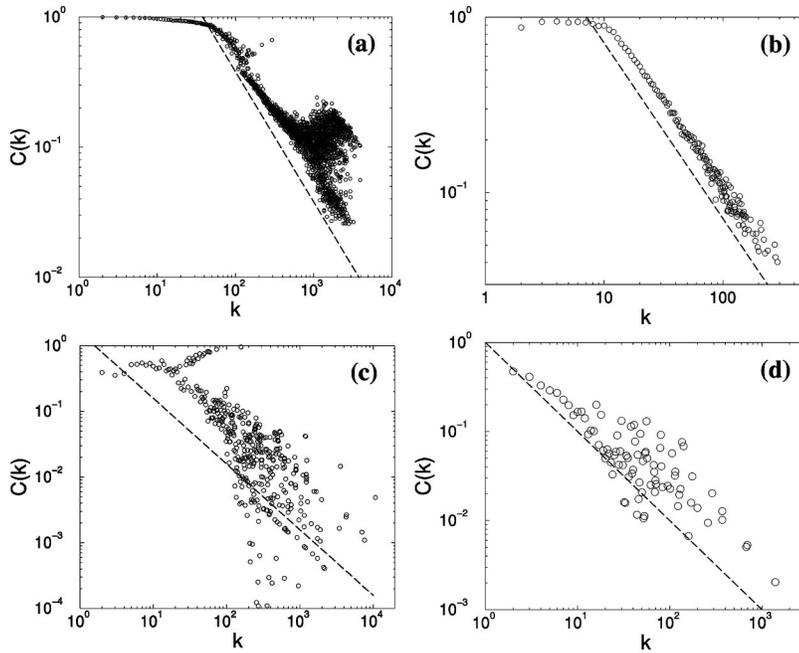


FIG. 3. The scaling of $C(k)$ with k for four large networks: (a) Actor network, two actors being connected if they acted in the same movie according to the www.IMDB.com database. (b) The semantic web, connecting two English words if they are listed as synonyms in the Merriam Webster dictionary [27]. (c) The World Wide Web, based on the data collected in Ref. [7]. (d) Internet at the autonomous system level, each node representing a domain, connected if there is a communication link between them. The dashed line in each figure has slope -1 , following Eq. (1).

works, the clustering coefficient of a node with k links follows the scaling law

$$C(k) \sim k^{-1}. \quad (1)$$

We argue that this scaling law quantifies the coexistence of a hierarchy of nodes with different degrees of clustering, and applies to the model of Figs. 1(a)–1(c) as well. Indeed, the nodes at the center of the numerous five-node modules have a clustering coefficient $C=1$. Those at the center of a 25-node module have $k=20$ and $C=3/19$, while those at the center of the 125-node modules have $k=84$ and $C=3/83$, indicating that the higher a node's degree, the smaller is its clustering coefficient, asymptotically following the $1/k$ law [Fig. 2(b)]. In contrast, for the scale-free model proposed in Ref. [12], the clustering coefficient is independent of k , i.e., the scaling law (1) does not apply [Fig. 2(b)]. The same is true for the random [9,10] or the various small world models [11,23], for which the clustering coefficient is independent of the nodes' degree.

Therefore, the discrete model of Fig. 1 combines within a single framework, the two key properties of real networks: their scale-free topology and high modularity, which results in a system-size independent clustering coefficient. Yet, the hierarchical modularity of the model results in the scaling law (1), which is not shared by the traditional network models. The question is, could hierarchical modularity, as captured by this model, characterize real networks as well?

III. HIERARCHICAL ORGANIZATION IN REAL NETWORKS

To investigate if such hierarchical organization is present in real networks, we measured the $C(k)$ function for several networks for which large topological maps are available. Next, we discuss each of these systems separately.

Actor Network. Starting from the www.IMDB.com data-

base, we connect any two actors in Hollywood if they acted in the same movie, obtaining a network of 392 340 nodes and 15 345 957 links. Earlier studies indicate that this network is scale-free with an exponential cutoff in $P(k)$ for high k [12,24,25]. As Fig. 3(a) indicates, we find that $C(k)$ scales as k^{-1} , indicating that the network has a hierarchical topology. Indeed, the majority of actors with a few links (small k) appear only in one movie. Each such actor i has a clustering coefficient equal to one, as all the actors i have links to are part of the same cast, and are therefore connected to each other. The high- k nodes include many actors who acted in several movies, and thus, their neighbors are not necessarily linked to each other, resulting in a smaller $C(k)$. At high k , the $C(k)$ curve splits into two branches, one of which continues to follow Eq. (1), while the other saturates. One explanation of this split is the decreasing amount of data points available in this region. Indeed, in the high- k region, the number of nodes having the same k is rather small. If one of these nodes corresponds to an actor who played only in a few movies with hundreds in the cast, it will have both high k and high C , considerably increasing the average value of $C(k)$. The k values, for which such high C nodes are absent continue to follow the k^{-1} curve, resulting in jumps between the high and small C values for large k . For small k , these anomalies are averaged out.

Language network. Recently, a series of empirical results have shown that the language, viewed as a network of words, has a scale-free topology [26–29]. Here, we study the network generated connecting two words to each other if they appear as synonyms in the Merriam Webster dictionary [27]. The obtained semantic web has 182 853 nodes and 317 658 links, and it is scale-free with degree exponent $\gamma=3.25$. The $C(k)$ curve for this language network is shown in Fig. 3(b), indicating that it follows Eq. (1), suggesting that the language has a hierarchical organization.

World Wide Web. On the WWW, two documents are con-

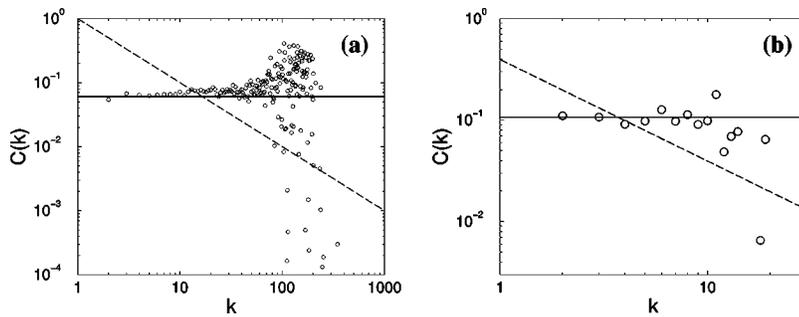


FIG. 4. The scaling of $C(k)$ for two large, nonhierarchical networks: (a) Internet at router level [35]. (b) The power grid of Western United States. The dashed line in each figure has slope -1 , while the solid line corresponds to the average clustering coefficient.

nected to each other if there is a URL pointing from one document to the other one. The sample we study, obtained by mapping out the `www.nd.edu` domain [7], has 325 729 nodes and 1 497 135 links, and it is scale-free with degree exponents $\gamma_{\text{out}}=2.45$ and $\gamma_{\text{in}}=2.1$, characterizing the out- and in-degree distribution, respectively. To measure the $C(k)$ curve, we made the network undirected. While the obtained $C(k)$, shown in Fig. 3(c), does not follow as closely the scaling law (1) as observed in the previous two examples, there is a clear evidence that $C(k)$ decreases rapidly with k , supporting the coexistence of many highly interconnected small nodes with a few larger nodes, which have a much lower clustering coefficient.

Indeed, the Web is full of groups of documents that all link to each other. For example, `www.nd.edu/~networks`, our network research dedicated site, has a high clustering coefficient, as the documents it links to have links to each other. The site is one of the several network-oriented sites, some of which point to each other. Therefore, the network research community still forms a relatively cohesive group, albeit less interconnected than the `www.nd.edu/~networks` site, thus having a smaller C . This network community is nested into the much larger community of documents devoted to statistical mechanics that has an even smaller clustering coefficient. Therefore, the k dependent $C(k)$ reflects the hierarchical nesting of the different interest groups present on the Web. Note that $C(k) \sim k^{-1}$ for the WWW was observed and briefly noted in Ref. [30].

Internet at the AS level. The Internet is often studied at two different levels of resolution. At the router level, we have a network of routers connected by various physical communication links. At the interdomain or autonomous system (AS) level, each administrative domain, composed of potentially hundreds of routers, is represented by a single node. Two domains are connected if there is at least one router that connects them. Both the router and the domain level topology have been found to be scale-free [31]. As Fig. 3(d) shows, we find that at the domain level, the Internet consisting of 65 520 nodes and 24 412 links [32], has a hierarchical topology as $C(k)$ is well approximated with Eq. (1). The scaling of the clustering coefficient with k for the Internet was earlier noted by Vazquez, Pastor-Satorras, and Vespignani (VPSV) [33,34], who observed $C(k) \sim k^{-0.75}$. VPSV interpreted this finding, together with the observation that the average nearest-neighbor connectivity also follows a power law with the node's degree, as a natural consequence of the *stub* and *transit* domains that partition the network in a hierarchical fashion into international connections, national

backbones, regional networks, and local area networks.

Our measurements indicate, however, that some real networks lack a hierarchical architecture, and do not obey the scaling law (1). In particular, we find that the power grid and the router level Internet topology have a k independent $C(k)$.

Internet at the router level. The router level Internet has 260 657 nodes connected by 1 338 100 links [35]. Measurements indicate that the network is scale-free [31,36] with degree exponent $\gamma=2.23$. Yet, the $C(k)$ curve [Fig. 4(a)], apart from some fluctuations, is largely independent of k , in strong contrast with the $C(k)$ observed for the Internet's domain level topology [Fig. 3(d)], and in agreement with the results of VPSV [33,34], who also note the absence of a hierarchy in router level maps.

Power Grid. The nodes of the power grid are generators, transformers, and substations and the links are high voltage transmission lines. The network studied by us represents the map of the Western United States, and has 4 941 nodes and 13 188 links [11]. The results again indicate that apart from fluctuations, $C(k)$ is independent of k .

It is quite remarkable that these two networks share a common feature: a geographic organization. The routers of the Internet and the nodes of the power grid have a well defined spatial location, and the link between them represent physical links. In contrast, for the examples discussed in Fig. 3, the physical location of the nodes was either undefined or irrelevant, and the length of the link was not of major importance. For the router level Internet and the power grid, the further are the two nodes from each other, the more expensive it is to connect them [36]. Therefore, in both systems, the links are driven by cost considerations, generating a distance driven structure, apparently excluding the emergence of a hierarchical topology. In contrast, the domain level Internet is less distance driven, as many domains, such as the AT&T domain, span the whole United States.

In summary, we offered evidence that for four large networks $C(k)$ is well approximated by $C(k) \sim k^{-1}$, in contrast to the k independent $C(k)$ predicted by both the scale-free and random networks. In addition, there is evidence for similar scaling in the metabolism [37] and protein interaction networks [38]. This indicates that these networks have an inherently hierarchical organization. In contrast, hierarchy is absent in networks with strong geographical constraints, as the limitation on the link length strongly constrains the network topology.

IV. STOCHASTIC MODEL AND UNIVERSALITY

The hierarchical model described in Fig. 1 predicts $C(k) \sim k^{-1}$, which offers a rather good fit to three of the four

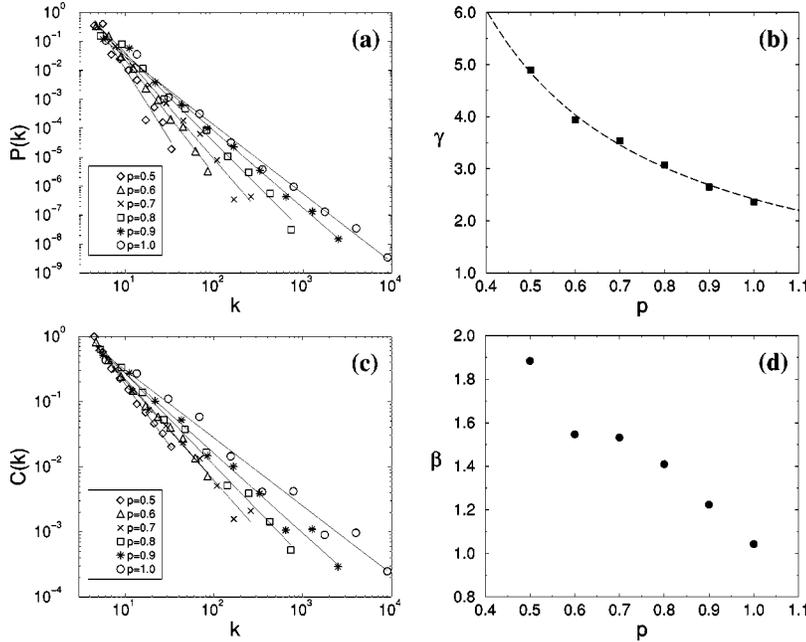


FIG. 5. The scaling properties of the stochastic model. (a) The degree distribution for different p values, indicating that $P(k)$ follows a power law with a p -dependent slope. (b) The dependence of the degree exponent γ on p , determined by fitting power laws to the curves shown in (a). The exponent γ appears to follow approximately $\gamma(p) \sim 1/p$ (dashed line). (c) The $C(k)$ curve for different p values, indicating that the hierarchical exponent β depends on p . (d) The dependence of β on the parameter p . The simulations were performed for $N=5^7(78125)$ nodes.

$C(k)$ curves shown in Fig. 3. The question is, is this scaling law (1) universal, valid for all hierarchical networks, or could different scaling exponents characterize the scaling of $C(k)$? Defining the hierarchical exponent β as

$$C(k) \sim k^{-\beta}, \quad (2)$$

where $\beta=1$ is a universal exponent, or can its value be changed together with γ ? In the following, we demonstrate that the hierarchical exponent β can be tuned as we tune some of the network parameters. For this, we propose a stochastic version of the model described in Fig. 1.

We start again with a small core of five nodes all connected to each other [Fig. 1(a)] and in step one ($n=1$), we make four copies of the five-node module. Next, we randomly pick a p fraction of the newly added nodes and connect each of them independently to the nodes belonging to the central module. We use preferential attachment [12,13] to decide, to which central node the selected nodes link to. That is, we assume that the probability that a selected node will connect to a node i of the central module is $k_i / \sum_j k_j$, where k_i is the degree of node i and the sum goes over all nodes of the central module. In the second step ($n=2$), we again create four identical copies of the 25-node structure obtained thus far, but we connect only a p^2 fraction of the newly added nodes to the central module. Subsequently, in each iteration n , the central module of size 5^n is replicated four times, and in each new module, a p^n fraction will connect to the current central module, requiring the addition of $(5p)^n$ new links.

As Fig. 5 shows, changing p alters the slope of both $P(k)$ and $C(k)$ on a log-log plot. In general, we find that increasing p decreases the exponents γ and β [Figs. 5(b), 5(d)]. The exponent $\beta=1$ is recovered for $p=1$, i.e., when all nodes of a module gain a link. While the number of links added to the

network changes at each iteration, for any $p \leq 1$, the average degree of the infinitely large network is finite. Indeed, the average degree follows:

$$\langle k \rangle_n = \frac{8}{5} \left(\frac{3}{2} + \frac{1-p^{n+1}}{1-p} \right), \quad (3)$$

which is finite for any $p \leq 1$.

Interestingly, the scaling of $C(k)$ is not a unique property of the model discussed above. A version of the model, where we keep the fraction of selected nodes, p , constant from iteration to iteration, also generates p dependent β and γ exponents. Furthermore, recently, several results indicate that the scaling of $C(k)$ is an intrinsic feature of several existing growing network models. Indeed, aiming to explain the potential origin of the scaling in $C(k)$ observed for the Internet, VSPV note that the fitness model [39,40] displays a $C(k)$ that appears to scale with k . While there is no analytical evidence for $C(k) \sim k^{-\beta}$ yet, numerical results [33,34] suggest that the presence of fitness does generate a hierarchical network architecture. In contrast, in a recent model proposed by Klemm and Eguiluz, there is analytical evidence that the network obeys the scaling law (1) [41]. In their model, in each time step, a new node joins the network, connecting to all *active* nodes in the system. At the same time, an active node is deactivated with probability $p \sim k^{-1}$. The insights offered by the hierarchical model can help understand the origin of the observed $C(k) \sim k^{-1}$. By deactivating the less connected nodes, a central core emerges to which all subsequent nodes tend to link to. New nodes have a large C and small k , thus they are rapidly deactivated, freezing into a large C state. The older, more connected, surviving nodes are in contact with a large number of nodes that have already disappeared from the active list, and they have small C [42].

Finally, Szabó, Alava, and Kertész have developed a rate equation method to systematically calculate $C(k)$ for evol-

ing network models [43]. Applying the method to a model proposed by Holme and Kim [44] to enhance the degree of clustering coefficient C seen in the scale-free model [12], they have shown that the scaling of $C(k)$ depends on the parameter p , which governs the rate, at which new nodes connect to the neighbors of selected nodes, bypassing preferential attachment. As for $p=0$, the Holme-Kim model reduces to the scale-free model, Szabó, Alava, and Kertész find that in this limit, the scaling of $C(k)$ vanishes. These models indicate that several microscopic mechanisms could generate a hierarchical topology, just as several models are able to create a scale-free network [1,2].

V. DISCUSSION AND OUTLOOK

The identified hierarchical architecture offers a different perspective on the topology of complex networks. Indeed, the fact that many large networks are scale-free is now well established. It is also clear that most networks have a modular topology, quantified by the high clustering coefficient they display. Such modules have been proposed to be a fundamental feature of biological systems [37,45], but have been discussed in the context of the WWW [18,46], and social networks as well [17,47]. The hierarchical topology offers a different avenue for bringing under a single roof these two concepts, giving a precise and quantitative meaning for the network's modularity. It indicates that we should not think of modularity as the coexistence of relatively independent groups of nodes. Instead, we have many small clusters that are densely interconnected. These combine to form larger, but less cohesive groups, which combine again to form even larger and even less interconnected clusters. This self-similar nesting of different groups or modules into each other forces a strict fine structure on real networks.

Most interesting is, however, the fact that the hierarchical nature of these networks is well captured by a simple quan-

tity, the $C(k)$ curve, offering us a relatively straightforward method to identify the presence of hierarchy in real networks. The law (1) indicates that the number and the size of the groups of different cohesiveness is not random, but follow rather strict scaling laws.

The presence of such a hierarchical architecture reinterprets the role of the hubs in complex networks. Hubs, the highly connected nodes at the tail of the power law degree distribution, are known to play a key role in keeping complex networks together, playing a crucial role from the robustness of the network [48,49] to the spread of viruses in scale-free networks [50]. Our measurements indicate that the clustering coefficient characterizing the hubs decreases linearly with the degree. This implies that while the small nodes are part of highly cohesive, densely interlinked clusters, the hubs are not, as their neighbors have a small chance of linking to each other. Therefore, the hubs play the important role of bridging the many small communities of clusters into a single, integrated network.

In many ways, our study offers only a starting point for understanding the interplay between the scale-free, hierarchical, and modular nature of real networks. While the $C(k)$ curves offer a tool to unearth the presence of a hierarchy, it is unclear that what are the minimal ingredients at the model level for such a hierarchy to emerge. Finally, the role of the geometrical factor, which appears to remove the hierarchy, needs to be elucidated. Further modeling and empirical studies should allow us to address these questions.

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- [1] R. Albert and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).
 - [2] S.N. Dorogovtsev and J.F.F. Mendes, *Adv. Phys.* **51**, 1079 (2002).
 - [3] H. Jeong, B. Tombor, R. Albert, Z. Oltvai, and A.-L. Barabási, *Nature (London)* **407**, 651 (2000).
 - [4] A. Wagner and D.A. Fell, *Proc. R. Soc. London, Ser. B* **268**, 1803 (2001).
 - [5] H. Jeong, S. Mason, A.-L. Barabási, and Z.N. Oltvai, *Nature (London)* **411**, 41 (2001).
 - [6] A. Wagner, *Mol. Biol. Evol.* **18**, 1283 (2001).
 - [7] R. Albert, H. Jeong, and A.-L. Barabási, *Nature (London)* **401**, 130 (1999).
 - [8] F. Liljeros, C.R. Edling, L.A.N. Amaral, H.E. Stanley, and Y. Åberg, *Nature (London)* **411**, 907 (2001).
 - [9] P. Erdős and A. Rényi, *Publ. Math. Debrecen* **6**, 290 (1959).
 - [10] B. Bollobás, *Random Graphs* (Academic Press, London, 1985).
 - [11] D.J. Watts and S.H. Strogatz, *Nature (London)* **393**, 440 (1998).
 - [12] A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999).
 - [13] A.-L. Barabási, R. Albert, and H. Jeong, *Physica A* **272**, 173 (1999).
 - [14] M.E.J. Newman, *Proc. Natl. Acad. Sci. U.S.A.* **98**, 404 (2001).
 - [15] M.E.J. Newman, *Phys. Rev. E* **64**, 016131 (2001).
 - [16] A.-L. Barabási, H. Jeong, Z. Néda, E. Ravasz, A. Schubert, and T. Vicsek, *Physica A* **311**, 590 (2002).
 - [17] M.S. Granovetter, *Am. J. Sociol.* **78**, 1360 (1973).
 - [18] G.W. Flake, S. Lawrence, and C.L. Giles, in *Proceedings of the Sixth International Conference on Knowledge Discovery and Data Mining* (ACM, Boston, MA, 2000), p. 150.
 - [19] L.A. Adamic and E. Adar (unpublished); See, <http://hpl.hp.com/shl/papers/web10/index.html>.
 - [20] A.-L. Barabási, E. Ravasz, and T. Vicsek, *Physica A* **299**, 559 (2001).
 - [21] S.N. Dorogovtsev, A.V. Goltsev, and J.F.F. Mendes, e-print cond-mat/0112143.
 - [22] S. Jung, S. Kim, and B. Kahng, *Phys. Rev. E* **65**, 056101 (2002).
 - [23] M.E.J. Newman, *J. Stat. Phys.* **101**, 819 (2000).
 - [24] R. Albert and A.-L. Barabási, *Phys. Rev. Lett.* **85**, 5234 (2000).

- [25] L.A.N. Amaral, A. Scala, M. Barthélemy, and H.E. Stanley, *Proc. Natl. Acad. Sci. U.S.A.* **97**, 11 149 (2000).
- [26] R. Ferrer i Cancho and R.V. Solé, *Proc. R. Soc. London, Ser. B* **268**, 2261 (2001).
- [27] S. Yook, H. Jeong, and A.-L. Barabási (unpublished).
- [28] M. Sigman and G. Cecchi, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 1742 (2002).
- [29] S.N. Dorogovtsev and J.F.F. Mendes, *Proc. R. Soc. London, Ser. B* **268**, 2603 (2001).
- [30] J.-P. Eckmann and E. Moses, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 5825 (2002).
- [31] M. Faloutsos, P. Faloutsos, and C. Faloutsos, *Comput. Commun. Rev.* **29**, 251 (1999).
- [32] See <http://moat.nlanr.net/infrastructure.html>.
- [33] A. Vázquez, R. Pastor-Satorras, and A. Vespignani, *Phys. Rev. E* **65**, 066130 (2002).
- [34] A. Vázquez, R. Pastor-Satorras, and A. Vespignani, e-print cond-mat/0206084.
- [35] R. Govindan and H. Tangmunarunkit, in *Proceedings of IEEE INFOCOM 2000, Tel Aviv, Israel* (IEEE, Piscataway, NJ, 2000), Vol. 3, p. 1371.
- [36] S.H. Yook, H. Jeong, and A.-L. Barabási, e-print cond-mat/0107417.
- [37] E. Ravasz, A.L. Somera, D.A. Mongru, Z.N. Oltvai, and A.-L. Barabási, *Science* **297**, 1551 (2002).
- [38] S. H. Yook *et al.* (unpublished).
- [39] G. Bianconi and A.-L. Barabási, *Europhys. Lett.* **54**, 436 (2001).
- [40] G. Bianconi and A.-L. Barabási, *Phys. Rev. Lett.* **86**, 5632 (2001).
- [41] K. Klemm and V.M. Eguiluz, *Phys. Rev. E* **65**, 036123 (2002).
- [42] Note, however, that as new nodes tend to connect to nodes that were added to the network shortly before them, the model generates a close to one-dimensional structure in time. See, A. Vázquez, Y. Moreno, M. Boguñá, R. Pastor-Satorras, and A. Vespignani, e-print cond-mat/0209183.
- [43] G. Szabó, M. Alava, and J. Kertész, e-print cond-mat/0208551.
- [44] P. Holme and B.J. Kim, *Phys. Rev. E* **65**, 026107 (2002).
- [45] L.H. Hartwell, J.J. Hopfield, S. Leibler, and A.W. Murray, *Nature (London)* **402**, C47 (1999).
- [46] S. Lawrence and C.L. Giles, *Nature (London)* **400**, 107 (1999).
- [47] D.J. Watts, P.S. Dodds, and M.E.J. Newman, *Science* **296**, 1302 (2002).
- [48] R. Albert, H. Jeong, and A.-L. Barabási, *Nature (London)* **406**, 378 (2000).
- [49] R. Cohen, K. Erez, D. ben Avraham, and S. Havlin, *Phys. Rev. Lett.* **86**, 3682 (2001).
- [50] R. Pastor-Satorras and A. Vespignani, *Phys. Rev. Lett.* **86**, 3200 (2001).