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A note on Taylor's stability rule

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Abstract. Analyzing the stability of the interface between two fluids of different densities which is accelerating normal to its surface, Taylor concluded a rule that "this surface is stable or unstable according to whether the acceleration is directed from the heavier to the lighter fluid or vice versa." We show by following Taylor's analysis that in the range where the acceleration of the interface is in the direction of the gravitational acceleration and its magnitude is smaller, this rule reverses direction. The interface is stable or unstable according to whether the acceleration is from the lighter to the heavier fluid or vice versa.

1. Introduction

Taylor [1] analyzes the stability of an infinite material interface between two inviscid fluids which is accelerated in the direction normal to its surface. Fixing coordinates at the accelerating interface and applying potential theory, he derives the complex frequency of the perturbation surface from which he concludes the rule that the surface is stable or unstable according to whether the acceleration is directed from the heavier to the lighter fluid or vice versa. Sharp [3] infers the onset of Taylor instability by reasoning from a thought experiment with the conclusion that: "If the heavy fluid pushes the light fluid, the interface is stable. If the light fluid pushes the heavy fluid, the interface is unstable". He then proceeds to comment that this criterion is among the most basic principles in the subject of interface instability. In analyzing the inequalities that follow directly from the complex frequency derived by Taylor, we show that when the acceleration normal to the interface is in the direction of the gravitational acceleration and its magnitude is smaller, the Taylor stability rule no longer applies – it reverses direction. The interface is stable or unstable according to whether the acceleration is from the lighter to the heavier fluid or vice versa.

Accordingly, when a liquid in a large open container is placed in an elevator descending at an acceleration smaller than the gravitational acceleration, the interface is stable although the direction of the acceleration of the air-liquid interface is from the lighter to the heavier fluid. When a raindrop falls from rest, its velocity increases, its drag, due to atmospheric friction, also increases, and consequently its acceleration becomes smaller than the gravitational acceleration. If the drop is large so that surface tension is negligible, the interface at its blunt leading edge is unstable, although the heavy fluid (water) is pushing the lighter fluid (air). The instability of raindrops was observed experimentally by Philipp Lenard as early as 1894.

The experimental results of Lewis [2], confirm Taylor's theory for accelerations which are larger than the gravitational acceleration, a range in which Taylor's stability rule applies. Neither in Sharp's survey [3] nor in the more recent book by Joseph et al.[4] has the exception to Taylor's rule presented in this note been considered.

2. Analysis

Taylor analyzes the stability of the infinite interface separating two superposed fluids of constant density ρ_1 and ρ_2, where ρ_1 occupies the positive half space, ρ_2 the negative half space, and the gravitational acceleration is directed from ρ_1 to ρ_2. The interface is given an acceleration a normal to its surface which is positive when directed from ρ_2 to ρ_1; and, a small perturbation of a wave number K. Applying potential theory and using coordinates fixed at the interface Taylor arrives at the following equation, (Eq. (5) in Taylor[1]):

$$n^2 = -K\left(g+a\right)\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \qquad (1)$$

where n is the complex frequency of the perturbation surface and g the magnitude of the gravitational acceleration.

It follows from Eq. (1) that the interface is stable when

$$\left(\rho_2 - \rho_1\right)(g+a) > 0 \qquad (2)$$

and unstable when

$$\left(\rho_2 - \rho_1\right)(g+a) < 0 \qquad (3)$$

Suppose $a > 0$. The acceleration of the interface is from ρ_2 to ρ_1 and according to inequalities (2) and (3) the interface is stable when $\rho_1 < \rho_2$ and unstable when $\rho_1 > \rho_2$; which is in accordance with Taylor's rule.

Suppose $a < 0$ and $|a| > g$. The acceleration is from ρ_1 to ρ_2 and according to inequalities (2) and (3) the interface is stable if $\rho_1 > \rho_2$ and unstable if $\rho_1 < \rho_2$, which is again in accordance with Taylor's rule.

Now suppose $a < 0$ and $|a| < g$. The acceleration is from ρ_1 to ρ_2 and according to inequalities (2) and (3) the interface is stable if $\rho_1 < \rho_2$ and unstable if $\rho_1 > \rho_2$, which is contrary to Taylor's rule.

An alternative intuitive derivation of Eqs. (2) and (3) can be obtained as follows: Fixing coordinates at the interface which is moving at an acceleration a normal to its surface, renders the interface at static equilibrium in D'Alembert's Principle sense. Accordingly, the pressure distributions normal to the interface are given in the positive half space by

$$p_1 = p_0 - \rho_1\left(g+a\right)y \qquad (4)$$

and in the negative half space by

$$p_2 = p_0 - \rho_2\left(g+a\right)y \qquad (5)$$

where p_0 is the pressure which is continuous across the interface and y is the coordinate measured normal to the interface .

Consider a fluid particle of volume ΔV which has been displaced from the negative side of the interface to the positive side. The body force on this volume is given by

$$F_y^g = -\rho_2\left(g+a\right)\Delta V \quad , \qquad (6)$$

the net pressure force acting on this volume as computed from the pressure distribution in the positive half space, Eq. (4), is given by

$$F_y^p = \rho_1 (g+a) \Delta V \ , \tag{7}$$

and thus, the net force acting on the fluid volume element is

$$F_{net} = (\rho_1 - \rho_2)(g+a)\Delta V \ . \tag{8}$$

Now, the interface is stable if the net force tends to return the particle back towards the interface i.e., $F_{net} < 0$; and, unstable if the net force tends to drive the particle away from the interface i.e., $F_{net} > 0$. Accordingly, we obtain that the interface is stable if

$$F_{net} = (\rho_1 - \rho_2)(g+a)\Delta V < 0 \tag{9}$$

and unstable if

$$F_{net} = (\rho_1 - \rho_2)(g+a)\Delta V > 0 \ . \tag{10}$$

Comparing Eqs. (9) and (10) with Eqs. (2) and (3) shows that they are equivalent.

As can be seen from Eqs. (2) and (3) the criteria of whether the interface is stable or unstable is independent of the wave number K. Taylor's linear stability analysis provides, in addition to the stability criteria, information regarding the instability growth-rate which is dependent on the wave number as given by Eq. (1) when n is real. As shown by Lewis [2], the theory holds well at the early stage of the growth of the instability.

We have thus shown that when the acceleration of the interface is in the range $-g < a < 0$, Taylor stability rule reverses and the interface is stable or unstable according to whether the acceleration is from the lighter to the heavier fluid or vice versa.

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