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COMPLETE STABILITY ANALYSIS OF NEUTRAL-TYPE FIRST ORDER TWO-TIME-DELAY SYSTEMS WITH CROSS-TALKING DELAYS*

RIFAT SIPAHI† AND NEJAT OLGAC†

Abstract. The stability robustness of first order linear time invariant dynamics of *neutral type* with multiple time delays against delay uncertainties is taken into consideration. We depart from a simpler investigation of Hale and Huang [*J. Math. Anal. Appl.*, 178 (1993), pp. 344–362], which studies the same problem for retarded-type systems. On this basis we further introduce two challenging features by including (a) terms that add neutral dynamics and (b) an additional term that introduces cross-talk between the multiple delays. To the best of the authors' knowledge, the stability posture of this class of systems can be treated only by a unique procedure. It is known as cluster treatment of characteristic roots (CTCR), which was recently developed for *retarded*-type dynamics. We first show the applicability of CTCR to the stability analysis of *neutral*-type multiple-delay dynamics. Next, we prove the well-known "small-delay" phenomenon for the dynamics at hand, interestingly, as a natural by-product of the CTCR paradigm. Finally, we present several case studies to display the steps and the strengths of CTCR. This deployment is scalable to treat similar problems with higher order dynamics, which have direct ramifications to some practical control applications.

Key words. neutral time-delayed dynamics, stability, multiple time delays, cluster treatment of characteristic roots (CTCR)

AMS subject classifications. 15A15, 15A09, 15A23

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1. Introduction.

We consider a general class of delayed differential equations (DDEs) from the stability robustness perspective, which has not been successfully investigated in the literature. The class is of first order *neutral type*, linear time invariant (LTI) time-delay systems with two cross-talking delays. The dynamics is written in conventional form [6], [7], [8] as

$$\begin{aligned} &\frac{d}{dt}[x(t) - ax(t-\tau_1) - bx(t-\tau_2)] \\ &= cx(t-\tau_1) + dx(t-\tau_2) + fx(t-\tau_1-\tau_2) + gx(t), \end{aligned} \tag{1.1}$$

where a, b, c, d, f, g are all real scalars as well as the dependent variable $x(t)$, $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$. We highlight the $fx(t-\tau_1-\tau_2)$ term as the "delay cross-talk" feature in the problem. The problem is to analyze the stability robustness of this system against time-delay uncertainties in the semi-infinite first quadrant of $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$. The characteristic equation of these dynamics is transcendental,

$$CE = s(1 - ae^{-\tau_1 s} - be^{-\tau_2 s}) - ce^{-\tau_1 s} - de^{-\tau_2 s} - fe^{-(\tau_1+\tau_2)s} - g = 0, \tag{1.2}$$

where arguments of CE, (s, τ_1, τ_2) are suppressed. The stability robustness question of this system reduces to finding (τ_1, τ_2) regions where all the characteristic roots

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of (1.2) remain in $\mathbf{C}^-$, the open left half of the complex plane. For a notational selection we represent the right half open plane by $\mathbf{C}^+$ and the imaginary axis by $\mathbf{C}^0$ in the rest of the text. The entire complex plane becomes the union of all three, $\mathbf{C} = \mathbf{C}^- \cup \mathbf{C}^0 \cup \mathbf{C}^+$.

It is clear from (1.2) that the highest power of s is accompanied by exponential terms (a and b nonzero) with time-delay signatures, qualifying it as a *neutral type* [2], [14], [6], [7], [8], [3], [1], [15], [20]. Furthermore, the term multiplying f introduces the cross-talk between the delays. That is, the delays τ_1 and τ_2 appear in summation form. We wish to make a clarifying statement here. Although the dynamics in (1.1) are selected as scalar (for ease of conveyance only), the procedure described in the rest of the paper is scalable to higher-dimensional systems (rather than scalar), for which such cross-talk among the delays appears quite naturally [19], [20]. For instance, if one considers the two-dimensional state-space representation of a retarded system as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}_1\mathbf{y}(t-\tau_1) + \mathbf{B}_2\mathbf{y}(t-\tau_2), \qquad y \in \mathbf{R}^2,\ \mathbf{A},\ \mathbf{B}_1,\ \mathbf{B}_2 \in \mathbf{R}^{2\times 2}, \tag{1.3}$$

it is trivial to see that the respective characteristic equation typically carries the signature of $e^{-(\tau_1+\tau_2)s}$ (i.e., cross-talking time delays) beyond the individual delay terms $e^{-\tau_1 s}$ and $e^{-\tau_2 s}$. The system is a first order vector equation, which can be decoupled into two second order scalar equations.

To the best of the authors' knowledge, no methodology exists in the literature today to resolve the stability robustness of such dynamics. For simplicity of the treatment (without loss of generality), we select scalar dynamics in this paper. In this regard, (1.2) is an extended version of the retarded-type characteristic equation treated in a cornerstone study [5] in "multiple-delay systems" literature, where a, b, f are all taken to be zero.

There are some practical ramifications of the class of dynamics we consider here. For instance, multiple time-delay feedback control is used to tune a vibration absorber (called the "delayed resonator") to two distinct excitation frequencies, in (3) in [22], with $n = 2$. The cited investigation, however, does not address the stability question of such systems. Furthermore, in (2) in [16] the authors experimentally demonstrate the viability of using acceleration feedback with time delay that renders neutral dynamics for the *delayed resonator* vibration absorber. As such, neutral dynamics with multiple delays becomes a practical phenomenon, of which the stability repercussions need further investigation. And the scalar form of these dynamics, as given in (1.1), is a meaningful starting platform to present the conceptual process we wish to follow.

It is well known that neutral-type DDEs exhibit drastic differences in their characteristic behavior compared to the retarded-type dynamics [14], [6], [7], [8], [15]. This difference may occur when either one or both components of the delay vector $\{\boldsymbol{\tau}\} = (\tau_1, \tau_2)$ transit from 0 to 0^+. We denote this transition as $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$ in the rest of the text. During this transition, the root continuity argument may collapse and infinitely many unbounded characteristic roots may appear in $\mathbf{C}^+$. Such a root discontinuity, however, never takes place anywhere else in the $\boldsymbol{\tau} \in \mathbf{R}^{2+}$ domain. Clearly, if it occurs, this behavior precludes stability for any $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$ even if the initial stability posture for $\boldsymbol{\tau} = \mathbf{0}$ is stable. This "small-delay phenomenon," or "strong stability condition," has been extensively studied in the literature [2], [14], [6], [7], [8], [3], [1], [15], and conditions guaranteeing root continuity in the transition of $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$ have been investigated with respect to the system properties. It is also known that one should first guarantee the root continuity for $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$ before performing the stability analysis on neutral DDEs. It is shown in [2] that to prevent

the small-delay phenomenon from occurring, the difference equation (also known as discrete kernel operator [14], [15]) consisting of the left-hand side of (1.1),

$$L(\tau_1, \tau_2) = 1 - ae^{-\tau_1 s} - be^{-\tau_2 s} = 0, \tag{1.4}$$

has to be *stable* for $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$. It is further proven [2] that this requires the parametric condition

$$|a| + |b| < 1. \tag{1.5}$$

This inequality constitutes the *necessary and sufficient condition* for neutral dynamics (1.1) to possess root continuity during the transition of $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$, but it is only a *necessary condition* for the system to exhibit stability at a point or in a region within $\boldsymbol{\tau} \in \mathbf{R}^{2+}$, $\boldsymbol{\tau} \neq \mathbf{0}$. This is a very critical nuance, which will be revisited throughout the text.

Dynamics that possess stable discrete kernel operator are called τ-*stabilizable* [14], [15], and we use this terminology in this text as well. Note that τ-stabilizability does not guarantee that for some values of $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$ this system will be stable. But conversely, if the system is not τ-stabilizable (i.e., it is τ-nonstabilizable), no $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$ can impart stability even if nondelayed dynamics, $\boldsymbol{\tau} = \mathbf{0}$, are stable. In other words, τ-stabilizability is a *necessary* but not *sufficient* condition for a stabilizing delay set $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$ to exist, as the name τ-stabilizability implies.

As the main motivation of this study, we state that there is no prior literature which declares the aimed "stability map" of this system in $\{\boldsymbol{\tau}\} \in \mathbf{R}^{2+}$ exhaustively and exactly (i.e., nonconservatively). Exhaustiveness means *all* the stable regions in any given segment of interest in (τ_1, τ_2) space attached to or detached from the origin $(0, 0)$. There is considerable literature treating this very problem, primarily using the Lyapunov stability perspective, which results in very conservative stability bounds in the domain of the time delays; see, for example, [10] and the references therein. Furthering this conservatism, these methods almost exclusively search for a stability region including the origin, $\{\boldsymbol{\tau}\} = \mathbf{0}$ (also called the *delay margin problem*). We distinguished the objective in this paper from the mentioned investigations, as we aim to determine all the stability regions exactly and exhaustively. To resolve this problem, we resort to a very recent conceptual development of the authors: the cluster treatment of characteristic roots (CTCR) [19], [18], [15], [20]. This procedure was originally introduced for retarded systems. In this work we implement the CTCR technique on multiple-delay neutral systems for the first time and obtain the corresponding stability picture.

The CTCR reveals the complete stability robustness against delay uncertainties. We also show that the t-stabilizability condition (i.e., the discrete kernel operator (1.4) to be stable or the parametric inequality (1.5) to hold) is obtained as a by-product of the CTCR. Interestingly, the CTCR approach also offers a very practical way of proving this necessary condition.

One can encounter a small number of investigations in the literature on DDEs with multiple delays. The existing reports consider simpler dynamics than (1.1) where a, b, f are all taken to be zero. These selections reduce the dynamics to a "retarded" class of time-delayed systems (TDS). For example, [5] addresses this question to determine the only stability region attached to the origin (a feature which arises from the simplified dynamics). [12] proposes some interesting lemmas for the location of the characteristic roots when additional conditions are imposed: $f = 0$, $\tau_1 = 1$, $1 < \tau_2 < 2$. The study indicates the overwhelming difficulties which arise for cases

with $\tau_2 > 2$. [11], [21] treat the stability question on second order systems, which are resonant when delays are zero, a different problem from the one treated here. All of these simplifications impose some strong restrictions for the respective methods to be expanded for the problem here. In [19], [15], [20] the treatment entails a novel concept (CTCR) for a very general class of retarded TDS cases overcoming such restrictions. The present text is motivated by the deployment of this new development to a new problem with the inclusion of $a\dot{\mathbf{x}}(t-\tau_1)$ and $b\dot{\mathbf{x}}(t-\tau_2)$ terms (which introduce the neutral behavior) and $f\dot{\mathbf{x}}(t-\tau_1-\tau_2)$ (i.e., delay cross-talk term). These additions complicate the exercise considerably. Furthermore, we will show that this deployment will reveal the τ-stabilizability condition (1.5) as an imbedded by-product.

In the following section we present a review of the enabling paradigm, CTCR. Section 3 discusses the small-delay phenomenon and re-proves the τ-stabilizability condition. In section 4 we display the stability analysis results for some case studies with different selections of a, b, c, d, f, g. Section 5 contains some observations and conclusions based on the new paradigm, CTCR.

2. Review of CTCR. The highlights of the CTCR framework, borrowed from [19], [20], are presented next. We restate that these previous investigations are for *retarded*-type multiple-delay systems. Neutral systems, however, with single delay only are addressed in [18], [14], [15]. We have shown earlier that for single-delay treatments [18], [13], [14], [15] CTCR can be deployed independently from the formation of the dynamics being of retarded or neutral type. The natural pursuit of research brings us to the topic of this paper: neutral-type dynamics with multiple time delays.

We first present some important features and definitions of the general class of systems represented by (1.1):

1. There is a countably infinite number of characteristic roots of (1.2) for a given delay set (τ_1, τ_2). It is impracticable to evaluate their distribution (i.e., their topology for all variations of the delay set $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$) in the quest of assessing the asymptotic stability. It is also known that for the dynamics to switch the stability posture, the characteristic roots should cross the imaginary axis, say at $s = \omega i$. Thus, for a successful stability analysis, one needs to detect exhaustively all the possible imaginary axis crossings. Let us denote the complete set of such crossing frequencies as $\mathbf{\Omega}$, and the corresponding root set as $\mathbf{S}_\Omega$:

$$\mathbf{\Omega} = \{\omega \,|\, CE(s = \omega i, \boldsymbol{\tau}) = 0,\ \boldsymbol{\tau} \in \mathbf{R}^{2+},\ \omega \in \mathbf{R}\}, \tag{2.1a}$$

$$\mathbf{S}_\Omega = \{s = \omega i \,|\, \omega \in \Omega\}. \tag{2.1b}$$

We also assume that this set, $\mathbf{\Omega}$, along with the generating delays, $\boldsymbol{\tau}$, is known completely at this stage.

2. Let us denote the correspondence between the delays (τ_1, τ_2) and ω with the notation $\langle \boldsymbol{\tau}, \omega \rangle$. Notice that, typically, there exist infinitely many characteristic roots ($s = \omega i$, $\omega \in \mathbf{\Omega}$, as $\mathbf{\Omega}$ represents a set of continuum) and the respective $\langle \boldsymbol{\tau}, \omega \rangle$ correspondence. Hence, one needs a treatment of these infinitely many roots simply by grouping them into sets, which show similar characteristics. We call this grouping the *root clustering* operation. To achieve this, however, those identifier characteristics, which we name the *clustering features*, need to be determined.

3. It is known that an imaginary root $s = \omega i$ with $\langle \boldsymbol{\tau}, \omega \rangle$, $\boldsymbol{\tau} = (\tau_1, \tau_2)$, correspondence will be repeated infinitely many times for the mesh points with equidistant grid size:

$$(\tau_{1j}, \tau_{2k}) = \left(\tau_1 + \frac{2\pi}{\omega} j,\ \tau_2 + \frac{2\pi}{\omega} k\right), \qquad j = 0, 1, 2, \ldots,\ k = 0, 1, 2, \ldots. \tag{2.2}$$

Considering the existence of root continuity, we state that *small* perturbations on τ yield *small* perturbations on ω, i.e.,

$$\langle \boldsymbol{\tau} + \boldsymbol{\epsilon}, \omega + \epsilon_c \rangle, \qquad 0 < |\boldsymbol{\epsilon}| \ll 1,\ 0 < |\epsilon_c| \ll 1. \tag{2.3}$$

Clearly, $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2)$ and ϵ_c are interdependent through the characteristic equation (1.2). Further elaboration on this point is kept outside the document to better streamline the discussions. One can detect infinitely many curves in the $\boldsymbol{\tau}$ domain, which traverse through these infinitely many mesh points (2.2) earmarked by ω. According to the D-subdivision method [4], these trajectories continuously partition the t domain into encapsulated regions in which the number of unstable roots, NU, remains fixed.

4. The *first clustering feature* follows the above comment: It is proven in [19], [15], [20] for much more general forms of the dynamics than (1.1) that there is a manageably small number of curves in (τ_1, τ_2) space which we call the "kernel curves," $\wp_0(\tau_1, \tau_2)$, where *all* the characteristic root sets can be found, containing at least one pair of imaginary roots. These imaginary roots, when determined for the entire domain of delays $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$, constitute the set $\mathbf{S}_\Omega$ as identified earlier (2.1). In summary, the kernel curves correspond to the complete set, $\mathbf{S}_\Omega$, in the sense of $\langle \boldsymbol{\tau}, \omega \rangle$.

The points on the kernel curves satisfy an important condition:

$$0 < \tau_k < \frac{2\pi}{\omega}, \qquad k = 1, 2. \tag{2.4}$$

This condition implies that the points on the kernel curves exhibit the smallest positive delay value for each delay complying with $\langle \boldsymbol{\tau}, \omega \rangle$. Assume that all such points on the kernel curves, call them generically "kernel points," $\boldsymbol{\tau}_{ker}$, are already known for *all* possible $\omega \in \boldsymbol{\Omega}$. By definition, the kernel curve formation is unique for a given system (1.1). The following notation encapsulates the complete representation of the kernel curve:

$$\wp_0 = \left\{ \boldsymbol{\tau} | \langle \boldsymbol{\tau}, \omega \rangle, \boldsymbol{\tau} \in \mathbf{R}^{2+}, \omega \in \boldsymbol{\Omega}, 0 \le \tau_k \le \frac{2\pi}{\omega}, k = 1, 2 \right\}. \tag{2.5}$$

For an exclusive stability analysis, this kernel curve set must be determined *completely* and *exhaustively*. The crucial feature of the kernel curves is that they represent *all* the possible imaginary roots of (1.2) for the entire $\boldsymbol{\tau} \in \mathbf{R}^{2+}$ domain. If there is a $\langle \boldsymbol{\tau}, \omega \rangle$ occurrence, either $\boldsymbol{\tau}$ is on the kernel curves or it can be reduced to a point on the kernel curves using j and k counters defined by (2.2) and (2.4) (one may even call it the "projection on the kernel"). That is, any and every delay set, $\boldsymbol{\tau}$, which results in a root $s \in \mathbf{S}_\Omega$ has to be represented in this "kernel set." The *first clustering feature*, therefore, appears very naturally; it is $\boldsymbol{\Omega}$, which represents the complete loci of $\boldsymbol{\tau}_{ker} \in \mathbf{R}^{2+}$ for which there exists at least a pair of imaginary characteristic roots of (1.2).

5. Kernel curves, $\wp_0(\tau_1, \tau_2)$, are, in fact, the loci of $\boldsymbol{\tau}$ described in (2.2) with the points represented by $j = k = 0$. Notice that there are ∞^2 (two-dimensional infinite) candidate points defined by (2.2) in the $\boldsymbol{\tau} \in \mathbf{R}^{2+}$ domain resulting in the same imaginary root, $\omega i \in \mathbf{S}_\Omega$. Those curves corresponding to $j > 0$ and $k > 0$ are called the "offspring curves," or "offspring" for short, and denoted by $\wp_{jk}(\tau_1, \tau_2)$,

where j and k identify the jth and kth generation offspring in τ_1 and τ_2, respectively. Consequently, the complete set containing the kernel curves and the offspring becomes $\wp(\tau_1, \tau_2)$:

$$\wp(\tau_1, \tau_2) = \wp_0(\tau_1, \tau_2) \cup \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \wp_{jk}(\tau_1, \tau_2). \tag{2.6}$$

6. It is critical to note that all the infinitely many trajectories of $\wp_{jk}(\tau_1, \tau_2)$ are created from the kernel curves, $\wp_0(\tau_1, \tau_2)$, via a nonlinear shifting term $2\pi/\omega$, as per (2.2). In addition to this, any kernel point on the trajectories of $\wp_0(\tau_1, \tau_2)$ defined by $j = k = 0$ imposes its ω signature identically onto its offspring ($j > 0$ and $k > 0$). Thus, $\mathbf{\Omega}$ remains *invariant* from kernel curves to offspring curves.

7. The kernel curves and the offspring constitute the complete (and exhaustive) distribution of (τ_1, τ_2) points where the characteristic equation $CE(s, \tau_1, \tau_2)$ has root sets containing at least one pair of imaginary roots. And outside the set $\wp(\tau_1, \tau_2)$ there is not a single point, which renders imaginary characteristic roots. These are the only locations in the (τ_1, τ_2) domain where the system (1.1) could transit from stable to unstable posture (or vice versa). These contours $\wp(\tau_1, \tau_2)$ must be determined exhaustively. Since $\wp(\tau_1, \tau_2)$ is completely generated from the kernel curves $\wp_0(\tau_1, \tau_2)$, it is sufficient for our purposes to determine these kernel curves exhaustively. We present a procedure for achieving this later in this section.

8. *Root tendency invariance property.* The root tendency along the τ_j, $j = 1, 2$, axis at the crossing of $s = \omega i$ is defined by $RT^{\tau_j}_{s=\omega i} = sgn(\Re(\frac{\partial s}{\partial \tau_j}|_{s=\omega i}))$. The root tendencies at $s \in \mathbf{S}_\Omega$ along τ_1 (or τ_2) across the corresponding points on a kernel curve and its offspring remain unchanged so long as τ_2 (or τ_1) is kept fixed. This is a proven feature for much more general constructs than (1.1) known as the "root tendency invariance" property; see [19, Proposition II], [15], [20]. This very strong property constitutes the *second clustering feature* along the contours defined by (10). Through this feature one can mark stabilizing (or destabilizing) transitions along the curves $\wp(\tau_1, \tau_2)$.

2.1. Steps of the CTCR procedure. Based on the above properties one can establish the complete stability robustness picture of the system by performing the following steps of the CTCR procedure:

1. Determine exhaustively the kernel and offspring curves, $\wp(\tau_1, \tau_2)$.

2. Start from a nondelayed system, $\boldsymbol{\tau} = \mathbf{0}$, evaluating $NU(0)$ via a trivial application of the Routh–Hurwitz method.

3. Following line segments in $\boldsymbol{\tau} \in \mathbf{R}^{2+}$, which are parallel to the individual coordinates τ_1 and τ_2, connect the origin ($\boldsymbol{\tau} = \mathbf{0}$) to a point of interest (τ_{1t}, τ_{2t}), where subscript t is used for target.

4. As this path crosses the kernel and offspring curves, increase NU by $+2$ (or -2) for the $RT = +1$ (-1), according to the D-subdivision method of El'sgol'ts [4].

5. Exhaustively identify the regions in $\boldsymbol{\tau} \in \mathbf{R}^{2+}$ where (τ_{1t}, τ_{2t}) results in $NU = 0$ as "stable" and the others ($NU > 0$) "unstable."

This completes the CTCR procedure and the stability robustness picture of this system.

Remember that the delays $\{\boldsymbol{\tau}\}$'s corresponding to the $s \in \mathbf{S}_\Omega$ set were assumed to be known in the first step of the above procedure. However, a methodology is still needed to create the topology in (2.1) and the corresponding kernel curves in (2.5) exhaustively. We discuss a *holographic mapping* procedure next to achieve this. For a

better understanding we will guide the reader through the details of the above steps in section 4 for the special dynamics at hand.

2.2. Exhaustive and complete determination of the kernel curves. This step is very critical to CTCR for obvious reasons. The complete contour of (τ_1, τ_2) which results in at least one imaginary root $s = \omega i$, $\omega \in \mathbf{R}$, must be determined. For this we use the Rekasius substitution, which was first introduced in [17] and also utilized in [19], [13], [15], [20]. It suggests the following representation for exponential terms:

$$e^{-\tau_j s} = \frac{1 - T_j s}{1 + T_j s}, \qquad T_j \in \mathbf{R},\ j = 1, 2. \tag{2.7}$$

This substitution becomes exact for $s = \omega i$, provided that the relation between T_i and τ_i,

$$\tau_{jk} = \frac{2}{\omega}[\tan^{-1}(\omega T_j) + k\pi], \qquad k = 0, 1, 2, \ldots, \tag{2.8}$$

holds. It represents an asymmetric mapping. A given ω and a T_j value correspond to infinitely many equidistant delays, τ_{jk}, $j = 1, 2$, $k = 0, 1, 2, \ldots$, where $\tau_{j,k+1} - \tau_{jk} = \frac{2\pi}{\omega}$ is the fixed grid size in the τ_j distribution. Notice that this grid size is identical for both τ_1 and τ_2 sets. In the other direction, however, the mapping is one-to-one, that is, each τ_j value maps into a single T_j value.

This asymmetric mapping between $\tau_j \leftrightarrow T_j$ transforms the transcendental CE of (1.2) into a new characteristic equation $CE_T(s, T_1, T_2)$, which is of fractional polynomial type. Multiplying this equation by $(1 + T_1 s)(1 + T_2 s)$, one obtains

$$\overline{CE}(s, T_1, T_2) = \sum_{j=0}^{3} b_j(T_1, T_2) s^j = 0. \tag{2.9}$$

Notice that $b_j(T_1, T_2)$ are some multinomials in two parameters T_1, T_2 except b_0, which is a constant. An interesting relation between the infinite-dimensional equation (1.2) and the third degree equation (2.9) is that they share the *same imaginary root sets completely*. The claim is that

$$\begin{aligned} \mathbf{\Omega} &= \mathbf{\Omega}[s | CE(s, \tau_1, \tau_2) = 0, (\tau_1, \tau_2) \in \mathbf{R}^{2+}] \cap \mathbf{C}^0 \\ &\equiv \overline{\mathbf{\Omega}}[s | \overline{CE}(s, T_1, T_2) = 0, (T_1, T_2) \in \mathbf{R}^2] \cap \mathbf{C}^0, \end{aligned} \tag{2.10}$$

where $\overline{\mathbf{\Omega}}$ represents the complete topology of root sets of $\overline{CE}$ in the $(T_1, T_2) \in \mathbf{R}^2$. The left-hand side of identity (2.10) consists of all the imaginary characteristic roots of CE for some $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$, and the right-hand side is the same imaginary root set for $\overline{CE}$ at some $(T_1, T_2) \in \mathbf{R}^2$. Every (T_1, T_2) pair, which results in an imaginary root $s = \omega i$ of $\overline{CE}(s, T_1, T_2)$, represents infinitely many (τ_{1j}, τ_{2k}) pairs as per (2.8). In other words, a unique $\boldsymbol{\tau}_{ker}$ and infinitely many offspring arise. For further mathematical details on this property, the reader is referred to [19], [15], [20].

The most beneficial point in transforming $CE(s, \tau_1, \tau_2)$ to $\overline{CE}(s, T_1, T_2)$ is obvious: The parametric equation (2.9) is much easier to study than (1.2). This is a tremendous reduction in the complexity of determining the imaginary characteristic roots of (1.2), as we will observe later. $\overline{\mathbf{\Omega}}$ can be determined completely for the entire space of $(T_1, T_2) \in \mathbf{R}^2$ as $(s = \omega i, \omega \in \overline{\mathbf{\Omega}})$. As per the claim in (2.10) this set of imaginary roots is identical to $\mathbf{S}_\Omega$. Based on these observations the following structured

TABLE 1
The Routh array for $\overline{CE}(s, T_1, T_2)$, arguments suppressed.

s^3	b_3	b_1
s^2	b_2	b_0
s^1	$\frac{b_2 b_1 - b_3 b_0}{b_2} = R_1$	
s^0	b_0	

steps are most natural. One should first find the projection of the kernel curves in (T_1, T_2) space. We will name it the *core curve* just to discriminate its domain (T_1, T_2) from the kernel curves in (τ_1, τ_2). The correspondence between the two sets of curves is discussed in the following segments. The kernel and offspring curves constitute the complete set of contours in (τ_1, τ_2) where the possible stability transition can occur according to the D-subdivision method of [4].

Let us now show how to determine the core curve of $\overline{CE}(s, T_1, T_2)$, i.e., those (T_1, T_2) that give rise to $s = \omega i$ of $\overline{CE}(s, T_1, T_2)$. The easiest procedure for finding all the imaginary roots of such characteristic polynomials is the classical Routh–Hurwitz method [9]. From the well-known rules of the Routh array of Table 1, the imaginary roots of the cubic equation (2.9) are encountered at any point (T_1, T_2) where

$$R_1(T_1, T_2) = 0 \tag{2.11}$$

is satisfied with an additional condition,

$$b_2(T_1, T_2) b_0 > 0. \tag{2.12}$$

At these points one can determine a crossing frequency $\omega = \sqrt{b_0 / b_2(T_1, T_2)} \in \mathbf{R}$.

The implicit expressions (2.11) and (2.12) represent the core curve completely. The full expansion of the Routh array for the system (1.1) is given in the appendix. Notice the formation of $R_1(T_1, T_2)$, which is a multinomial in T_1 and T_2 of degrees 2 and 2, respectively. Therefore, for a given value of T_2 (T_1), there can be at most 2 (2) real T_1 (T_2). That is, the core curve has at most two separate segments. Since there is one-to-one mapping between the core curves in (T_1, T_2) and the kernel curves in (τ_1, τ_2) as proven in [19], [15], [20], this system can possess at most two separate contours for the kernel curves in (τ_1, τ_2), although these two contours may be split into separate segments according to the positivity condition in (2.4). The next step in CTCR is to numerically map these core curves into kernel curves via (2.8) and further to offspring in (τ_1, τ_2) space via (2.2). This completes the determination process of the entire set of kernel and offspring curves, $\wp(\tau_1, \tau_2)$.

With $\wp(\tau_1, \tau_2)$ at hand the second clustering feature can now be utilized. Namely, some segments of the kernel curves are identified as stabilizing transitions as opposed to destabilizing transitions. Ultimately, the D-subdivision methodology is implemented to declare a unique stability map of the system in (τ_1, τ_2) space. This map, in fact, constitutes an exact (nonconservative) declaration of the stability robustness picture against uncertain delays (τ_1, τ_2). And to the best of our knowledge, this is unique.

3. An interesting result on $\boldsymbol{\tau}$-stabilizability. In this section, we further discuss τ-stabilizability and show that it is, in fact, a natural by-product of the CTCR procedure. Several rigorous and cumbersome proofs of the τ-stabilizability condition (1.5) can be found in the literature [6], [7], [8]. As was stated earlier, if this necessary condition is not satisfied, even an infinitesimally small time delay $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$

destabilizes the system, although the nondelayed ($\boldsymbol{\tau} = \mathbf{0}$) system may start asymptotically stable. Infinitely many unbounded unstable roots appear, the root continuity argument collapses in the vicinity, and consequently the D-subdivision method [4], which is really the basis of CTCR, does not apply anymore. The CTCR procedure is, however, still valid except for its final step involving D-subdivision deployment. When applied for small delays, the CTCR procedure indeed gives rise to the τ-stabilizability property. This is the highlight of the discussions in this section.

LEMMA 3.1. *The necessary condition for the system in* (1.1) *to be stable for any* $\boldsymbol{\tau} \in \mathbf{R}^{2+}$ *is the* τ*-stabilizability condition given in* (1.5).

Proof. In order to examine the stability transition of (1.1) as τ varies $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$, we first check the stability posture of the nondelayed dynamics, which is written as

$$CE(s, \boldsymbol{\tau} = \mathbf{0}) = s(1 - a - b) - (c + d + f + g) = 0. \tag{3.1}$$

For this system to represent any dynamics, $1 - a - b$ must be nonzero. Furthermore, if $(1 - a - b)(c + d + f + g)$ is negative (positive), the nondelayed system has one stable (unstable) pole. We next investigate the $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$ transition. During this transition, we wish to observe no change in the stability posture of the system. The contrary occurrence would mean $\omega \to \infty$ (i.e., the Riemann sphere crossing; see [15]), which invites infinitely many unbounded unstable roots. This instability posture is impossible to recover from within the $\tau_1 > 0$, $\tau_2 > 0$ domain. The reason is simply that even for some $RT = -1$ points in the quadrant, $\tau_1 > 0$, $\tau_2 > 0$, there will only be a pair (or finite number of pairs in some degenerate cases) of roots returning to $\mathbf{C}^-$, since the root continuity is now in effect. We need infinitely many such crossings to take place in order to recover from the destabilizing effect caused by the small delay. Thus, $\tau_j : 0 \to \epsilon_j$, $0 \ll \epsilon_j < 1$, has to exert no Riemann sphere crossings for any stable behavior to exist.

If there is such a crossing for small delays, we can use the Rekasius transformed equation (2.9) along with $\boldsymbol{\tau} \to \mathbf{T}$ correspondence given in (2.8). Notice that for small delays, (2.8) becomes

$$\tan\left(\frac{\tau_j \omega}{2}\right) = T_j \omega, \qquad j = 1, 2, \tag{3.2}$$

which implies that $\tau_j : 0 \to \epsilon_j$ corresponds to $T_j : 0 \to \epsilon_j/2$. In short, the τ_j and T_j variations are of the same order of magnitude. In the two-dimensional delay domain such a transition can be represented selecting a directional slope m, which is defined as

$$m = \tau_2/\tau_1 = T_2/T_1, \qquad m \in [0, \infty). \tag{3.3}$$

The "no crossing" requirement should be independent of m. Using the relation (3.3), equation (2.9) becomes

$$\overline{CE}(s, \mathbf{T}) = \overline{CE}(s, T_1, mT1) = 0. \tag{3.4}$$

Taking the limit for $0 < T_1 \ll 1$ on (3.4), we arrive at the following by dropping higher powers of T_1 and favoring the lowest power only in the coefficients of s:

$$\begin{aligned} &\lim_{0<T_1\ll 1} \overline{CE}(s, T_1, mT_1) = (1 + a + b) m T_1^2 s^3 \\ &+ (1 + a - b + m\{1 - a + b\}) T_1 s^2 + (1 - a - b) s + (-c - d - f - g) = 0. \end{aligned} \tag{3.5}$$

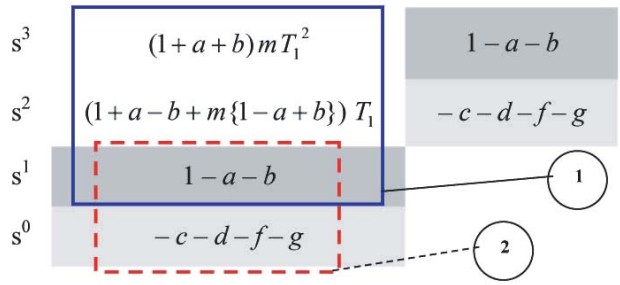

FIG. 1. *The Routh array on* (3.5) *with termwise limit* $T_1 \to \epsilon_1/2$.

We wish to find the conditions on the system parameters a, b, c, d, f, g so that (3.5) does not exhibit an imaginary root. If one forms the respective Routh array and incorporates the small T_1 approximation (i.e., favoring the lowest degree terms) for each term during the formation of this array, Figure 1 is obtained. The first column should exhibit no sign change. It consists of two blocks:

1. The coefficients of the quadratic expression for the discrete kernel operator (1.4) after Rekasius substitution. That is,

$$\begin{aligned} \overline{L}(s, T_1, T_2) &= (1 + T_1 s)(1 + T_2 s) - a(1 - T_1 s) - b(1 - T_2 s) \\ &= (1 + a + b)mT_1^2 s^2 + (1 + a - b + m\{1 - a + b\})T_1 s + (1 - a - b) = 0. \end{aligned} \tag{3.6}$$

2. The block, which is found by the coefficients of s in the nondelayed characteristic equation (3.1). If these two coefficients agree (disagree) in sign, the nondelayed dynamics are stable (unstable), respectively. In turn, it implies that the nondelayed system has no (one) root in $\mathbf{C}^+$.

If we wish to observe no change of stability posture from $T_1 : 0 \to \epsilon_1/2$ transition, independent of m, the terms in block 1 should agree in sign. This proposition yields the conditions

$$c_1 = 1 + a + b > 0, \tag{3.7a}$$

$$c_2 = 1 + a - b + (1 - a + b)m > 0, \tag{3.7b}$$

$$c_1 = 1 - a - b > 0. \tag{3.7c}$$

It is trivial to show that these conditions reduce to a simple form $|a| + |b| < 1$, which is the τ-stabilizability condition (1.5) as shown in the literature [6], [7], [8]. The admissible region in the space of (a, b) is given in Figure 2. If (a, b) satisfy (3.7), it simply guarantees that there will not be a crossing while $\boldsymbol{\tau}_{(1,2)} : 0 \to 0^+$. This property also implies that the system may be stable for some values of $(\tau_1, \tau_2) \in \mathbf{R}^{2+}$. □

Summary observations:

1. Using CTCR, we arrive at the same necessary condition for τ-stabilizability (1.5) as declared in [6], [7], [8] via a simple mathematical manipulation (Figure 1). This shows that τ-stabilizability is a natural by-product of the CTCR procedure.

2. As can be seen from Figure 2, τ-stabilizability is independent of τ_1 and τ_2 and is dictated only by the parametric selection of (a, b) pairs.

4. Example case studies. We present three case studies here for some combinations of parametric possibilities of (1.2). The first case is handled following a step-by-step pursuit of CTCR to assist the reader. The others are offered in their final outcome with some observations.

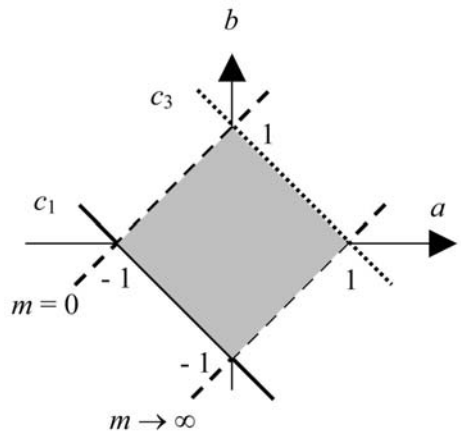

FIG. 2. *Depiction of conditions in* (3.7) *and* τ*-stabilizability region.*

4.1. Case I. Let us take $a = -0.8$, $b = -0.15$, $c = -1.5$, $d = 2$, $f = -5$, $g = 1$ in (1.2). It is clear that these dynamics are τ-stabilizable since the condition in (1.5) is satisfied, i.e., $|a| + |b| < 1$; therefore we proceed to the stability analysis. The characteristic equation is

$$s(1 + 0.8\, e^{-\tau_1 s} + 0.15\, e^{-\tau_2 s}) + 1.5\, e^{-\tau_1 s} - 2\, e^{-\tau_2 s} + 5\, e^{-(\tau_1+\tau_2) s} - 1 = 0, \tag{4.1}$$

which is stable with a pole at $s = -1.7949$ for $\{\boldsymbol{\tau}\} = \mathbf{0}$. After Rekasius substitution, we obtain the corresponding equation to (2.9) as

$$T_1 T_2 s^3 + (90T_1T_2 + 33T_2 + 7T_1)s^2 + (-50T_2 - 190T_1 + 39)s + 70 = 0. \tag{4.2}$$

The Routh array conditions for the complete core curve and the corresponding imaginary roots, following (2.11), (2.12) and the descriptions in the appendix, are

$$\frac{(-171T_2 - 13.3)T_1^2 + (-31.8T_2 - 45T_2^2 + 2.73)T_1 + 12.87T_2 - 16.5T_2^2}{(90T_2 + 7)T_1 + 33T_2} = 0, \tag{4.3}$$

$$b_2(T_1, T_2)b_0 = (90T_2 + 7)T_1 + 33T_2 > 0. \tag{4.4}$$

Figure 3 displays the viable curves satisfying (4.3) and (4.4) in (T_1, T_2) space. The core curve is those sections of the thick black curve which lie in the background region (shaded). Its determination, however, is exhaustive. Notice that the R_1 expression is quadratic in both T_1 and T_2; see also the appendix, (A.1). Therefore, for one value of T_1 there are at most two real T_2's, and vice versa. That is, at most two separate contours may constitute the core curve. The corresponding kernel curve in (τ_1, τ_2) space is generated numerically using (2.8), as seen in Figure 4 (thick line, shown in red in the electronic version). This figure also contains the complete range of crossing frequencies, $\mathbf{\Omega}$. Accordingly the system under consideration can only possess imaginary roots within the range of $1.6 \le \omega \le 51.12$ *rad/s* which defines the set $\mathbf{\Omega}$. The kernel curves contain all (τ_1, τ_2) points, which create the imaginary root set of the system entirely. That is, these dynamics can have only those imaginary characteristic roots generated by this kernel curve, and *no point in* $\boldsymbol{\tau} \in \mathbf{R}^{2+}$ *space* can render an imaginary root at ω not belonging to $\mathbf{\Omega}$. Corresponding offspring curves to the kernel

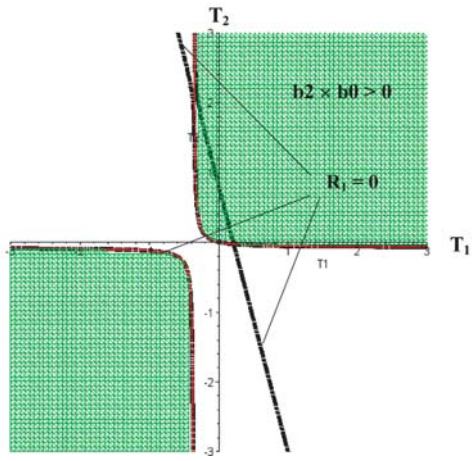

FIG. 3. $\{\mathbf{T}\}$ *domain representation, equality condition* (4.3)*, and the inequality condition in* (4.4)*.*

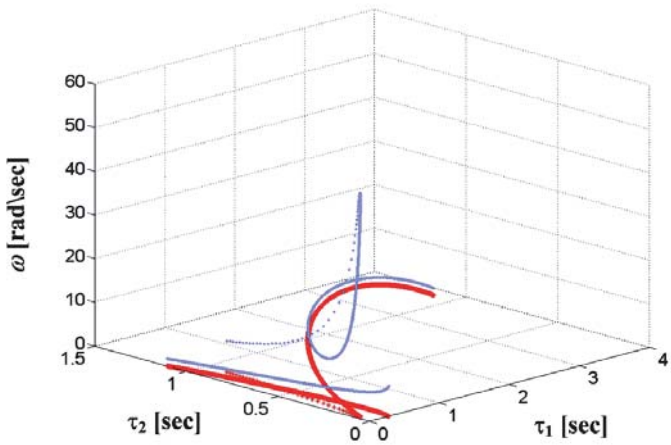

FIG. 4. *The complete kernel formation (thick line, shown in red in the electronic version) of Case* I *and the only possible* ω*'s (thin line, shown in blue in the electronic version).*

are obtained using (2.2) and are displayed in Figure 5 (thin line, shown in blue in the electronic version). They also generate exactly the same set of imaginary roots as the kernel curves, nothing more and nothing less. This concludes the formation of the first clustering feature. All the points (τ_1, τ_2), which render imaginary characteristic roots, are in Figure 5 as kernel and offspring curves.

We now look at the kernel curves and determine the root tendencies. For instance, at points A_1, A_2, and A_3, $RT^{\tau_2} = -1$ (i.e., stabilizing) and it is invariant from kernel curves to all the offspring curves. Using the D-subdivision method of [4] one can determine the number of unstable roots NU in each region in Figure 5. They are sparingly shown in the figure. Obviously wherever $NU = 0$, the stable region appears, which is shaded in the figure. In Figure 5, we also recognize that the dynamics are τ_2-independent stable for $\tau_1 < 0.0595$ s. Just to give an idea of the efficiency of the CTCR procedure, we state the CPU time of creating Figure 5 (on a 3.2 MHz Pentium 4 with 512 MB RAM) to be 25 seconds.

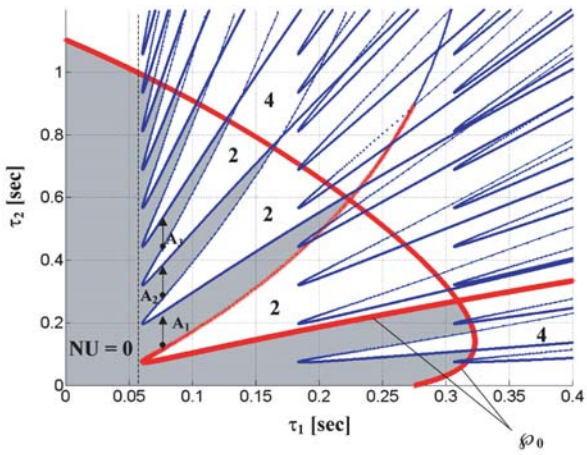

FIG. 5. *Complete stability picture of Case* I.

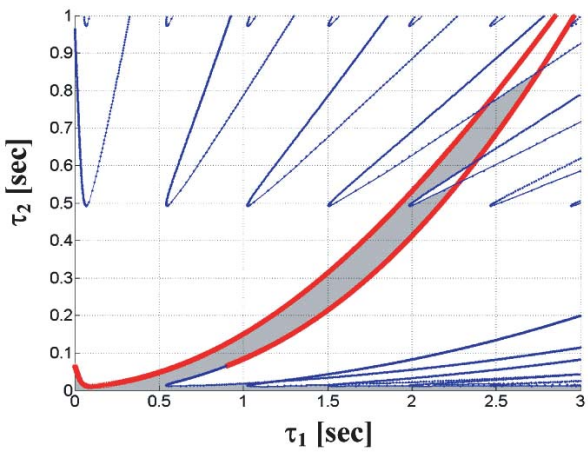

FIG. 6. *Complete stability picture of Case* II.

4.2. Case II. The system with parameters $a = 0.05$, $b = 0.9$, $c = 0.25$, $d = -1$, $f = -1.25$, $g = -1$ is again τ-stabilizable, i.e., $|a| + |b| < 1$. In Figure 6, we suppress the intermediary steps and give the stability picture only. The stable region is shaded ($NU = 0$). We observe that for $\tau_2 < 0.0109$ s the system becomes τ_1-independent stable. Also, we find (following the same logic as in Case I) that the only resonance frequencies of these dynamics are within the range $0.58 \leq \omega \leq 12.97$ rad/s, which is the set $\mathbf{\Omega}$.

4.3. Case III. Take $a = -0.35$, $b = 0.4$, $c = -12$, $d = -18$, $f = 4$, $g = -6$ as the set of parameters. The system is again τ-stabilizable. Without the intermediary steps, we show in Figure 7 the stability picture only. The stable region is shaded ($NU = 0$). We observe that for $\tau_2 < 0.0139$ s the system becomes τ_1-independent stable, and for $\tau_1 < 0.0525$ s the system is τ_2-independent stable. For this case, the only possible resonance frequencies are within the set $\mathbf{\Omega} : \{1.62 \leq \omega \leq 48.06\ rad/s\}$.

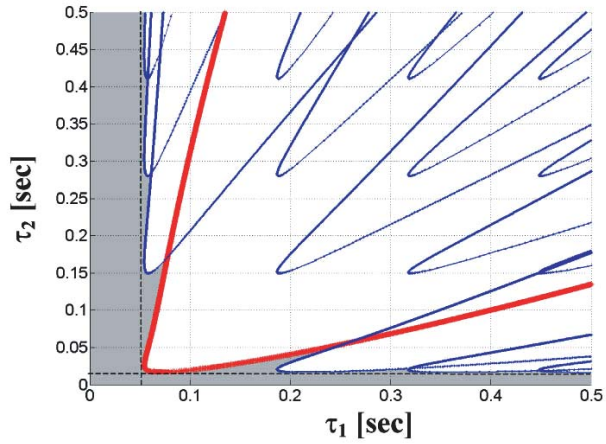

FIG. 7. *Complete stability picture of Case* III.

5. Conclusion. A general class of first order LTI scalar, two-time-delay systems of *neutral type* is studied for their stability posture. The unique features in this work are twofold. First, we have the *additional terms* that convert a cornerstone study on retarded-type dynamics by Hale and Huang [5] into a neutral type. Second, the dynamics are exacerbated by a *cross-talk term in the delays*. Both these added dynamics make the stability analysis considerably more complex. A new framework, the cluster treatment of characteristic roots (CTCR), is deployed to overcome these added complexities. CTCR offers a systematic and numerically very efficient procedure for the creation of complete stability maps of such systems. Moreover, CTCR is also deployed to re-prove the well-known necessary condition of the small delay phenomenon of neutral dynamics. It turns out that the CTCR procedure automatically generates this condition as a by-product. The end results are demonstrated in numerical case studies.

Appendix. Take the most general form of (1.2). Making the substitution of (2.7), creating (2.9), and deploying the Routh array as in Table 1, one obtains the corresponding terms to (2.11) and (2.12) as

$$\begin{aligned} R_1(T_1,T_2) = [(\{2gc-2fd-c^2+d^2-g^2+f^2\}T_2 + ad-af+ag+bf+g \\ -bg+bc-bd-ac-c+d-f)T_1^2 + (\{-d^2-g^2+c^2+2gd-2fc+f^2\}T_2^2 \\ +\{2bd-2af+2ac-2f-2bf-2bg+2g-2c-2bc-2ad-2ag-2d\}T_2 \\ -1-b^2+2b+a^2)T_1 + (-d+af-ac+g+bc-f+c-bf+ad \\ +bg-bd-ag)T_2^2 + (b^2+2a-a^2-1)T_2]/b_2(T_1,T_2), \end{aligned} \tag{A.1}$$

$$\begin{aligned} b_2(T_1,T_2) = (\{c+d-f-g\}T_2+1+a-b)T_1 + (1-a+b)T_2, \\ b_0 = -(c+d+f+g). \end{aligned} \tag{A.2}$$

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