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# Exact Upper and Lower Bounds of Crossing Frequency Set and Delay Independent Stability Test for Multiple Time Delayed Systems

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**Abstract:** A general class of linear time invariant (LTI) multiple time-delay system is investigated in order to (i) obtain exact lower and upper bounds of crossing frequency set (CFS), and (ii) test the necessary and sufficient conditions of delay independent stability (DIS). The method commences by deploying Rekasius substitution for the transcendental terms in the characteristic function, reducing it into a finite dimensional one. After substitution, utilization of elimination theory allows one to achieve the two nontrivial objectives (i)-(ii). The approach is new and novel as it does not require any parameter sweeping and graphical display; it is exact and can test necessary and sufficient conditions of DIS over only a single variable polynomial. A case study is provided to show the effectiveness of the proposed method. *Copyright © IFAC 2009*

**Keywords:** Time delay systems, Crossing frequency set, Stability independent of delay test, Stability, Multiple time delays, Rekasius substitution.

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## 1. INTRODUCTION

Time Delay Systems (TDS) arise in many applications from diverse areas such as biology (MacDonald, 1978; Queinnec et al., 2007), economy (Forrester, 1961; Mak et al., 1976; Delice and Sipahi, 2008), machine tool chatter (Asl and Ulsoy, 2003; Sipahi, 2008), population dynamics (Kuang, 1993), communication systems (Niculescu et al., 2003; Schöllig et al., 2007); see Erneux (2009) and the references therein for other applications.

Investigation of asymptotic stability of TDS with respect to delays is essential, however analyzing asymptotic stability is a nontrivial task even for the linear time invariant (LTI) cases, primarily due to the infinite dimensional nature of TDS. Considerable number of techniques along these lines exist for stability analysis of LTI-TDS (Stépán, 1989; Gu et al., 2003; Sipahi and Olgac, 2005; Michiels and Niculescu, 2007); nevertheless most of the existing work have limitations on either system dimension or number of time delays, or on both. We see that frequency sweeping methodology (Chen and Latchman, 1995) stands out for its applicability to robustness and geometric approach for computing stability switching boundaries (SSB) in 2D (Gu et al., 2005) and 3D delay space (Sipahi and Delice, 2009). To quote Gu and Niculescu (2003) on the frequency sweeping methodology: “This idea motivates the frequency sweeping method, which can also take advantage of many robust stability ideas formulated in recent years”. With this idea, many papers are published where delay independent stability (DIS) necessary (Chen and Latchman, 1995; Gu et al., 2003) and necessary and sufficient conditions are checked (Chen et al., 2008); or SSB are extracted for systems stability of which depends on

delays (Gu et al., 2005; Almodaresi and Bozorg, 2008; Sipahi and Delice, 2009).

The general class of LTI multiple time delayed system (MTDS) considered here is described by the differential equation:

$$\dot{\mathbf{x}}(t) = \mathbf{A} \mathbf{x}(t) + \sum_{\ell=1}^L \mathbf{B}_{\ell} \mathbf{x}(t - \tau_{\ell}), \quad (1)$$

where  $\mathbf{A} \in \mathbb{R}^{N \times N}$ ,  $\mathbf{B}_{\ell} \in \mathbb{R}^{N \times N}$  are the constant matrices;  $\mathbf{x}(t) \in \mathbb{R}^{N \times 1}$  is the state vector;  $\tau_{\ell}$  are the positive pure delays. The stability analysis of (1) requires to investigate the eigenvalues of (1) that are on the imaginary axis, i.e.  $s = j\omega$ , for some particular delay values (Datko, 1978). It is these eigenvalues which may cross the imaginary axis at  $s = j\omega$  and may cause stability reversals/switches at those particular delay values which define SSB.

In the case of  $L = 1$ , crossing frequency set (CFS) of all  $\omega$  solutions which generates SSB has finite number of elements. On the other hand when  $L > 1$ , the CFS becomes a union of continuous curves (Niculescu, 2001). Moreover, it is known that the upper bound of this CFS is finite (Hale and Verduyn Lunel, 1993). Many studies used this idea and swept  $\omega$  in a range starting from zero up to a conservative upper bound and solved the aforementioned eigenvalue problem numerically to find the corresponding delay values, and ultimately the SSB (Gu et al., 2005; Almodaresi and Bozorg, 2008; Sipahi and Delice, 2009). Graphical display in these analyses is common practice for checking whether larger  $\omega$  values reveal any solution.

Computation times in extracting SSB can be significantly shortened with the availability of the precise upper and lower bounds of CFS. The purpose of this paper is to compute these upper  $\bar{\omega}$  and lower  $\underline{\omega}$  bounds of CFS and to test the necessary and

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sufficient conditions of DIS in (1). For MTDS, crossing frequency range may exist in multiple distinct intervals, and without knowing  $\omega$  and  $\bar{\omega}$ , it is hard to assure the capture of  $[\omega, \bar{\omega}]$  range in a computation. One should be very conservative when choosing the frequency range, however missing an admissible frequency interval may still occur. Missing frequency ranges causes completely misleading and incomplete stability maps due to unconsidered parts of SSB. Moreover, larger crossing frequency values may render SSB closer to the origin of delay space; hence missing large frequency values is detrimental since some regions near the origin which are thought to be stable can be actually unstable. Finding the exact upper and lower bounds of CFS eliminates the aforementioned problems and also prevents unnecessary frequency sweeping.

TDS researchers have studied DIS conditions as well as extraction of stability maps (Chen et al., 1995; Niculescu and Chen, 1999; Gu et al., 2003). They defined the DIS condition for single delay  $L = 1$  as: (i)  $\mathbf{A}$  matrix is asymptotically stable and (ii) any vector induced matrix norm  $\|\cdot\|$ ,  $\|(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}_1\| < 1$ ,  $\forall \omega \geq 0$ . For MTDS, characteristic function is taken as  $a_0(s) + \sum_{k=1}^m a_k(s)e^{-\tau_k s}$  (Chen et al., 2008) and DIS condition is defined based on frequency sweeping conditions. Notice that, characteristic function in (Chen et al., 2008) is one of the most complicated problems solved so far, however, a frequency sweeping test to address the DIS condition of (1) does not exist to our best knowledge. Moreover, sweeping frequency to a conservative upper bound for DIS test is cumbersome and time consuming. In contrast, one can check DIS condition of (1) *directly* without frequency sweeping and graphical display by means of the methodology presented in this paper. This new methodology is in principle able to deliver a single variable polynomial, roots of which declare DIS of (1). In this sense, our approach is practical and more inclusive since (1) has no limitation on the ranks or entries of system matrices.

There are other techniques to test DIS of TDS, however, Kamen (1980); Thowsen (1982); Hertz et al. (1984); Gu et al. (2001); Wei et al. (2008) are only applicable for single delay case and Wu and Ren (2004); Wang and Hu (1999) are feasible for two delay cases. Notice that in Wu and Ren (2004), Euler formula is utilized for exponential functions and linearity is preserved since characteristic function has no commensurate degrees; and Wei et al. (2008) transforms frequency sweeping conditions in Hale et al. (1985) to easily testable algebraic conditions by utilizing elimination theory. In all these papers, extensions to  $L > 2$  cases is restrictive due to the fact that the number of available equations is less than the number of unknowns in the stability problem.

Our methodology not only addresses the DIS problem for  $L > 2$ , but it is also applicable for specifically considered delay domain which may contain fewer delays than  $L$ . This is especially needed since visualization of stability maps is possible only in 2D and 3D domains (Sipahi and Delice, 2009). For instance, one can compute lower and upper bounds of CFS in two or three delay domains or check DIS conditions in these pertaining domains for a system with four delays. Moreover, in some cases, computation times needed to extract SSB can be independent of system dimension and the number of delays (Sipahi and Delice, 2009). In such cases, the exact upper and lower bounds become dominantly important from numerical efficiency. Therefore, knowledge of lower and upper bounds of CFS shortens the computation time significantly and increases tractability of the stability problem. Consequently,

stability maps can be identified without sweeping frequency conservatively.

Notations used in the text are standard. We use bold face font for matrices, vectors and sets. Open right half, open left half and imaginary axis of complex plane  $\mathbb{C}$  are represented by  $\mathbb{C}_+$ ,  $\mathbb{C}_-$  and  $j\mathbb{R}$ , respectively.  $\mathbb{R}$ ,  $\mathbb{R}_+$ ,  $\mathbb{Z}_+$  denotes the set of real numbers, positive real numbers and the set of positive integer numbers, respectively. We use  $s \in \mathbb{C}$  for the Laplace variable;  $\Re(s)$  for the real part of  $s$  and  $\Im(s)$  for the imaginary part of  $s$ .  $\sup$  stands for supremum of a set;  $|\mathbf{M}|$  for determinant of a square matrix  $\mathbf{M}$  and  $R_\ell(p_1, p_2)$  for resultant of multinomials  $p_1(T_1, T_2, \dots, T_\ell)$  and  $p_2(T_1, T_2, \dots, T_\ell)$  with eliminating  $T_\ell$ .  $\boldsymbol{\tau} = \{\tau_\ell\}_{\ell=1}^L$  is the delay vector;  $\mathbf{T} = \{T_\ell\}_{\ell=1}^L$  is the pseudo-delay vector;  $z_\ell = e^{-\tau_\ell s}$  and  $c_\ell$  is the commensurate degree of  $\tau_\ell$ . We omit arguments when no confusion occurs.

## 2. PRELIMINARIES

Characteristic function of the system in (1) is given by:

$$f(s, \boldsymbol{\tau}) = \sum_{k=0}^K P_k(s) e^{-s \sum_{\ell=1}^L m_{k\ell} \tau_\ell}, \quad (2)$$

where  $P_k$  are polynomials in terms of  $s$  with real coefficients and  $K, m_{k\ell} \in \mathbb{Z}_+$ ,  $\{m_{0\ell}\}_{\ell=1}^L = 0$ . Since delays cannot pervade through the highest order derivative of the state, equation (2) represents a *retarded* class LTI-TDS (Stépán, 1989). The characteristic function (2) possesses infinitely many roots due to the presence of transcendental terms.

*Definition 1.* For a given  $\boldsymbol{\tau}$ , MTDS (1) is asymptotically stable if and only if

$$f(s, \boldsymbol{\tau}) \neq 0, \quad \forall s \in \mathbb{C}_+ \cup j\mathbb{R}, \quad (3)$$

and (1) is DIS if the condition (3) holds for all  $\boldsymbol{\tau} \in \mathbb{R}_+^L \cup \{\mathbf{0}\}$ .

Definition 1 is general depiction of asymptotic stability and DIS notion, however verifying condition (3) in (2) for all  $s \in \mathbb{C}_+ \cup j\mathbb{R}$  and  $\boldsymbol{\tau} \in \mathbb{R}_+^L$  is impossible. Exploiting the continuity property of the roots (Datko, 1978; Niculescu, 2001) on  $\mathbb{C}$  may reduce the difficulty of checking the stability condition. Stability may change only when the roots cross the imaginary axis, since the imaginary axis is the boundary separating the stable,  $\mathbb{C}_-$ , and unstable regions,  $\mathbb{C}_+$  on the  $\mathbb{C}$ . In other words, stability may change only when  $\Re(s) = 0$ . For detecting the stability transitions, one should analyze the characteristic function on the imaginary axis, by setting  $s = j\omega$ ,  $\omega \in \mathbb{R}_+$ . All  $\omega$  values, where  $s = j\omega$  is a root of (2) for some positive delays, define CFS:

$$\Omega = \{\omega \in \mathbb{R}_+ \mid f(j\omega, \boldsymbol{\tau}) = 0, \quad \forall \boldsymbol{\tau} \in \mathbb{R}_+^L\}. \quad (4)$$

At this point, we can convert the infinite dimensional characteristic function (2) to a finite dimensional characteristic function via the exact Rekasius transformation (Rekasius, 1980),

$$z_\ell = \frac{1 - T_\ell s}{1 + T_\ell s}, \quad T_\ell \in \mathbb{R}, \quad s = j\omega, \quad \ell = 1, \dots, L. \quad (5)$$

Upon substitution of (5) into (2) and with manipulation to remove the fractions, we obtain the transformed characteristic function:

$$g(\omega, \mathbf{T}) = \left( f(s, \boldsymbol{\tau}) \Big|_{z_\ell = \frac{1 - jT_\ell \omega}{1 + jT_\ell \omega}, \ell=1, \dots, L} \right) \prod_{\ell=1}^L (1 + jT_\ell \omega)^{c_\ell}, \quad (6)$$

which is a function of  $\omega$  and  $T_1, \dots, T_L$  only.

*Corollary 1.* Let  $\bar{\Omega}$  be the CFS of  $g(\omega, \mathbf{T})$ . The identity  $\bar{\Omega} \equiv \Omega$  holds.

*Proof 1.* See proof in (Sipahi and Olgac, 2005).  $\square$

*Remark 1.* Since  $\bar{\Omega} \equiv \Omega$  holds, we prefer to study  $\bar{\Omega}$  instead of  $\Omega$ . This is central for the main results below.

Before concluding Section 2, it is helpful to recall a fundamental property, which we need in the proof of Theorem 4.

*Property 1.* (Hale and Verduyn Lunel, 1993) Since the characteristic function (2) is *retarded* class dynamics, the supremum of the set  $\Omega$  is always upper bounded.

*Remark 2.* If  $\omega = 0$  is a root of (2), then it can be checked and treated by Fazelinia et al. (2007). In such cases, the lower bound of CFS is automatically zero,  $\underline{\omega} = 0$  which is a trivial case in our development. In this paper, we focus on nontrivial and thus nonzero lower and upper bounds of CFS, Section 3.

### 3. MAIN RESULT

In the sequel, we present the theoretical framework which enables a practical and direct computation of the finite lower and upper bounds of CFS. Furthermore, DIS condition naturally arises in the context of our development. For this purpose, we first define the resultant and discriminant, and next present the theorems.

The characteristic equation (6) can be written as:

$$g(\omega, \mathbf{T}) = g_{\Re}(\omega, \mathbf{T}) + j g_{\Im}(\omega, \mathbf{T}) = 0. \quad (7)$$

When (7) holds, its real  $g_{\Re}$  and imaginary part  $g_{\Im}$  should be concurrently zero:

$$g_{\Re} = \sum_{i=0}^{c_L} a_i(T_1, \dots, T_{L-1}) T_L^i = 0, \quad a_{c_L} \neq 0, \quad (8)$$

and

$$g_{\Im} = \sum_{i=0}^{c_L} b_i(T_1, \dots, T_{L-1}) T_L^i = 0, \quad b_{c_L} \neq 0. \quad (9)$$

Now, we utilize the elimination theory to eliminate  $T_L$ , without loss of generality, from the two polynomials  $g_{\Re} = 0$  and  $g_{\Im} = 0$  (Gelfand et al., 1994). A  $2c_L$ -order Sylvester matrix is constructed by eliminating  $T_L$ , and its determinant is

$$R_L(g_{\Re}, g_{\Im}) = \begin{vmatrix} a_0 & a_1 & \dots & a_{c_L} & 0 & 0 & 0 \\ 0 & a_0 & a_1 & \dots & a_{c_L-1} & a_{c_L} & 0 & 0 \\ \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & a_{c_L-1} & a_{c_L} \\ 0 & 0 & 0 & \dots & \dots & \dots & b_{c_L-1} & b_{c_L} \\ 0 & 0 & b_0 & b_1 & \dots & \dots & b_{c_L-1} & b_{c_L} & 0 \\ \dots & \dots \\ b_0 & b_1 & \dots & \dots & b_{c_L} & 0 & 0 & 0 \end{vmatrix}. \quad (10)$$

When  $g_{\Re} = 0$  and  $g_{\Im} = 0$  polynomials have common zeros, the resultant is zero.

*Corollary 2.* All the  $T_1, \dots, T_{L-1}$  roots of the resultant are determined by  $\mathbf{V} = \{(T_1, \dots, T_{L-1}) \in \mathbb{C} \mid R_L(g_{\Re}, g_{\Im}) = 0\}$ , and let all  $\mathbf{T} \in \mathbb{R}^L$  roots of  $g(\omega, \mathbf{T}) = 0$  be defined by  $\bar{\mathbf{V}} = \{\mathbf{T} \in \mathbb{R}^L \mid g(\omega, \mathbf{T}) = 0\}$ , then it is easy to see that the projections of all the points in  $\bar{\mathbf{V}}$  onto  $T_1, \dots, T_{L-1}$  space are a subset of  $\mathbf{V}$ .

*Proof 2.* Proof follows from the fact that the singularity of Sylvester's matrix,  $R_L(g_{\Re}, g_{\Im}) = 0$ , is a necessary condition for  $g_{\Re}$  and  $g_{\Im}$  to have common roots.  $\square$

*Corollary 3.* The following conditions are equivalent:

$$i) \frac{\partial \omega}{\partial \tau} = 0, \quad ii) \frac{\partial \omega}{\partial T} = 0.$$

*Proof 3.* Invoking the chain rule,

$$\frac{\partial \omega}{\partial \tau} = \frac{\partial \omega}{\partial T} \cdot \frac{\partial T}{\partial \tau} = 0, \quad (11)$$

leads to either  $\partial \omega / \partial T = 0$  or  $\partial T / \partial \tau = 0$ . Using the relationship  $T = \tan(\tau \omega / 2) / \omega$  (Sipahi and Olgac, 2005), one can compute the derivative of  $T$  with respect to  $\tau$  as

$$\frac{\partial T}{\partial \tau} = \frac{1}{2} [1 + (\omega T)^2] \neq 0. \quad (12)$$

Hence the corollary follows.  $\square$

Next, we provide the definition of discriminant.

*Definition 2.* Let  $H_1$  and  $H_2$  be two polynomials in terms of  $\mu_1 \dots, \mu_n$ . Without loss of generality, resultant  $R_n$  is computed via (10) by eliminating  $\mu_n$ . The resultant of  $R_n$  and  $\partial R_n / \partial \mu_{n-1}$  is called the *discriminant*,

$$D_{n-1} \triangleq R_{n-1}(R_n, \partial R_n / \partial T_{n-1}),$$

of the polynomials  $H_1$  and  $H_2$  (Sturmfels, 2002).

After resultant and discriminant concepts, and Corollary 2-3, we can present the theorems which reveal DIS condition as well as the exact upper and lower bounds of CFS.

*Theorem 4.* Compute the discriminant  $L - 1$  times eliminating  $T_i$ ,  $i = L - 1, L - 2, \dots, 1$  successively. Minimum and maximum real positive roots of the final discriminant that corresponds to  $\mathbf{T} \in \mathbb{R}$  yields the finite and exact lower and upper bounds of crossing frequency set.

*Proof 4.* Minima and maxima occur when derivative of  $\omega$  with respect to each entry of  $\tau$  or equivalently  $\mathbf{T}$ , from Corollary 3, is zero. We resort the implicit function theorem (Courant, 1988),

$$\frac{\partial \omega}{\partial T_{L-1}} = -\frac{\partial R_L / \partial T_{L-1}}{\partial R_L / \partial \omega} = 0, \quad (13)$$

which leads to  $\partial R_L / \partial T_{L-1} = 0$  assuming<sup>2</sup> that  $\partial R_L / \partial \omega \neq 0$ . In light of Corollary 2, common solutions are obtained from the resultant of  $R_L = 0$  and  $\partial R_L / \partial T_{L-1} = 0$ , which is the discriminant by definition, that is  $D_{L-1} = R_{L-1}(R_L, \partial R_L / \partial T_{L-1}) = 0$ . Continuing in a similar manner, we compute

$$\frac{\partial \omega}{\partial T_{L-2}} = -\frac{\partial D_{L-1} / \partial T_{L-2}}{\partial D_{L-1} / \partial \omega} = 0, \quad (14)$$

which leads to  $D_{L-2} \triangleq R_{L-2}(D_{L-1}, \partial D_{L-1} / \partial T_{L-2}) = 0$ . Repeating the same procedure, one can compute,  $D_{L-3} \dots D_1$  where  $D_1 \triangleq R_1(D_2, \partial D_2 / \partial T_1) = 0$ . Consequently, the *polynomial*,  $D_1(\omega)$ , is a function of  $\omega$  only and its minimum and maximum real positive roots that corresponds to  $\mathbf{T} \in \mathbb{R}$  give rise to exact lower  $\underline{\omega}$  and upper  $\bar{\omega}$  bounds of CFS, respectively. These roots are obviously finite due to Property 1 and finite coefficients of  $D_1$  (Prasolov, 2004).  $\square$

Notice that  $R_L$  is called the resultant and  $D_{L-1} \dots D_1$  are called the discriminant as per definition. Calculation of  $R_L$  and  $D_{L-1} \dots D_1$  follows the same logic, however their arguments are different.

<sup>2</sup> In addition to  $R_L = 0$  and  $\partial R_L / \partial T_{L-1} = 0$  conditions; if  $\partial R_L / \partial \omega$  is also zero, "singular point", e.g., a double point, cusp or isolated point occurs (Courant, 1988). Such degeneracies are left for a more comprehensive study for future.

*Theorem 5.* MTDS is stable independent of delays in the  $L$ -D delay domain if and only if  $D_1(\omega)$  in Theorem 4 has no real roots excluding zero, and  $\mathbf{A} + \sum_{\ell=1}^L \mathbf{B}_\ell$  is Hurwitz stable.

*Proof 5.* Having no real roots of  $D_1$  indicates CFS is empty set. Since CFS generates SSB, the condition  $\bar{\Omega} = \emptyset$  does not yield any SSB or vice versa. As a result, if there is no SSB, whole  $L$ -D delay domain exhibits the delay free system's stability behavior which is stable by construction.  $\square$

*Remark 3.* Let  $H_1$  and  $H_2$  be two polynomials with orders  $h_1$  and  $h_2$ . In order to make the approach more tractable, companion matrix technique can be utilized for elimination of variables (Barnett, 1973). Since, companion matrix dimension  $h_1 \times h_1$  is smaller than Sylvester matrix dimension  $(h_1 + h_2) \times (h_1 + h_2)$ , computation time can be considerably shortened, and the approach can be further simplified if the resultant is computed by companion matrix technique in each step or in desired steps.

#### 4. CASE STUDY

Illustrative example in this section is taken from Sipahi et al. (2008) where the system has three delays. Firstly, lower and upper bounds of CFS are computed on 3D delay domain,  $\Omega_{3D} \in [\omega_{3D}, \bar{\omega}_{3D}]$ , next  $\tau_3 = 1$  is kept fixed, without loss of generality, and lower and upper bounds of CFS are computed on 2D delay domain,  $\Omega_{2D} \in [\omega_{2D}, \bar{\omega}_{2D}]$ . Once these bounds are known, one needs to sweep  $\omega$  only in  $\Omega_{3D}$  ( $\Omega_{2D}$ ) to extract the stability maps exhaustively on 3D (2D) delay domain. It is worthy to note that  $\Omega_{2D} \subseteq \Omega_{3D}$ . State matrices in (1) are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7.5 & -19.25 & -17.75 & -7 \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0.3 & 0 \\ 0 & -0.45 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0.15 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and the corresponding characteristic function is

$$f(s, \tau) = s^4 + (7.0 + 0.45z_1)s^3 + (3.15z_1 + 5.1375z_2 + 0.6z_3 + 17.75)s^2 + (10.238z_1 + 1.1250z_2 + 4.2z_3 + 1.0125z_1z_2 - 1.5975z_2z_3 + 19.25)s + 1.0125z_1^2 - 0.2025z_1z_2z_3 + 1.35z_2z_3 + 7.5,$$

where  $z_i$  denotes  $e^{-\tau_i s}$ . Following Theorem 4, we calculate

$$D_1(\omega) = \sum_{k=0}^{190} \alpha_{2k} \omega^{2k},$$

where  $\alpha_{2k} \in \mathbb{R}$  and it is omitted for conciseness. From  $D_1(\omega)$ , it is easy to confirm that  $\Omega_{3D} \in [0.3521, 2.0101]$ . The calculated  $\Omega_{3D}$  coincides with Figure 5 found in the cited work. We also calculate  $\Omega_{2D}$  as  $[0.9573, 1.9747]$  by taking  $\tau_3 = 1$ . Notice that Rekasius substitution is deployed only to  $z_1$  and  $z_2$  when computing  $\Omega_{2D}$ . Consequently, it is necessary and sufficient to sweep the frequency from 0.9573 to 1.9747 to extract the stability map in Figure 6a of the cited work. The computation time to compute these bounds by using MAPLE's *Resultant* routine in MATLAB is approximately 3 seconds.

#### 5. CONCLUDING REMARKS

A novel approach for revealing exact lower and upper bounds of crossing frequency set (CFS) for the most general linear time invariant multiple time delayed system (MTDS) is developed. The methodology also enables delay independent stability (DIS) test *directly* without sweeping any parameter or using graphical display. To achieve this with necessary and sufficient conditions, Rekasius substitution and elimination theory are utilized. With our new approach, frequency sweeping algorithms become significantly faster. We demonstrate the effectiveness of the new method in a case study.

#### REFERENCES

- Almodaresi, E. and Bozorg, M. (2008). Stability crossing surfaces for systems with three delays. In *Proceedings of the 17th World Congress*, 13342–13347. Seoul, Korea.
- Asl, F.M. and Ulsoy, A.G. (2003). Analysis of a system of linear delay differential equations. *ASME Journal of Dynamic Systems, Measurement, and Control*, 125(2), 215–223.
- Barnett, S. (1973). Matrices, polynomials, and linear time-variant systems. *IEEE Transactions on Automatic Control*, 18(1), 1–10.
- Chen, J. and Latchman, H.A. (1995). Frequency sweeping tests for asymptotic stability independent of delay. *IEEE Transactions on Automatic Control*, 40(9), 1640–1645.
- Chen, J., Niculescu, S.I., and Fu, P. (2008). Robust stability of quasi-polynomials: Frequency-sweeping conditions and vertex tests. *IEEE Transactions on Automatic Control*, 53(5), 1219–1234.
- Chen, J., Xu, D., and Shafai, B. (1995). On sufficient conditions for stability independent of delay. *IEEE Transactions on Automatic Control*, 40(9), 1675–1680.
- Courant, R. (1988). *Differential and integral calculus*, volume 2. Interscience Publishers, New York.
- Datko, R. (1978). A procedure for determination of the exponential stability of certain differential-difference equations. *Quarterly of Applied Mathematics*, 36, 279–292.
- Delice, I.I. and Sipahi, R. (2008). Supply chain dynamics with decision making and production delays - stability analysis and optimum controller selection. In *2008 ASME International Mechanical Engineering Congress and Exposition*. Boston, USA.
- Erneux, T. (2009). *Applied Delay Differential Equations*, volume 3 of *Surveys and Tutorials in the Applied Mathematical Sciences*. Springer-Verlag, New York.
- Fazelinia, H., Sipahi, R., and Olgac, N. (2007). Stability analysis of multiple time delayed systems using 'Building Block' concept. *IEEE Transactions on Automatic Control*, 52(5), 799–810.
- Forrester, J.W. (1961). *Industrial Dynamics*. MIT Press, Cambridge, MA.
- Gelfand, I.M., Kapranov, M.M., and Zelevinsky, A.V. (1994). *Discriminants, Resultants, and Multidimensional Determinants*. Mathematics: Theory & Applications. Birkhäuser, Boston.
- Gu, K., Kharitonov, V.L., and Chen, J. (2003). *Stability of Time-Delay Systems*. Birkhäuser, Boston.
- Gu, K. and Niculescu, S.I. (2003). Survey on recent results in the stability and control of time-delay systems. *ASME Journal of Dynamic Systems, Measurement, and Control*, 125(2), 158–165.

- Gu, K., Niculescu, S.I., and Chen, J. (2005). On stability crossing curves for general systems with two delays. *Journal of Mathematical Analysis and Applications*, 311(1), 231–253.
- Gu, N., Tan, M., and Yu, W. (2001). An algebra test for unconditional stability of linear delay systems. In *Proceedings of the 40th IEEE Conference on Decision and Control*, volume 5, 4746–4747. Orlando, Florida USA.
- Hale, J.K., Infante, E.F., and Tsen, F.P. (1985). Stability in linear delay equations. *Journal of Mathematical Analysis and Applications*, 105(2), 533–555.
- Hale, J.K. and Verduyn Lunel, S.M. (1993). *An Introduction to Functional Differential Equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York.
- Hertz, D., Jury, E.I., and Zeheb, E. (1984). Stability independent and dependent of delay for delay differential systems. *Journal of The Franklin Institute*, 318(3), 143–150.
- Kamen, E. (1980). On the relationship between zero criteria for two-variable polynomials and asymptotic stability of delay differential equations. *IEEE Transactions on Automatic Control*, 25(5), 983–984.
- Kuang, Y. (1993). *Delay Differential Equations with Applications in Population Dynamics*. Academic Press, Boston.
- MacDonald, N. (1978). *Time Lags in Biological Models*, volume 27 of *Lecture Notes in Biomathematics*. Springer-Verlag, Berlin.
- Mak, K.L., Bradshaw, A., and Porter, B. (1976). Stabilizability of production-inventory systems with retarded control policies. *International Journal of Systems Science*, 7(3), 277–288.
- Michiels, W. and Niculescu, S.I. (2007). *Stability and Stabilization of Time-Delay Systems: An Eigenvalue-Based Approach*, volume 12 of *Advances in Design and Control*. SIAM, Philadelphia.
- Niculescu, S.I. (2001). *Delay Effects on Stability: A Robust Control Approach*, volume 269 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin.
- Niculescu, S.I. and Chen, J. (1999). Frequency sweeping tests for asymptotic stability: A model transformation for multiple delays. In *Proceedings of the 38th IEEE Conference on Decision and Control*, 4678–4683. Phoenix, USA.
- Niculescu, S.I., Taoutaou, D., and Lozano, R. (2003). Bilateral teleoperation with communication delays. *International Journal of Robust and Nonlinear Control*, 13(9), 873–883.
- Prasolov, V.V. (2004). *Polynomials*, volume 11 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin.
- Queinnec, I., Tarbouriech, S., Garcia, G., and Niculescu, S.I. (eds.) (2007). *Biology and Control Theory: Current Challenges*, volume 357 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin Heidelberg.
- Rekasius, Z.V. (1980). A stability test for systems with delays. In *Proceedings Joint Automatic Control Conference*, TP9-A. San Francisco, CA.
- Schöllig, A., Münz, U., and Allgöwer, F. (2007). Topology-dependent stability of a network of dynamical systems with communication delays. In *European Control Conference*, 1197–1202. Kos, Greece.
- Sipahi, R. (2008). *Stability Analysis of LTI Multiple Time Delay Systems - Cluster Treatment of Characteristic Roots*. VDM Verlag Dr. Müller, Germany.
- Sipahi, R. and Delice, I.I. (2009). Extraction of 3D stability switching hypersurfaces of a time delay systems with multiple fixed delays. *Automatica*, 45(6), 1449–1454.
- Sipahi, R., Fazelinia, H., and Olgac, N. (2008). Generalization of cluster treatment of characteristic roots for robust stability of multiple time-delayed systems. *International Journal of Robust and Nonlinear Control*, 18(14), 1430–1449.
- Sipahi, R. and Olgac, N. (2005). Complete stability robustness of third-order LTI multiple time-delay systems. *Automatica*, 41(8), 1413–1422.
- Stépán, G. (1989). *Retarded Dynamical Systems: Stability and Characteristic Functions*, volume 210 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, co-publisher John Wiley & Sons, Inc., New York.
- Sturmfels, B. (2002). *Solving systems of polynomial equations*, volume 97 of *Conference Board of the Mathematical Sciences regional conference series in mathematics*. American Mathematical Society, Providence, Rhode Island.
- Thowsen, A. (1982). Delay-independent asymptotic stability of linear systems. *IEE Proceedings D Control Theory & Applications*, 129(3), 73–75.
- Wang, Z.H. and Hu, H.Y. (1999). Delay-independent stability of retarded dynamic systems of multiple degrees of freedom. *Journal of Sound and Vibration*, 226(1), 57–81.
- Wei, P., Guan, Q., Yu, W., and Wang, L. (2008). Easily testable necessary and sufficient algebraic criteria for delay-independent stability of a class of neutral differential systems. *Systems & Control Letters*, 57(2), 165–174.
- Wu, S. and Ren, G. (2004). Delay-independent stability criteria for a class of retarded dynamical systems with two delays. *Journal of Sound and Vibration*, 270(4-5), 625–638.