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STABILITY ROBUSTNESS OF RETARDED LTI SYSTEMS WITH SINGLE DELAY AND EXHAUSTIVE DETERMINATION OF THEIR IMAGINARY SPECTRA*

RIFAT SIPAHI† AND NEJAT OLGAC‡

Abstract. In this paper we consider the stability robustness of the general class of vector LTI (linear time invariant) equations with a single delay, $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau)$, $\mathbf{x} \in \mathbf{R}^n$. The robustness is against the uncertain, but constant delay, $\tau \in \mathbf{R}^+$. We first present a set of novel propositions and state that the solution must start from the complete knowledge of imaginary spectra of the system, and the corresponding delays. The propositions claim that such spectra form a set of manageably small number of members, and this number is upper bounded by n^2 regardless of the composition of $\mathbf{A}$ and $\mathbf{B}$ matrices. They also claim that the infinite-dimensional system at hand has an outstanding discipline regarding these imaginary spectra. This discipline invites the recently developed concept called the cluster treatment of characteristic roots (CTCR). The CTCR procedure requires a complete and precise determination of the imaginary spectra of the system. There are many procedures in the literature to achieve this. They are, in fact, some variations of the five main methods of different levels of precision and complexity. There is, however, no study known to the authors for presenting a comparison among these methods. This paper addresses this need. We first offer an overview of each of the five methods and then compare their numerical performances over an example case study.

Key words. linear time-delayed dynamics, quasi polynomial, imaginary spectra detection, robust stability

AMS subject classifications. 15A15, 15A09, 15A23

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1. Problem statement and an explicit function for stability.

The stability of linear time invariant retarded time delayed systems (LTI–TDS) has been a very active research topic during the past several decades [1, 2, 3, 4, 5]. Numerous contributions by renowned investigators can be found in the literature on the subject. Although at present the focus of attention in the time delayed systems (TDS) community is directed toward much more complex dynamics (such as parametric uncertainties, robustness, time-varying time delays, nonlinear TDS), the LTI–TDS has an undisputed knowledge base which offers plenty of insight into some realistic problems. Furthermore it is the authors' belief that the LTI–TDS field still remains rich with challenging and unsolved problems. Some existing methods, for instance, present new knowledge, which have not been recognized until recently [4, 6, 7, 26]. Some others suggest variations on the earlier techniques—to overcome some subtle and hidden impracticalities—mainly from a numerical deployment point of view [8].

The general dynamics in question is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau), \tag{1.1}$$

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where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{A}$ and $\mathbf{B} \in \mathbf{R}^{n\times n}$ are known matrices with ranks n and p ($\leq n$), respectively, and $\tau \in \mathbf{R}^+$, which is the only free parameter in (1.1). The question is to determine the stability outlook of the system in the semi-infinite τ domain. The characteristic equation of the system is

$$CE(s,\tau) = \det(s\,\mathbf{I} - \mathbf{A} - \mathbf{B}\,e^{-\tau s}) = 0, \tag{1.2}$$

and it contains time delays of a commensurate nature with degree up to p; i.e., there exist exponential terms $e^{-k\tau s}$, $k = 0, 1, \ldots, p$, in (1.2), where $p = \text{rank}(\mathbf{B}) \leq n$. The system is infinite dimensional and as such possesses infinitely many characteristic roots, also known as the spectrum $\sigma(\tau)$ of the system. The question of stability for systems of this class translates into some conditions on τ to guarantee that all the infinitely many characteristic roots lie on the stable left half of the complex plane ($\mathbf{C}^-$). In pursuit of this, we present a remark and two relevant propositions next, partially following [6, 15] which contain the highlights of the CTCR paradigm.

Remark 1. If a delay, τ_0, results in a spectrum, which contains an imaginary root, i.e., $s = \omega i \in \sigma(\tau)$, there exist infinitely many delays

$$\tau_k = \tau_0 \mp \frac{2\pi}{\omega}k, \quad k = 0, 1, 2, \ldots, \tag{1.3}$$

which impart the same $s = \omega i \in \sigma(\tau_k)$. In short, the intersection of these infinitely many spectra contains a pair of imaginary roots,

$$(\mp\omega i) \in \bigcap_{k=0}^{\infty} \sigma(\tau_k). \tag{1.4}$$

Notice that these τ_k values corresponding to a single $s = \omega i$ are separated by $2\pi/\omega$. We name the smallest positive of them the *kernel delay value*, τ_{ker}. τ_{ker} begets all the other delays as per (1.3), and we call them the *offspring delay values*. Briefly, the set of infinitely many delay values

$$\left\{\tau_{ker} + \frac{2\pi}{\omega}k,\ k = 0, 1, 2, \ldots\right\} \text{ all result in } s = \omega i \in \sigma\left(\tau_{ker} + \frac{2\pi}{\omega}k\right). \tag{1.5}$$

PROPOSITION 1. *For the entire $\tau \in \mathbf{R}^+$ domain,*

(i) *there is a manageably small number, say m, of imaginary roots, $s = \omega i$, of* (1.1)*;*

(ii) *this number is upper bounded by n^2, i.e., $m \leq n^2$, regardless of the composition of the constant matrices $\mathbf{A}$ and $\mathbf{B}$.*

We will present the proof of Proposition 1 later in the text. We wish to state here, however, that this proposition brings an extraordinary mathematical confinement to the otherwise vastly distributed delay values creating a spectrum, which contains an imaginary root, i.e., $\{\tau > 0 \mid \sigma(\tau) \cap \mathbf{C}^0 \neq 0\}$, where $\mathbf{C}^0$ represents the imaginary axis.

PROPOSITION 2. *Assume that all m of the imaginary roots $\{\omega\} \in \mathbf{R}^{m+}$ and all corresponding* kernel delay values *$\{\tau_{ker}\} \in \mathbf{R}^{m+}$ are known. The* root tendency*, which is defined as*

$$RT|_k = sgn\left[\text{Re}\left(\left.\frac{\partial s}{\partial \tau}\right|_{\substack{s=\omega i \\ \tau = \tau_{ker} + \frac{2\pi}{\omega}j}}\right)\right], \quad j = 0, 1, 2, \ldots, \tag{1.6}$$

is invariant with respect to the counter j.

Proof of Proposition 2. The characteristic equation of (1.2) can be rewritten as

$$CE(s,\tau) = \sum_{k=0}^{p} a_k(s) e^{-k\tau s} = 0, \tag{1.7}$$

where $p = \text{rank}(\mathbf{B})$, $p \leq n$, is the degree of commensuracy in the dynamics and $a_k(s)$ are polynomials of degree $n-k$. A variational form of this equation is

$$\sum_{k=0}^{p} ([a_k'(s) - a_k(s)k\tau]e^{-k\tau s}ds - a_k(s)kse^{-k\tau s}d\tau) = 0, \tag{1.8}$$

where $a_k'(s) = da_k(s)/ds$. The root sensitivities at a point $s = \omega i$ corresponding to the delays given in (1.5) are derived from (1.8) as

$$\left.\frac{ds}{d\tau}\right|_{\substack{s=\omega i \\ \tau=\tau_{ker}+\frac{2\pi}{\omega}j}} = \left.\frac{\sum_{k=0}^{p} a_k(s)kse^{-k\tau s}}{\sum_{k=0}^{p}[a_k'(s) - a_k(s)k\tau]e^{-k\tau s}}\right|_{\substack{s=\omega i \\ \tau=\tau_{ker}+\frac{2\pi}{\omega}j}}. \tag{1.9}$$

It is trivial to show that

$$sgn\left[\text{Re}\left(\left.\frac{ds}{d\tau}\right|_{\substack{s=\omega i \\ \tau=\tau_{ker}+\frac{2\pi}{\omega}j}}\right)\right] = sgn[H(s)|_{s=\omega i}], \tag{1.10}$$

where

$$H(s) = \text{Im}\left(\left.\frac{\sum_{k=0}^{p} a_k'(s)e^{-k\tau s}}{\sum_{k=0}^{p} a_k(s)ke^{-k\tau s}}\right|_{\substack{s=\omega i \\ \tau=\tau_{ker}+\frac{2\pi}{\omega}j}}\right).$$

To obtain (1.10), one can divide the numerator and the denominator of (1.9) by $\sum_{k=0}^{n} a_k(s)kse^{-k\tau s}$ and study the real part of the new expression. Expression (1.10) is obviously independent of the counter j. We call this feature the *root tendency invariance property.* □

Proposition 2 simply implies that the root transition at ωi is either destabilizing ($RT = +1$) or stabilizing ($RT = -1$) for a given τ_{ker} and its offspring $\tau_{ker} + \frac{2\pi}{\omega}j$, $j = 0, 1, 2, \ldots$, which render this particular imaginary root.

Using Propositions 1 and 2, one can create an explicit function of τ for the number of unstable roots (i.e., the number of roots in the set $\sigma(\tau) \cap \mathbf{C}^+$) as follows:

$$NU(\tau) = NU(0) + \sum_{k=1}^{m} \left[\Gamma\left(\frac{\tau - \tau_{ker}}{\Delta\tau_k}\right) \cdot U(\tau, \tau_{ker}) \cdot RT_k\right]_k, \tag{1.11}$$

where $NU(0)$ is the number of unstable roots when $\tau = 0$, $U(\tau, \tau_{ker})$ = step function in τ with the step taking place at τ_{ker}:

$$U(\tau, \tau_{ker}) = \begin{cases} 0, & 0 < \tau < \tau_{ker}, \\ 1 \quad \text{for} & \tau \geq \tau_{ker},\ \omega = 0, \\ 2, & \tau \geq \tau_{ker},\ \omega \neq 0. \end{cases} \tag{1.12}$$

$\Gamma(x)$ = ceiling function of x, and Γ returns the smallest integer greater than or equal to x. The $[\,]_k$ notation in (1.11) is simply used to re-emphasize that τ_{ker} needs to be changed by the counter k. The $NU(\tau)$ expression requires only knowledge of the following:

(i) NU(0);
(ii) τ_{ker}, $k = 1, \ldots, m$, the smallest τ values corresponding to each ω (m of them) as per Proposition 1;
(iii) $\Delta\tau_k = 2\pi/\omega_k$, $k = 1, \ldots, m$;
(iv) $RT|_k$, $k = 1, \ldots, m$, the invariant root tendencies, as per Proposition 2.

The τ intervals, where $NU = 0$, render stable behavior for the system. $NU > 0$, on the other hand, would mean instability. Since the *kernel delay set* and the corresponding m crossing roots are all known, expression (1.11) exhaustively defines all the stable regions of the system in the delay space, $\tau \in \mathbf{R}^+$.

It is obvious from (1.11) that the stability robustness declaration is possible only if and only if the m members of the $\{\tau_{ker}\}$ kernel delay set and the corresponding m crossing roots $s = \omega i$ are exhaustively and accurately determined. We devote the rest of the paper to reviewing the five distinctive methodologies to achieve this. We also include their comparisons based on accuracy and efficiency. These methodologies are

(a) the Schur–Cohn method (Hermite matrix formation) [3, 11];
(b) elimination of transcendental terms [5];
(c) the matrix pencil–Kronecker sum method [3, 12];
(d) the Kronecker multiplication/elementary transformation method [13];
(e) Rekasius substitution [14].

2. Brief review of the methodologies. In this section we revisit the five main methodologies mentioned to prepare for the comparative work. Let us take the expanded form of (1.2) and explain each method based on that.

(a) Schur–Cohn criterion as per [3, 11]. The formation starts with rewriting (1.7), multiplying it by $e^{k\tau s}$, $k = 0, 1, \ldots, p-1$. This generates p equations in terms of $e^{k\tau s}$, $k = -p, \ldots, -1, 0, 1, \ldots, p-2, p-1$, which are $2p$ linearly independent terms. Next let us consider the companion equation, $CE(-s, \tau) = 0$, which is also satisfied for $s = \omega i$ due to the fact that the imaginary characteristic root ωi always appears as a complex conjugate pair,

$$\overline{CE}(s,\tau) = CE(-s,\tau) = \sum_{k=0}^{p} a_k(s) e^{-k\tau s} = \sum_{k=0}^{p} \overline{a}_k(s) e^{k\tau s}, \tag{2.1}$$

where $\overline{f}(s) = f(-s)$ is indeed the conjugate operation when $s = \omega i$. We then multiply $\overline{CE}(s,\tau)$ with $e^{-k\tau s}$, $k = 1, 2, \ldots, p$, generating additional p equations in terms of the same $2p$ linearly independent terms $e^{k\tau s}$, $k = -p, -p+1, \ldots, p-1$. Both of these sets of p equations can be combined into a single matrix equation as

(2.2)

$$\begin{bmatrix}
0 & 0 & 0 & \ldots & a_0 & a_1 & a_2 & \ldots & \ldots & a_p \\
0 & 0 & \ldots & a_0 & a_1 & a_2 & \ldots & & a_p & 0 \\
\vdots & & \ddots & \ddots & \vdots & \vdots & & \ddots & & \vdots \\
\vdots & \ddots & \ddots & & \vdots & \vdots & \ddots & & & \vdots \\
a_0 & a_1 & a_2 & & a_{p-1} & a_p & \ldots & \ldots & \ldots & 0 \\
0 & 0 & & 0 & \overline{a}_p & \overline{a}_{p-1} & \overline{a}_{p-2} & \ldots & \ldots & \overline{a}_0 \\
0 & 0 & \ldots & \overline{a}_p & \overline{a}_{p-1} & \overline{a}_{p-2} & \ldots & & \overline{a}_0 & 0 \\
\vdots & & \ddots & \ddots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \ddots & \ddots & & \vdots & \vdots & & & \ddots & \vdots \\
\overline{a}_p & \overline{a}_{p-1} & \ldots & \ldots & \overline{a}_1 & \overline{a}_0 & 0 & \ldots & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
e_{p-1} \\ e_{p-2} \\ \vdots \\ \vdots \\ \vdots \\ e_1 \\ e_0 \\ \vdots \\ \vdots \\ e_{-p}
\end{bmatrix}
= \mathbf{A}_1 \mathbf{E}_1 = \mathbf{0},$$

where e_k represents $e^{k\tau s}$ as shorthand notation to prevent cluttering the equation, and $\mathbf{A}_1$, $\mathbf{E}_1$ are evidently the $(2p \times 2p)$ matrix and the $(2p \times 1)$ vector, respectively. If one rewrites this equation by rearranging the exponential terms, it can be cast in the form

(2.3)
$$\begin{bmatrix} a_0 & 0 & 0 & \dots & 0 & a_p & a_{p-1} & \dots & a_2 & a_1 \\ a_1 & a_0 & 0 & \dots & 0 & 0 & a_p & a_{p-1} & \dots & a_2 \\ \vdots & & \ddots & \ddots & \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & & \vdots & \vdots & \ddots & & & \vdots \\ a_{p-1} & a_{p-2} & \dots & \dots & a_0 & 0 & 0 & \dots & 0 & a_p \\ \overline{a}_p & 0 & \dots & \dots & 0 & \overline{a}_0 & \overline{a}_1 & \dots & \dots & \overline{a}_{p-1} \\ \overline{a}_{p-1} & \overline{a}_p & \dots & \dots & \dots & 0 & \overline{a}_0 & \dots & \dots & \overline{a}_{p-2} \\ \vdots & & \ddots & \ddots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & & \vdots & \vdots & & & \ddots & \vdots \\ \overline{a}_1 & \overline{a}_2 & \dots & \dots & \overline{a}_p & 0 & \dots & \dots & 0 & \overline{a}_0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ \vdots \\ e_{p-1} \\ e_{-p} \\ e_{-p+1} \\ \vdots \\ \vdots \\ e_{-1} \end{bmatrix} = \mathbf{A}_2\mathbf{E}_2 = \mathbf{0}.$$

Obviously for a nontrivial solution of $\mathbf{E}_2$, the $\mathbf{A}_2$ matrix must be singular:

$$\det \mathbf{A}_2(s) = 0. \tag{2.4}$$

This matrix, $\mathbf{A}_2$, is known to be the Schur–Cohn matrix. Notice the favorable fragmentation of $\mathbf{A}_2$ into four $p \times p$ segments in the rearranged form as

$$\mathbf{A}_2 = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2^H & \Lambda_1^H \end{bmatrix}, \tag{2.5}$$

where Λ^H implies the Hermitian of Λ, and (2.5) presents a compact form adopted by [3, 11].[1] Λ_1 and Λ_2 are evidently matrices from (2.3).

This method suggests that if (1.7) has any imaginary root pair $s = \mp\omega i$, it should also satisfy (2.4). Consequently, the question of finding all the imaginary roots of (1.2) reduces to finding the imaginary roots of (2.4), which is a polynomial of s with degree $2np$. Notice that the original system (1.1) is a retarded time delayed system. Thus the nondelayed term of the characteristic equation (1.7), $a_0(s)$, is an nth degree polynomial of s. So the problem is cast into determining the purely imaginary roots of the $2np$ degree polynomial equation (2.4), which can produce maximum np pairs of imaginary roots. For cases when $p = \text{rank}(\mathbf{B}) = n$, the supremum of this number becomes n^2.

Evaluation of the $\det \mathbf{A}_2(s)$, however, needs a symbolic operation, while $2p$ terms are multiplied and added $2p$ times for expanding the determinant. Each of these $2p$ multiplications would create some round-off errors, eventually resulting in a polynomial of s with erroneous coefficients. This operation ultimately yields poor precision in determining the desired imaginary roots.

An alternative to this symbolic evaluation of a determinant is to determine the eigenvalues of a constant matrix (see Theorem 2.1 of [3]). Namely,

$$\det[\omega\,\mathbf{I} - \mathbf{P}] \equiv \det(\mathbf{P}_n^{-1}) \det(\mathbf{A}_2)|_{s=\omega i}, \tag{2.6}$$

[1] We wish to make a remark on the formulation error of equations (3.79) and (3.80) in [11], which need to be corrected according to (2.3) above.

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{I} \\ -\mathbf{P}_n^{-1}\mathbf{P}_0 & -\mathbf{P}_n^{-1}\mathbf{P}_1 & \dots & \dots & \dots & -\mathbf{P}_n^{-1}\mathbf{P}_{n-1} \end{bmatrix}, \tag{2.7}$$

$$\mathbf{P}_k = \begin{pmatrix} i^k\mathbf{T}_k & i^k\mathbf{H}_k \\ (-i)^k\mathbf{H}_k^T & (-i)^k\mathbf{T}_k^T \end{pmatrix}, \quad k = 0, 1, 2, \dots, n,$$

$$\mathbf{H}_k = \begin{bmatrix} a_{pk} & a_{p-1,k} & \dots & a_{1k} \\ 0 & \ddots & \ddots & a_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{pk} \end{bmatrix}, \quad \mathbf{T}_k = \begin{bmatrix} a_{0k} & 0 & \dots & 0 \\ a_{1k} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{p-1,k} & \dots & a_{1k} & a_{0k} \end{bmatrix} \tag{2.8}$$

with $k = 0, 1, \dots, n-1$ and $a_j(s) = \sum_{k=0}^{p} a_{jk}s^k$ which are the terms defined in (1.7). Notice that (2.6) indicates that the imaginary roots of $\det \mathbf{A}_2 = 0$ are identical to the real eigenvalues of $\mathbf{P}$, which is a constant matrix. So the numerical procedure is now converted into a simpler and more precise one, i.e., "real eigenvalue" determination of a constant matrix.

(b) Elimination of transcendental terms (as introduced by [5] and utilized in [16, 17, 18, 19, 20, 21]). This procedure follows a starting premise similar to that in the Schur–Cohn methodology in (a). If $CE(s,\tau)$ of (1.7) has an imaginary root, then, correspondingly, $\overline{CE}(s,\tau)$ of (2.1) should have the same root. Multiplying (2.1) by $e^{-p\tau s}$, we obtain

$$e^{-p\tau s}\overline{CE}(s,\tau) = \sum_{k=0}^{p} \overline{a}_k(s)e^{(k-p)\tau s} = 0. \tag{2.9}$$

One can then eliminate the highest commensuracy term (i.e., $e^{-p\tau s}$) between (1.7) and (2.9), yielding a new equation

$$CE_1(s,\tau) = \sum_{k=0}^{p-1} a_k^{(1)} e^{-k\tau s} = 0, \tag{2.10}$$

which is of commensuracy degree $p-1$. If one repeats this procedure of eliminating the highest degree commensuracy terms p times successively, one arrives at

$$CE_p(s) = a_0^{(p)}(s) = 0, \tag{2.11}$$

an algebraic characteristic equation in which no transcendentality remains. One can show that $a_0^{(p)}(s)$ is a polynomial of degree $n2^p$, of which purely imaginary roots are in question. Notice that due to the successive substitution of "s" with "$-s$" during the manipulations, the imaginary roots of the original characteristic equation CE are preserved, although the degree of the s terms in polynomials $CE_i(s)$ continuously increases. Ultimately there remains only $n2^p$ finite roots of (2.11) instead of the

infinitely many roots of the original equation (1.7). It is guaranteed that only the imaginary roots of these two equations are identical. Therefore searching for the imaginary roots of (2.11) is a sufficient procedure for the mission. The practical usage of this analytically elegant procedure in the literature is very limited [16, 18, 20, 21] because of the round-off errors it invites during the successive evaluation of $CE_p(s)$, albeit in alphanumeric form.

One can further prove that the $CE_p(s)$ and $\det \mathbf{A}_2(s)$ in (2.4) have a common factor, which is the complete polynomial of $\det \mathbf{A}_2(s)$, because they are obtained based on the same fundamental premise of $s \to -s$ substitution, and they represent the same system of (1.1). When numerically executed, however, this factorization disappears due to successive round-off errors. Ultimately one finds two sets of roots, which may be close to one another but not identical.

Clearly, for $n = p = 1, 2$ the degrees of the polynomials of (2.4) and (2.11) are identical, and they are equal to 2. For these cases, the equations are indeed identical. For $n = p > 2$, which implies the case of full rank $\mathbf{B}$ matrix ($p = n$), $n2^n > 2n^2$, and clearly the procedure in (a) is a much more favorable proposition for determining the purely imaginary roots. Notice that the $n2^n - 2n^2$ excess roots may also contain some false imaginary roots, which should not appear at all. We suppress the proofs of these statements, but we will revisit them for example case studies later.

(c) Matrix pencil–Kronecker sum method introduced in [3, 12]. This procedure departs from (1.2), which is rewritten as

$$\det[s\mathbf{I} - (\mathbf{A} + \mathbf{B}z)] = 0, \quad z = e^{-\tau s}. \tag{2.12}$$

Using the argument that if $s = \omega i$ is a root of (2.12), we see that so is $s = -\omega i$ when z is replaced with $1/z$. One can say that the eigenvalues of $\mathbf{A} + \mathbf{B}z$ and $\mathbf{A} + \mathbf{B}z^{-1}$ must be $s = \mp\omega i$, which can also be expressed using the property of the Kronecker sum of matrices (see the appendix for a definition). A commonly known property of this operation is that the eigenvalues of the Kronecker sum of two matrices are equal to the pairwise sum of the individual eigenvalues of the matrices [22, 23]. That is, at least one of the eigenvalues of the following matrix has to be zero:

$$(\mathbf{A} + \mathbf{B}z) \oplus (\mathbf{A} + \mathbf{B}z^{-1}), \tag{2.13}$$

where $\oplus$ represents the Kronecker summation. In other words,

$$\det[(\mathbf{A} + \mathbf{B}z) \oplus (\mathbf{A} + \mathbf{B}z^{-1})] = 0. \tag{2.14}$$

This equation gives rise to a polynomial in z of degree $2n^2$ for $n = p$. One needs to solve the $2n^2$ roots of (2.14) and determine those which have the unity magnitude $|z| = 1$. Only those roots represent $s = \omega i$ eigenvalues as per (2.12). For the roots which satisfy this condition, one next solves the imaginary roots $s = \omega i$ from (2.12). Notice that by substituting z as a complex number in (2.12), one obtains a polynomial with complex but constant coefficients. Therefore, most of the neat features of ordinary polynomials with constant coefficients disappear. For instance, there is no guarantee of the complex conjugate feature of the roots. Therefore to decide whether an imaginary root $s = \omega i + \epsilon$, $\epsilon << 1$, is really an imaginary root, except that it is displaced infinitesimally due to numerical/computational error, is not a trivial task. This particular point alone, from the numerical deployment point of view, suggests a drawback.

Similarly to the methodology explained for (a), one can convert the symbolic determinant evaluation of (2.14) into an equivalent eigenvalue determination of a

constant matrix; see Theorem 3.1 of [3] or [24]. In this new form, a generalized eigenvalue problem appears as

$$\det[z\mathbf{U}-\mathbf{V}] = z^{n^2}\det[(\mathbf{A}+\mathbf{B}z)\oplus(\mathbf{A}+\mathbf{B}z^{-1})] = 0, \tag{2.15}$$

where

$$\mathbf{U} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix}_{2n^2\times 2n^2}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{B}_0 & -\mathbf{B}_1 \end{bmatrix}_{2n^2\times 2n^2},$$

and $\mathbf{B}_0 = \mathbf{I}\otimes\mathbf{B}^T$, $\mathbf{B}_1 = \mathbf{A}\otimes\mathbf{A}^T$, $\mathbf{B}_2 = \mathbf{B}\otimes\mathbf{I}$, all of which have dimensions of $(n^2\times n^2)$. Again the generalized eigenvalue operation is numerically a much more reliable and efficient operation than evaluating the roots of the determinant in (2.14). We now wish to utilize these findings to prove Proposition 1.

Proof of Proposition 1. It is clear that (2.15) results in a polynomial of z with degree $2n^2$. Consequently, there can be at most n^2 pairs of complex conjugate unitary solutions for z, and that is the upper bound of the number of root crossings claimed in Proposition 1. Similar arguments can be made following the discussions in section 2(a) immediately after (2.5). The maximum possible number of imaginary eigenvalues for $\mathbf{A}_2$ in (2.3) is n^2 (when $p=n$). □

(d) Kronecker multiplication/elementary transformation method [13]. Before we proceed with this method, we wish to define a critical elementary vectorization transformation, $\xi : \mathbf{C}^{n\times n} \to \mathbf{C}^{n^2\times 1}$ [22, 23]. The ξ operation converts a matrix $\mathbf{M}^{(n\times n)} = [\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n]^T_{n\times n}$ into $(\xi\mathbf{M})^{(n^2\times 1)} = [\mathbf{m}_1^T, \mathbf{m}_2^T, \ldots, \mathbf{m}_n^T]^T_{1\times n^2}$ and the multiplication of three $n\times n$ matrices $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ into a Kronecker product of dimension $n^2\times n^2$ [22, 23], which is given as

$$\xi(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3) = (\mathbf{P}_1\otimes\mathbf{P}_3^T)\xi\mathbf{P}_2, \tag{2.16}$$

where $(\bullet)^T$ denotes the transpose of $(\bullet)$. The aim is to form a $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$ product, which will then be mapped into the right-hand side of (2.16). This mapping, as explained below, brings convenience to solving the pure imaginary roots of dynamics (1.1).

The procedure departs from (1.1), for which a solution of the form $\mathbf{x}(t) = e^{st}\mathbf{v}$ is suggested, where $(s, \mathbf{v}^{(n\times 1)})$ is an eigenvalue-eigenvector pair (both of which are complex in general). Differentiating this expression and substituting in (1.1), one gets

$$(s\mathbf{I}-\mathbf{A}-\mathbf{B}e^{-\tau s})\mathbf{v} = 0, \tag{2.17}$$

which can be rewritten as

$$(s\mathbf{I}-\mathbf{A})\mathbf{v} = e^{-\tau s}\mathbf{B}\mathbf{v}. \tag{2.18}$$

If $s=\omega i$ is a root of (2.18), then so is its conjugate $s=-\omega i$. We can express this by conjugating the complex equation in (2.18),

$$\mathbf{v}^*(-s\mathbf{I}-\mathbf{A}^T) = e^{\tau s}\mathbf{v}^*\mathbf{B}^T, \tag{2.19}$$

where $(\bullet)^*$ denotes the conjugate transpose of $(\bullet)$. One can now multiply (2.18) and (2.19) side by side to get

$$(s\,\mathbf{I}-\mathbf{A})\;\mathbf{V}\;(s\,\mathbf{I}+\mathbf{A}^T) \;=\; -\mathbf{B}\,\mathbf{V}\,\mathbf{B}^T \quad \text{with } \mathbf{V}^{(n\times n)} \;=\; \mathbf{v}\mathbf{v}^*. \tag{2.20}$$

Equation (2.20) is exactly in the form to be transformed by using ξ as defined in (2.16). It returns

$$\begin{aligned} &\{(s\,\mathbf{I} - \mathbf{A}) \otimes (s\,\mathbf{I} + \mathbf{A}) + \mathbf{B} \otimes \mathbf{B}\}\,\xi\mathbf{V} = 0 \\ &\Rightarrow \boldsymbol{\lambda}(s)\,\xi\mathbf{V} = 0, \end{aligned} \tag{2.21}$$

where $\boldsymbol{\lambda}(s)$ is evidently the matrix operation. For nontrivial solutions of $\xi\mathbf{V} \neq \mathbf{0}$, the only way to satisfy (2.21) is to set

$$\det \boldsymbol{\lambda}(s) = 0. \tag{2.22}$$

We can conclude that the desired imaginary roots are determined by solving the roots of a $2n^2$ degree polynomial (2.22). However, one should take notice that this root-finding algorithm for higher dimensions ($2n^2 > 10$) becomes numerically unreliable due to repeated round-off errors in the determinant expansion procedure unless the operation is performed using a very large number of significant digits. In that case, however, excessive computational cost will appear. In order to circumvent this difficulty using lower precision calculations, one can expand (2.21) using Kronecker product identities as defined in [22, 23]. The outcome of this is a matrix polynomial as follows:

$$\boldsymbol{\lambda}(s) = \mathbf{G}_0\, s^2 + \mathbf{G}_1\, s + \mathbf{G}_2, \tag{2.23}$$

where $\mathbf{G}_0 = \mathbf{I} \otimes \mathbf{I}$, $\mathbf{G}_1 = \mathbf{I} \otimes \mathbf{A} - \mathbf{A} \otimes \mathbf{I}$, $\mathbf{G}_2 = \mathbf{B} \otimes \mathbf{B} - \mathbf{A} \otimes \mathbf{A}$. Then this matrix polynomial can be linearized [24] by the fact that $\mathbf{G}_0$ is an invertible matrix. The linearized form of (2.23) is expressed as

$$\mathbf{F} = \begin{bmatrix} \mathbf{0} & \mathbf{T}_0 \\ -\mathbf{T}_2 & -\mathbf{T}_1 \end{bmatrix} \quad \text{with } \mathbf{T}_0 = \mathbf{I} \otimes \mathbf{I},\ \mathbf{T}_1 = \mathbf{G}_0^{-1}\mathbf{G}_1,\ \mathbf{T}_2 = \mathbf{G}_0^{-1}\mathbf{G}_2, \tag{2.24}$$

where the zeros of $\det(\boldsymbol{\lambda}(s)) = 0$ are the eigenvalues of $\mathbf{F}$, i.e., $\det(\boldsymbol{\lambda}(s)) = \det(s\mathbf{I} - \mathbf{F}) = 0$. The imaginary roots of dynamics (1.1) have to be among the eigenvalues of $\mathbf{F}$, which can be computed by many practical and efficient routines such as the *eig*($\mathbf{F}$) subroutine of MATLAB or *eigenvalues*($\mathbf{F}$) subroutine of Maple. Eventually, one can arrive at a numerically more reliable set of imaginary roots of dynamics (1.1), as demonstrated in the example section.

(e) Rekasius substitution as introduced in [14] and utilized by [6, 8, 25]. This critical procedure is an exact substitution of transcendental terms in (1.7) with

$$e^{-\tau s} = \frac{1 - Ts}{1 + Ts} \quad \text{when } s = \omega\, i \ \textit{only}, \tag{2.25}$$

where $T \in \Re$ and

$$\tau = \frac{2}{\omega}\,[\tan^{-1}(\omega\, T) \mp \ell\pi\,], \quad \ell = 0\,, 1\,, \ldots. \tag{2.26}$$

This exact substitution creates a new characteristic equation,

$$CE\,(s\,, T) = \sum_{k=0}^{p} a_k(s) \left(\frac{1 - T\,s}{1 + T\,s}\right)^k = 0. \tag{2.27}$$

Multiplying (2.27) by $(1+T\,s)^p$, one obtains

$$\sum_{k=0}^{p} a_k(s)\,(1+T\,s)^{p-k}(1-T\,s)^k \;=\; 0. \tag{2.28}$$

Considering that $a_k(s)$ are ordinary polynomials, (2.28) is nothing other than a polynomial in s with parameterized coefficients in T. Since the system in (1.1) is of retarded type, the highest degree term of s is n and it is in $a_0(s)$. Equation (2.28), therefore, contains a polynomial of s with degree $n+p$. The problem is to determine *all* $T \in \mathbf{R}$ values, which cause imaginary roots of $s = \omega\, i$ for this equation. This can be done by forming the Routh array of (2.28) and setting the only term in the s^1 row to zero [9, 10, 26]. It can be shown that this polynomial is of degree $n\,p$ in T, of which only the real roots are searched. Once these roots are determined, the corresponding crossing frequencies ($s = \omega\, i$) can be found using the auxiliary equation, which is formed by the s^2 row of the Routh array [9, 26]. Notice that the s^2 row has two terms, which are functions of T. They must agree in sign for those T values to yield imaginary roots. Final results are exhaustive in detecting all the imaginary characteristic roots we set out to solve, and their number is upper bounded by n^2 (again in support of Proposition 1).

We wish to discuss two limiting cases here. The first has to do with the extrema of the T domain, namely $T = \mp\infty$. It is easy to check whether these unbounded T values are of interest to us or not. That is, if $T \to \mp\infty$ and there is a finite imaginary root s, the corresponding $e^{-\tau s} \to -1$ (from (2.25)) leaves $CE\,(s\,,\,\tau\,)$ as a simple polynomial of s. The roots of this polynomial can be found easily. If any one of these roots is purely imaginary, that root becomes part of the solution of interest.

The second limiting case is for the imaginary root(s) appearing at the origin, $s = \omega\, i$ with $\omega = 0$, which makes $e^{-\tau s} = 1$ as per (2.25). Consequently one needs to check if

$$\sum_{k=0}^{p} a_k(0) = 0 \tag{2.29}$$

holds from (1.7). If it does, there should be at least one stationary root at $s = 0$, which remains there for all $\tau \in \mathbf{R}^+$. It is easy to determine the multiplicity of this root for some τ values simply by checking if the successive derivatives of (1.7) with respect to s are also zero for the same τ values. For instance, if $s = 0$ is a double root, the second root may cross over the imaginary axis, altering the stability of the system, as the stationary one remains fixed at the origin.

Both of these limiting cases can appear, regardless of the methods (a)–(e). That is, $e^{-\tau s} \to \pm 1$ can occur as a property of the system at hand.

3. A numerical case study and comparative observations. We now take an example case study to display a comparison among the five methodologies we discussed above. Consider the numerical example in [6] which has $p = \text{rank}(\mathbf{B}) = 3 = n$:

$$\mathbf{A} = \begin{pmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{pmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}. \tag{3.1}$$

The respective characteristic equation is

$$
\begin{aligned}
CE\,(s\,,\ \tau) \ &= \ s^3 + 6\,s^2 + 45.5\,s + 111 \ + \ (0.9\,s^2 - 116.8\,s - 22.1)\,e^{-\tau s} \\
&\quad + (90.9\,s - 185.1)\,e^{-2\tau s} + \ 119.4\,e^{-3\tau s} = 0,
\end{aligned} \tag{3.2}
$$

in which $a_k(s)$, $k = 0, \ldots, 3$, expressions are readily identified as represented in (1.7). For this system the following exhaustive list of (τ_{ker}, ω) is given in [6]. We cross-check this list, using a sufficiently high number of digits in performing the methodologies (a)–(e), and obtain Table 3.1.

TABLE 3.1
The fundamental delays and the imaginary spectra.

τ_{ker}	ω
7.208	0.8404
0.8725	2.1109
0.1859	2.9123
0.1624	3.0351
0.2219	15.5032

Here the notation $(\tau_{ker,j}, \omega_j)$ implies the minimum positive delay, $\tau_{ker,j}$, defined by (1.3) and the corresponding root at $\omega_j\, i$. In (3.2) we now start utilizing the five methodologies as described earlier. The main point of comparison is to be able to precisely declare the complete Table 3.1. Computational efficiency may also be considered as a basis for comparison, and we will address this point later in the text. In order to conserve space, the numerical results are given in truncated forms at the fourth decimal except where necessary for the arguments.

(a) Schur–Cohn procedure. $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ matrices of (2.5) are readily formed using the terms of $a_0(s), \ldots, a_3(s)$. Notice that the degree of $\det(\mathbf{A}_2)$ of (2.4) is $2np = 18$. Thus the mission is reduced to determining the purely imaginary roots of

$$
\begin{aligned}
\det(A_2) \ &= \ s^{18} + \ 10^{-8}\ s^{17} + 165.81\ \ s^{16} + 0.3\ \ 10^{-4}\, s^{15} - 18681.53\ s^{14} \\
&\quad + 0.022\ s^{13} - 248693.82\ \ s^{12} + \ 4.5\ s^{11} + 35141.36\ \ s^{10} + 330.83\ s^9 \\
&\quad + 0.41\ 10^{10}\ s^8 + 8671.27\ s^7 + \ 0.94\ \ 10^{11}\ \ s^6 + \ 72932.09\ s^5 + 0.71\ \ 10^{12}\ s^4 \\
&\quad + 148284.13\ s^3 + 0.19\ \ 10^{13} s^2 + 29249.72\ s\ \ + 0.1\ 10^{13} = 0
\end{aligned} \tag{3.3}
$$

which displays two major obstacles as follows:

(i) Equation (3.3) is expected to have only even powers of s. Due to the accumulated numerical error in the symbolic manipulations, one cannot achieve this even with 60-digit precision (for this exercise). The trial-based determination of the significant digits needed is an important hindrance. For comparison purposes, we consider the numerical errors at the level of up to 1/4 of significant digits as acceptable accuracy. For a 20-digit operation, any error of 10^{-15} or less is considered to be zero.

(ii) Evaluation of the imaginary roots of (3.3) results in the form of $s = \omega\, i + \varepsilon$, $\varepsilon << 1$, lending themselves to another question for the number of significant digits needed. For instance, using 20 *digits* of precision in Maple, (3.3) gives the following apparently imaginary roots:

$$
\begin{aligned}
&0.21\ 10^{-7} \mp 0.8404i, \quad 0.28\ 10^{-6} \mp 2.1109i, \quad -0.55\ 10^{-8} \mp 15.5032i, \\
&-0.13\ 10^{-5} \mp 2.9123i, \quad 0.69\ 10^{-6} \mp 3.0351i.
\end{aligned}
$$

As can be seen, the numerical errors in real parts (which are supposed to be zero) are at least of the order of 10 magnitudes larger than the computational precision, and they all violate the 1/4-digit rule. This point makes it necessary to adjust the accuracy in calculations.

An alternative but more reliable procedure along this line was described earlier over (2.6)–(2.8). Accordingly, the real eigenvalues of a matrix are evaluated instead of the roots of a polynomial. Using this procedure, one obtains the five crossing frequencies (which correspond to the real eigenvalues of $\mathbf{P}$) as $0.8404 - 0.44\ 10^{-17}\, i$, $2.1109{+}0.66\ 10^{-16}\, i$, $2.9123{+}0.63\ 10^{-15}\, i$, $3.0351{+}0.68\ 10^{-15}\, i$, $15.5032{+}0.72\ 10^{-17}\, i$, still with 20-digit precision. These numbers are all acceptably close to the desired results under the proposed 1/4-digit rule.

(b) Elimination of transcendental terms. Starting from the characteristic equation (3.2) and following the three steps as described in section 2(b), one can sequentially eliminate $e^{-3\,\tau\, s}$, $e^{-2\,\tau\, s}$, and $e^{-\tau\, s}$. Notice that these steps preserve the purely imaginary roots of the characteristic equation. The final form should contain only even powers of s:

$$\begin{aligned} CE_p(s) &= s^{24} + 220.81\ s^{22} - 8823.73\ s^{20} - 0.12\ 10^{7}\ s^{18} - 0.27\ 10^{8}\ s^{16} \\ &+ 0.39\ 10^{10}\ s^{14} + 0.32\ 10^{12}\ s^{12} + 0.89\ 10^{13}\ s^{10} + 0.12\ 10^{15}\ s^{8} \\ &+ 0.81\ 10^{15}\ s^{6} + 0.28\ 10^{16}\ s^{4} + 0.44\ 10^{16}\ s^{2} + 0.2\ 10^{16} = 0. \end{aligned} \tag{3.4}$$

The imaginary roots of the original system have to be among the roots of this polynomial. We again use 20 significant digits of precision in Maple during the symbolic manipulations. The same level of precision is used in the following steps of root finding. (Cautionary note: Even for the case of up to 18 digits of precision in Maple, one can still observe odd powered terms appearing in (3.4).)

Equation (3.4) results in 24 symmetric roots with respect to the origin, of which only the purely imaginary ones are important. They are

$$\begin{array}{llll} \mp 5.9977i, & \mp 3.9460i, & \mp 1.8587i, & \mp 0.8404i, \\ \mp 2.1109i, & \mp 2.9123i, & \mp 3.0351i, & \mp 15.5032i. \end{array} \tag{3.5}$$

It is easy to demonstrate that the first three roots on the first row are faulty findings and do not represent true crossings. To see that, one can substitute them into the original characteristic equation (3.2) just to show that they do not satisfy this equation. On the other hand, the remaining five roots do. They also yield $|\, e^{-\tau\,\omega\, i}\,| = 1$ from which one can determine the respective time delays.[2]

One can also observe that the difference between the degrees of (3.3) and (3.4) is 6. In fact the relation between the two polynomials is described as

$$CE_p(s,\ \tau)\ =\ P_1(s)\ \det(\mathbf{A}_2), \tag{3.6}$$

where the degree $(P_1) = 6$, and its roots result in the three false pairs of imaginary roots for the specific numerical example, as mentioned above.

[2]Incidentally, the procedure described in [5, eq. (48)] yields incorrect time delays due to the accumulation of numerical error. Primarily an ill conditioning occurs due to some coefficients in (2.11) being 15–16 orders of magnitude apart from one another.

(c) Matrix pencil–Kronecker sum application. Deploying (2.14) we obtain

$$
\begin{aligned}
\det\left[(\mathbf{A}+\mathbf{B}\,z)\oplus(\mathbf{A}+\mathbf{B}\,z^{-1})\right] \;=\; & (0.17\;10^{7}\;z^{18}-141137.96\;z^{17}\\
& +0.23\;10^{8}\;z^{16}-0.17\;10^{9}\;z^{15}+0.52\;10^{9}\;z^{14}-0.81\;10^{9}\;z^{13}+0.62\;10^{9}\;z^{12}\\
& -0.32\;10^{8}\;z^{11}-0.43\;10^{9}\;z^{10}+0.56\;10^{9}\;z^{9}-0.43\;10^{9}\;z^{8}-0.32\;10^{8}\;z^{7}\\
& +0.62\;10^{9}\;z^{6}-0.81\;10^{9}\;z^{5}+0.52\;10^{9}\;z^{4}-0.17\;10^{9}\;z^{3}+0.22\;10^{8}\;z^{2}\\
& -141137.96\;z+0.17\;10^{7})/z^{9}=0.
\end{aligned}
\tag{3.7}
$$

We are seeking the roots of (3.7) with magnitude equal to 1. There is a major difficulty, specifically in deciding the tolerance level of $|\,z\,| = 1$. This difficulty disappears, however, when higher precision (above 20 digits) is used. For example, with 20 digits of precision, the errors in the magnitudes of the five roots become precisely zero. They are listed in Table 3.2, together with their corresponding five imaginary roots, which are obtained by replacing $z = e^{-\tau\,\omega\,i}$, $|\,z\,| = 1$, in (3.2) and solving for ω.

TABLE 3.2
The tolerances in evaluating z's and the imaginary spectra.

$\lvert 1-z\rvert$	Imaginary spectra
0	$-0.57\;10^{-19}+0.8404i$
0	$0.47\;10^{-20}+2.1109i$
0	$-0.73\;10^{-19}+2.9123i$
0	$0.5\;10^{-19}+3.0352i$
0	$-10^{-21}+15.5032i$

Notice that the cumulative effects of numerical error in imaginary spectra are all acceptable in 1/4-digit sense. However, with up to 13 digits of precision, one observes relatively large accumulated errors (violating the 1/4-digit rule) in the computation of imaginary roots. Therefore if a root of z displays $|\,z\,| - 1 = \varepsilon$, it is quite difficult to assess whether we should take it as an indicator of crossing or not. Increasing the digits higher than 14 (obviously including 20) this concern disappears for this case study.

When the more reliable procedure is followed, as mentioned in section 2(c), $\mathbf{U}$ and $\mathbf{V}$ matrices are trivially obtained from their definitions. The generalized eigenvalues of the $(\mathbf{U},\mathbf{V})$ pair, which have unitary magnitudes (obtained with 20-digit precision again) are listed in Table 3.3.

TABLE 3.3
The tolerances in evaluating z's and the corresponding imaginary spectra.

$\lvert 1-z\rvert$	Imaginary spectra
$-0.14\;10^{-17}$	$-0.25\;10^{-17}\mp 0.8404i$
$-0.34\;10^{-17}$	$0.27\;10^{-17}\mp 2.1109i$
$0.12\;10^{-16}$	$0.18\;10^{-16}\mp 2.9123i$
$-0.78\;10^{-17}$	$0.31\;10^{-16}\mp 3.0351i$
$0.4\;10^{-18}$	$-0.95\;10^{-18}\mp 15.5032i$

Interestingly, on this implementation one can use 9-digit precision and can still obtain acceptable errors looking at the real parts of the imaginary roots, as shown in Table 3.4. This table indicates that the generalized eigenvalue computation can be

performed with a much smaller number of digits, such as 9, without sacrificing the accuracy of the roots.

TABLE 3.4
The tolerances in evaluating z's and the imaginary spectra.

$\lvert 1-z\rvert$	Imaginary spectra
0	$-0.15\ 10^{-7} \mp 0.8404i$
-10^{-8}	$0.66\ 10^{-8} \mp 2.1109i$
-10^{-8}	$0.4\ 10^{-7} \mp 2.9123i$
$-0.18\ 10^{-7}$	$-0.42\ 10^{-7} \mp 3.0351i$
0	$0.27\ 10^{-7} \mp 15.5032i$

(d) Kronecker multiplication/elementary transformation method. Following the procedure given in section 2(d), one obtains (2.22), again with 20-digit precision, as

$$\begin{aligned}\det\ \boldsymbol{\lambda}(s) \ =\ & s^{18} + 0.3\ 10^{-9}\ s^{17} + 165.81\ s^{16} + 0.99\ 10^{-6}\ s^{15} - 18681.53\ s^{14} \\ & + 0.0007\ s^{13} - 248693.48\ s^{12} + 0.16\ s^{11} + 35183.49\ s^{10} + 12.97\ s^{9} \\ & + 0.41\ 10^{10}\ s^{8} + 370.2\ s^{7} + 0.94\ 10^{11}\ s^{6} + 3348.51\ s^{5} + 0.71\ 10^{12}\ s^{4} \\ & + 7209.8\ s^{3} + 0.19\ 10^{13}\ s^{2} + 1480\ s + 0.1\ 10^{13} = 0.\end{aligned} \tag{3.8}$$

Purely imaginary roots of this equation are the expected crossing frequencies as given in Table 3.5. With this precision level, the accumulated errors in the real parts are relatively large (in the 1/4-digit sense) in magnitude as can be seen in this table. As mentioned in section 2(d), in order to improve the accuracy of the imaginary roots of (1.1), we pursue computing the eigenvalues of matrix $\mathbf{F}$, eig($\mathbf{F}$). For this computation, 10- and 20-digit precision are used separately. The results are given in Table 3.6, with the error terms in the real parts being acceptably small. Thus we conclude that this method can produce the imaginary roots accurately even with 10-digit operations.

TABLE 3.5

Imaginary spectra
$0.1\ 10^{-8} \mp 0.8404i$
$0.13\ 10^{-7} \mp 2.1109i$
$-0.82\ 10^{-7} \mp 2.9123i$
$0.53\ 10^{-7} \mp 3.0351i$
$-0.1\ 10^{-9} \mp 15.5032i$

TABLE 3.6
Imaginary spectra.

10-digit precision	20-digit precision
$0.8847\ 10^{-15} \mp 0.8404i$	$0.88\ 10^{-18} \mp 0.8404i$
$0.111\ 10^{-13} \mp 2.1109i$	$-0.1\ 10^{-18} \mp 2.1109i$
$-0.1856\ 10^{-12} \mp 2.9123i$	$0.19\ 10^{-16} \mp 2.9123i$
$0.172\ 10^{-12} \mp 3.0351i$	$-0.19\ 10^{-16} \mp 3.0352i$
$-0.355\ 10^{-14} \mp 15.5032i$	$0.66\ 10^{-17} \mp 15.5032i$

(e) Rekasius substitution. In this procedure, at least 13 digits of precision are required in order to obtain the crossing frequencies as listed in the beginning of this section. The characteristic equation of (3.2) is rewritten after the Rekasius transformation as

$$\begin{aligned} CE(s,T) \;=\; & T^3 s^6 + (5.1\,T^3 + 3\,T^2)\,s^5 + (253.2\,T^3 + 17.1\,T^2 + 3\,T)\,s^4 \\ & + (-171.4\,T^3 + 162.4\,T^2 + 18.9\,T + 1)\,s^3 \\ & + (898.4\,T^2 - 71.2\,T + 6.9)\,s^2 + (137.8\,T + 19.6)\,s + 23.2 = 0 \end{aligned} \tag{3.9}$$

for which we need all $T \in \Re$ values yielding the imaginary roots. The numerator of the only term of the s^1 row of the respective Routh array in (3.9) is set to zero for this:

$$\begin{aligned} & T^3\,(0.4004\;10^7\,T^9 - 0.5418\;10^6\;T^8 - 0.1060\;10^7\;T^7 - 0.7869\;10^5\;T^6 \\ & \quad - 0.1501\;10^5\;T^5 + 0.1216\;10^4\;T^4 + 0.4011\;10^3\;T^3 \\ & \quad - 0.1025\;10^2\,T^2 + 0.1197\;T - 0.1120) = 0. \end{aligned} \tag{3.10}$$

Notice that we are seeking the real roots of (3.10), and three of them are at $T = 0$. The remaining roots, i.e., 9 ($= n^2$) of them, have to be solved. Out of these nine roots only five happen to be real and all five satisfy the sign agreement condition in the s^2 row of the Routh array (see the note in section 2(e)). They concur with the results of [6] which uses a slightly different procedure to arrive at these T values. An auxiliary equation is formed by the terms of the s^2 row generating the crossing frequencies, ω. The T versus ω correspondence is shown in (3.11):

$$[T] = \begin{bmatrix} -0.4269 \\ -0.1332 \\ 0.0828 \\ 0.0952 \\ 0.6232 \end{bmatrix} \rightarrow [\omega] = \begin{bmatrix} 15.5032 \\ 0.8404 \\ 3.0351 \\ 2.9123 \\ 2.1109 \end{bmatrix}. \tag{3.11}$$

Note that by using T and ω, one can also find the respective delays τ as per (2.26).

4. Comparative comments and conclusions. The five procedures described above reduce the problem at hand to the following solutions:

(a) Schur–Cohn: Roots of the $2n^2$ degree complex polynomial are solved for purely imaginary roots.
(b) Elimination of transcendental terms: The $n\,2^n$ degree real polynomial is solved for imaginary roots.
(c) Kronecker sum: The $2n^2$ degree polynomial is solved for complex roots with $|\,z\,| = 1$.
(d) Kronecker multiplication: The $2n^2$ degree polynomial is solved for imaginary roots.
(e) Rekasius: The n^2 degree polynomial is solved for real roots ($T \in \mathbf{R}$).

From the perspective of developmental steps, the least involved path appears to be in (e). Nevertheless, utilizing the appropriate matrix operations, methods (a), (c), and (d) prove to be equally potent, producing the desired results. In fact for the example case, method (d) generates the results with the smallest number of significant digits among the five. Method (b) is the one that requires special attention, avoiding the false solutions. As the dimension n increases, method (b) becomes more problematic to process, leaving the other four methods as plausible paths to follow.

We wish to mention two attractive features of method (e): First, the degree of the polynomial in question is considerably smaller than all the other four. Second, we seek the real roots (T) only, not the complex ones. This is a great relief from a numerical perspective, as complex (particularly purely imaginary) roots are very hard to detect when numerical errors creep in. A similar hardship appears when the test of $|z| = 1$ is performed in method (c).

We should also make note of the fact that methods (b) and (e) have a symbolic segment in the deployment, as the remaining three methods are numerical. Upon the request of one of the reviewers, we performed a study of efficiency on all of the methods (i.e., the CPU times consumed) including the symbolic as well as numerical operations. These CPU costs did not show noteworthy variations from one method to another. For instance, the tabulations given in section 3 for all five methods were obtained within 1 sec. on a PC with an Intel Centrino 1.6 MHz processor with 512 MB RAM. It is clear that this efficiency measure is not as important as the accuracy of the findings unless the operation is repeated a large number of times for some parametric studies. But this subject is outside the scope of the present paper.

It is also noteworthy that the authors are engaged in expanding the procedures described here to systems with multiple independent delays. In fact, [10] is a document which utilizes procedure (e) for systems with two independent delays. Several other studies on the deployment of procedure (c) for multiple delay cases are also in progress.

Appendix. Let $\mathbf{A} = [a_{ij}] \in \mathbf{R}^{k \times \ell}$, $\mathbf{B} = [b_{ij}] \in \mathbf{R}^{m \times n}$. Then the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \otimes \mathbf{B}$, is defined as follows:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & & \vdots \\ a_{n1}\mathbf{B} & \cdots & a_{nn}\mathbf{B} \end{bmatrix} \in \mathbf{R}^{km \times \ell n}. \tag{A.1}$$

If $k = \ell$ and $m = n$, then the Kronecker sum of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \oplus \mathbf{B}$, is defined by

$$\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I}_m + \mathbf{I}_k \otimes \mathbf{B} \in \mathbf{R}^{km \times km}. \tag{A.2}$$

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