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# Specification and formal verification of fuzzy information processing for the case of edge detection

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# SPECIFICATION AND FORMAL VERIFICATION OF FUZZY INFORMATION PROCESSING FOR THE CASE OF EDGE DETECTION

A Thesis Presented

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# Abstract

This thesis explores how an information fusion system can be verified before it is built. Towards this aim, a system is first specified mathematically in a formal language. The specification then can be analyzed by formulating various conjectures (theorems) and proving (or disproving) them using an automatic theorem prover. In our work we use Metaslang as the formal specification language and Specware, which is a tool that supports category theory based algebraic specification of software. One of the most important aspects in information fusion is the uncertainty of the fused decisions that are based on uncertain sources. Out of many possible uncertainty types we focus on fuzzy set theory. We have specified a library of fuzzy set theory that can be used to build specifications of fuzzy information processing systems. As an example, a (fuzzy) edge detection algorithm was implemented and then verified using two theorem provers - Snark and Isabelle. With the help of this approach, reasoning about the impact of the uncertainty of input information is specified formally in every step of the fusion process and can be verified before the system is coded in a programming language.

# Acknowledgements

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# Contents

# List of Figures

# List of Tables

# Chapter 1

# INTRODUCTION

## 1.1 Introduction

Among the various paradigmatic changes in science and mathematics in this century, one such change concerns the concept of uncertainty. In science, this change has been manifested by a gradual transition from the traditional view, which insists that uncertainty is undesirable in science and should be avoided by all possible means, to an alternative view, which is tolerant of uncertainty and insists that science cannot avoid it.

According to the traditional view, science should strive for certainty in all manifestations (precision, specificity, sharpness, consistency, etc.); hence, uncertainty (imprecision, nonspecificity, vagueness, inconsistency, etc.) is regarded as unscientific. According to the alternative (modern) view, uncertainty is considered essential to science; it is not only an unavoidable plague, but it has, in fact, a great utility [1].

In order to develop an information fusion system, it is required to deal with uncertainty in a formal way. Formal verification tools, such as theorem provers and software development systems, help to build specifications for an information fusion system and verify their correctness mathematically before they are built.

Specware is a formal specification tool and an automated software development system [12] developed by Kestrel Institute. This tool is used in this thesis. It is known that building correct and clear specifications is difficult. But Specware can help by exploiting Category Theory, which includes algebraic theories, and allowing for building specifications progressively from smaller to larger components.

SNARK, is an automated theorem-proving program developed in Common Lisp [16], and Isabelle is a generic proof assistant that allows mathematical formulas to be expressed in a formal language and provides support for proving those formulas in a logical calculus [15], Isabelle is also used in this thesis. It will be demonstrated that a specification in Specware can be translated to Isabelle using the Specware interface and verified using some proving capabilities of Isabelle.

In this thesis, fuzzy set theory is used to handle uncertainty. So fuzzy information fusion system specifications are built. Fuzzy set, fuzzy number, fuzzy arithmetic operations, and various fuzzification methods are specified and stored in a specification library. The library is then used to build an edge detection algorithm. Proof obligations are then extracted and proved using both Isabelle and SNARK.

### Why Is Fuzzy Set Used?

From the beginning of modern science until the end of the nineteenth century, uncertainty was generally viewed as undesirable in science and the idea was to avoid it. This attitude gradually changed with the emergence of statistical mechanics at the beginning of the twentieth century. To deal with the unmanageable complexity of mechanical processes on the molecular level, statistical mechanics resorted to the use of statistical mechanics, probability theory has been successfully applied in many other areas of science. However, in spite of

its success, probability theory is not capable of capturing uncertainty in all its manifestations [2]. So these limitations are part of the reason why fuzzy sets are used in this thesis.

## 1.2 Thesis Overview

The thesis is organized as follows. In Chapter 2, necessary aspects of category theory are presented. In Chapter 3, the tool Specware and the Metaslang language are explained. In Chapter 4, theorem prover Isabelle is introduced. In Chapter 5, the setup of Specware environment is presented. In Chapter 6, fuzzy sets and fuzzy reasoning operators are explained. In Chapter 7, fuzzy system modeling and an example - fuzzy edge detection - are presented. Conclusions and future works are proposed in Chapter 8.

# Chapter 2

# CATEGORY THEORY

## 2.1 Introduction

Category theory is a relatively young branch of mathematics, stemming from algebraic topology, and designed to describe various structural concepts from different mathematical fields in a uniform way. Indeed, category theory provides a bag of concepts (and theorems about those concepts) that form an abstraction of many concrete concepts in diverse branches of mathematics, including computing science. Hence it will come as no surprise that the concepts of category theory form an abstraction of many concepts that play a role in algorithmics [6]. So category theory is gracefully concise and simple language for describing information fusion systems.

Followings include some definitions that describe Category Theory and taken from [5]

## 2.2 Definition Of Category

A category $C$ of

- a collection of $Obj$ called $C$-objects;
- a collection of $Arw$ called $C$-arrows;

• operations assigning to each C-arrows $f$ a $C$-object $dom\ f$ (the domain of $f$) and a $C$-object $cod\ f$ (the "codomain " of f). If $a = dom\ f$ and $b = cod\ f$ this is displayed as

$$f : a \rightarrow b \text{ or } a \xrightarrow{f} b$$

•an operation, " $\circ$ ", called composition, assigning to each pair $\langle \mathsf{g}, \mathsf{f} \rangle$ of $C$-arrows with $dom\ g = cod\ f$, a $C-$arrow $g \circ f : dom\ f \rightarrow cod\ g$, the composite of $f$ and $g$ such that the Associative Law holds : Given the configuration

$$\begin{aligned} a \quad &\rightarrow \quad b \rightarrow c \rightarrow d \\ & \qquad \text{of } C - \text{objects and } C - \text{arrows, then} \\ h \circ (g \circ f) \quad &= \quad (h \circ g) \circ f \end{aligned}$$

• an assignment to each $C$-object, $b$, a $C$-arrow, $id_{b:b \rightarrow b}$, called the identity arrow on $b$, such that the Identity Law holds : For any $C$-arrows $f : a \rightarrow b$ and $g : b \rightarrow c$

$$id_b \circ f = f \quad and \quad g \circ id_b = g$$

## 2.3 The Category Of Signatures

In algebraic specifications, the structure of a specification is defined in terms of an abstract collection of values, called sorts and operations over those sorts. This structure is called a signature. A signature describes the structure that an implementation must have to satisfy the associated specification; however, a signature does not specify the semantics of the specification. The semantics are added later via axioms.

### 2.3.1 Signature

A signature $\Sigma = \langle S, \Omega \rangle$, consists of a set $S$ of sorts and a set $\Omega$ of operation symbols, defined over $S$. Associated with each operation's symbol is a sequence of sorts called its rank. For example, $f : s_1, s_2, ..., s_n \rightarrow s$ indicates that $f$ is the name of an n-ary function, taking arguments of sorts $s_1, s_2, ..., s_n$ and producing a result of sort $s$. A nullary operation symbol, $c :\rightarrow s$ is called a constant of sort $s$.

For signatures, the $C$-arrows are called *signature morphisms*. Signatures and their associated signature morphisms form the category, Sign.

### 2.3.2 Signature Morphism

Given two signatures $\Sigma = \langle S, \Omega \rangle$ and $\acute{\Sigma} = \langle \acute{S}, \acute{\Omega} \rangle$ a signature morphism $\sigma : \Sigma \rightarrow \acute{\Sigma}$ is a pair of functions $\langle \sigma_s : S \rightarrow \acute{S}, \sigma_\Omega : \Omega \rightarrow \acute{\Omega} \rangle$, mapping sorts to sorts and operations to operations such that the sort map is compatible with the ranks of the operations, i.e., for all operation symbols $f : s_1, s_2, ..., s_n \rightarrow s$ in $\Omega$ the operation symbol $\sigma_\Omega(f) : \sigma_s(s_1), \sigma_s(s_2), ..., \sigma_s(s_n) \rightarrow \sigma_s(s)$ is in $\acute{\Omega}$. The composition of two signature morphisms, obtained by composing the functions comprising the signature morphisms, is also a signature morphism. The identity signature morphism on a signature maps each sort and each operation onto itself. Signatures and signature morphisms form a category, Sign, where the signatures are the $C$-objects and the signature morphisms are the $C$-arrows.

## 2.4 The Category of Specifications

To model the semantics, signatures are extended with axioms that define the intended semantics of the signature operations. A signature with associated axioms is called a specification.

### 2.4.1 Specification

*A specification $SP$ is a pair $\langle \Sigma, \Omega \rangle$ consisting of a signature $\Sigma = \langle S, \Phi \rangle$ and a collection $\Phi$ of $\Sigma$-sentences (axioms).*

A specification only defines the semantics required of a valid implementation. In fact, for most specifications, there are a number of implementations that satisfy the specification.

As signatures have signature morphisms, specifications also have specification morphisms.

### 2.4.2 Specification Morphisms

Specification morphisms are signature morphisms that ensure that the axioms in the source specification are theorems (are provable from the axioms) in the target specification. Showing that the axioms of the source specification are theorems in the target specification is a proof obligation that must be shown for each specification morphism. Specification and specification morphisms enable the creation and modification of specifications that correspond to valid signatures within the category Sign.

*A specification morphism from a specification $SP = \langle \Sigma, \Phi \rangle$ to a specification $SP' = \langle \Sigma', \Phi' \rangle$ is a signature morphism $\sigma.\Sigma \rightarrow \Sigma'$ such that for every model $M \in Mod[SP']$, $M|_\sigma \in Mod[SP]$. The specification morphism is also denoted by the same symbol, $\sigma{:}\Sigma \rightarrow \Sigma'$*

## 2.5 Diagrams

A diagram in a category $C$ is a collection of vertices and directed edges, consistently labeled with objects and arrows of $C$, where "consistently" means that if an edge in the diagram is labeled with an arrow and has domain $A$ and codomain $B$, then the endpoints of this edge must be labeled with $A$ and $B$.

Diagrams are often used for stating and proving properties of categorical constructions. Such properties can often be expressed by saying that a particular diagram commutes.

# Chapter 3

# SPECWARE OVERVIEW

## 3.1 What is Specware?

It is a tool for building and manipulating a collection of related specifications. Specware can be considered [13] :

•*a design tool*, because it can represent and manipulate designs for complex systems, software or otherwise

•*a logic*, because it can describe concepts in a formal language with rules of deduction

•*a programming language*, because it can express programs and their properties

•*a database*, because it can store and manipulate collections of concepts, facts, and relationships

It can be used to develop domain theories, develop code from specifications and develop specifications from code [13].

Specware is designed with the idea that large and complex problems can be specified by combining small and simple specifications. The problem specifications may be further refined into a working system by the controlled stepwise introduction of implementation design decisions, in such a way that the refined specifications and ultimately the working code is a provably correct refinement

of the original problem specification [13]

Specware uses notions and procedures based on category theory and related mathematics to manipulate specifications [10]. The advantage of category theory as the foundation of Specware is that it enables the production of a well-defined stepwise refinement from an abstract specification to concrete implementation. Specification morphisms preserve the structure of one specification through the translation to another specification and preserve theorems across the specifications [11].

Specware produces logical inference using external theorem provers such as; SRI's Snark and Isabelle. External provers are connected to Specware through logic morphisms, which relate logics to each other. While Snark is an automated theorem prover, Isabelle is more interactive theorem prover that user can move forward step by step.

For this project, Specware 4.2.5 version was used.

## 3.2 MetaSlang

MetaSlang is the specification language of Specware. It is a version of high-order logic like PVS[1] and HOL[2].

MetaSlang is a functional language which is valuable for proving properties regarding functions. It includes syntactic constituents for describing functional semantics within a specification as well as constructs for describing composition, refinement, code generation, and proof capabilities. Specification constituents include types, expressions, and axioms which can be used to describe domain-specific formalisms [14]. All about MetaSlang in this section is a basic explanation. The *Specware Language Manual* contains a detailed

[1]PVS is a specification language integrated with support tools and a theorem prover. For more information: http://pvs.csl.sri.com/

[2]HOL is a programming environment in which theorems can be proved and proof tools implemented. For more information http://hol.sourceforge.net/

description of the MetaSlang grammar, including a BNF description.

### 3.2.1 Specs

A specification is a finite presentation of a theory in higher-order logic [11]. Specifications, or specs, provide the means to describe abstract concepts of the problem domain. A basic specification consists of a set of specification constituents. The first one is *types*, which indicate a set of values syntacticcally. The second element is *operations* (or *ops*, for short), which denote an element of the set denoted by its type [13]. The last is *axioms* and *definitions*, which define actions of types and operations. A spec can be expanded by importing other specs. This process can be useful for more complicated problems. Specs are also the objects used in morphisms which define the *part-of* or *is-a* relationships between two specs. Morphisms allow for refinement of specs and provide the utility to take simple abstract specifications and refine them to more concrete, complex specifications [14].

```
FuzzySet = spec
declaration
...
endspec
```

Figure 3.1: Sample spec.

### 3.2.2 Types

A type is a syntactic entity that denotes a set of values. Types are collections or sets of objects and expressions that characterize those objects. Specware has several built-in types such as Boolean, Integer, etc. These can be used in specifications directly. They are imported automatically for every spec by

Specware. Besides the built-in types, users can define new types or combine existing types to build more complex types. Some examples are shown in Figure 3.2.

```
type Nz = {i:Nat | i ~= 0}
type Image = Integer * Integer -> Nat
type Fuzzy_set a = a -> Uni_intvl
```

Figure 3.2: Some type declarations.

### 3.2.3 Operations(ops) and Definitions(defs)

An operation or op is a syntactic symbol accompanied by a type. An op is an element of the set that is denoted by its type. A specification has a number of built-in operations, such as logical connectives.

Ops can be polymorphic. While a monomorphic op denotes an element of the set denoted by the type of the op, a polymorphic op denotes a function that, given a set for each parameter type of the polymorphic type of the op, returns an element of the set obtained by applying to such parameter sets the function denoted by the type of the op [19].

```
% Monomorphic op
op delta :  Nz
% Polymorphic op
op inf :  Intvl -> Nat
op fequal :  Fuzzy_number * Fuzzy_number -> Uni_intvl
```

Figure 3.3: Some op declarations.

The behavior and constraint of an op can be shown by definitions. If definition is used for an op it must match the signature of the op declaration.

Also there is a possibility to combine defs and ops using a construct. Examples show that we can use defs and ops in a combined way.

```
% Op declaration
op [a] fuzzy_complement : Fuzzy_set a -> Fuzzy_set a
% Op definition
def [a] fuzzy_complement(f1) = fn d -> 1 - f1(d)
```

Figure 3.4: Op declaration and definition (separate).

```
% Op declaration and definiton
op min (x1: Nat, x2: Nat): Nat =
if x1 <= x2 then x1 else x2
```

Figure 3.5: Op declaration and definition (combined).

An op definition can be considered as a special notation for an axiom. However defining ops axiomatically may not be always a good idea since an axiom is automatically assumed to be true and has no obligations while defs have proof obligations associated with them. So it is encouraged to use defs as much as possible.

```
% Op declaration
op uni_min : Uni_intvl * Uni_intvl -> Uni_intvl
% Axiomatically op definition
axiom uni_min_def is
fa(u1: Uni_intvl, u2: Uni_intvl)
uni_min(u1,u2) = (if u1 <= u2 then u1 else u2)
```

Figure 3.6: Axiomatically op definiton.

### 3.2.4 Claims: Axioms, Conjectures, and Theorems

There are three kinds of claims that are usable in Speceware: axioms, conjectures, and theorems. Axioms, conjectures and theorems are formulas in a specification that use types and ops. They are a term of type Boolean.

In detail, an axiom is a logical sentence that is asserted to hold for all the objects and functions discussed by the specification while a theorem is a provable consequence of all the axioms of the theory. Conjectures are generated automatically by Specware, but the user can also create conjectures as well [14].

```
% Alpha-cut Theorem
theorem alpha_cut_closed is
fa(f:Fuzzy_number, a:Uni_intvl, S:Set_of_Nat)
(alpha_cut(f, a) = S <=> (fa(d:Nat)(d in?  S) =>
(ex(d1:Nat, d2:Nat)(d1<=d2) && (d1<=d) && (d<=d2))))
```

Figure 3.7: A Theorem sample.

### 3.2.5 Lambda-Forms

Lambda calculus can be used to define functions that take inputs and produces outputs. A form used to describe a function it is called a *lambda expression*. In Specware, lambda forms are represented by the symbol "fn"; the value of a lambda-form is a partial or total function. For instance, the meaning of a given lambda-form of type S → T is the function f mapping each inhabitant x of S to a value y of type T, where y is the return value of x for the match of the lambda-form. If the match accepts each x of type S, function f is total; otherwise it is partial, and undefined for those values x rejected [13]. A simple usage of lambda-forms can be seen below.

```
op fuzzy_add(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat)(z: Nat):
Uni_intvl =
sup(map (fn (x,y) -> min(f1 x, f2 y)) (fn (x,y) -> x + y = z))
```

## 3.3 Specware Shell

Specware specifications are processed in Specware Shell. Only units in specifications can be processed with commands. However, Specware shell can be run without a text editor, it can be run on XEmacs environment. Working on Xemacs environment has some advantages like; when an unit is processed if an error occurs XEmacs shows where the error is. Also Xemacs has syntax highlighting option that helps the user to edit specifications easily.

Processing a unit causes the recursive processing of the units referenced in that unit's term. For instance, if spec A extends spec B which in turn extends spec C, then when A is processed, B and C are also processed. There must be no circularities in the chain of unit dependencies [18].

When a unit is processed, processing results are saved to an internal cache by Specware. When the user tries to get Specware to process the same unit, Specware checks the cache which carries the unit's identifier and avoids recomputations if there is not any change on the file.

The Specware Shell consist of some commands that are used for basic file operations such as cd and ls. Also there are some other commands that are special to Specware such as show and proc.

## 3.4 Specware and SNARK

SNARK (SRI's New Automated Reasoning Kit) is an automated theorem-proving program developed in Common Lisp. Its principal inference rules are resolution and paramodulation. SNARK's style of theorem proving is similar to Otter's [16].

Specware comes packaged with the SNARK first-order theorem prover. In order to get SNARK to prove theorems, first the user should add proof terms in specifications and then these specifications should be processed by Specware.

Specware will report an error if the claim to be proved does not occur in the spec, or if not all claims following occur in the spec before the claim to be proved [18]. SNARK saves all information about execution process in a log file. However, log files can be more complicated for some users, they can be helpful to understanding why the proofs succeeded or failed.

## 3.5 Specware and Isabelle

Isabelle is a generic proof assistant. It allows mathematical formulas to be expressed in a formal language and provides tools for proving those formulas in a logical calculus. The main application is the formalization of mathematical proofs and in particular formal verification, which includes proving the correctness of computer hardware or software and proving properties of computer languages and protocols [15].

Specware interface allows the use of Isabelle to formulate and prove obligations. The interface is essentially just an emacs command that converts a Specware spec to an Isabelle theory, along with extensions in the Specware syntax to allow Isabelle proof scripts to be embedded in Specware specs, and to allow the user to specify translation of Specware ops and types to existing

Isabelle constants and types [17]. User can translate Specware declarations, definitions, axioms and theorems to the corresponding Isabelle versions by using gen-obligations command.

```
* gen-obligations FuzzyEdgeDetection#FUZZY_NUMBER
"/home/user/Specware-4-2-5/Codes/Isa/
FuzzyEdgeDetection_FUZZY_NUMBER.thy"
```

Figure 3.8: Usage of gen-obligations command.

```
type Fuzzy_number = Fuzzy_set Nat
type Set_of_Nat = FiniteSet Nat
axiom normality is
fa(f:Fuzzy_number)
height(f) = 1
theorem alpha_cut_closed is
fa(f:Fuzzy_number, a:Uni_intvl, S:Set_of_Nat)
(alpha_cut(f, a) = S <=> (fa(d:Nat)(d in?  S) =>
(ex(d1:Nat, d2:Nat)(d1<=d2) && (d1<=d) && (d<=d2))))
```

Figure 3.9: Sample Specware Specification.

A Specware definition may translate into one of four different kinds of Isabelle definitions: defs, recdefs and the newer funs and functions. Simple recursion on coproduct constructors translates to fun, but more complicated recursion is usually translated to fun. Some recursion still translates to recdef because the fun and function support is new, but the user can force translation to function. Non-recursive functions are translated to defs, except in some cases they are translated to fun which allows more pattern matching [17].

```
types Fuzzy_number = "nat Fuzzy_set"
types Set_of_Nat = "nat Set__FiniteSet"
theorem normality_Obligation_subtype:
\<lbrakk>Fun_PR between_zero_one_p (f::nat \<Rightarrow>
Uni_intvl);
\<forall>(x::int).
x = int (height f)
\<longrightarrow> between_zero_one_p (nat x) \<and> x
\<ge> 0\<rbrakk> \<Longrightarrow>
height f \<ge> 0"
by auto
axioms normality:
"\<lbrakk>Fun_PR between_zero_one_p f\<rbrakk>
\<Longrightarrow> height f = 1"
theorem alpha_cut_closed:
"\<lbrakk>Fun_PR between_zero_one_p (f::nat \<Rightarrow>
Uni_intvl); between_zero_one_p (a::nat); finite
S\<rbrakk> \<Longrightarrow> (alpha_cut(f, a) =
(S::Set_of_Nat)) = (\<forall>(d::nat).  d \<in> S
\<longrightarrow> (\<exists>(d1::nat) (d2::nat).  d1
\<le> d2 \<and> (d1 \<le> d \<and> d \<le> d2)))"
```

Figure 3.10: Isabelle obligation theory translated from sample Specware specification above.

# Chapter 4

# ISABELLE

## 4.1 Introduction

Isabelle and Specware can be run separately under their own Xemacs jobs. However, it is possible to run them under the same XEmacs. At the present time, Specware and Isabelle cannot be run together under Windows. This only can be done under Linux by using *IsabelleSpecware* command.

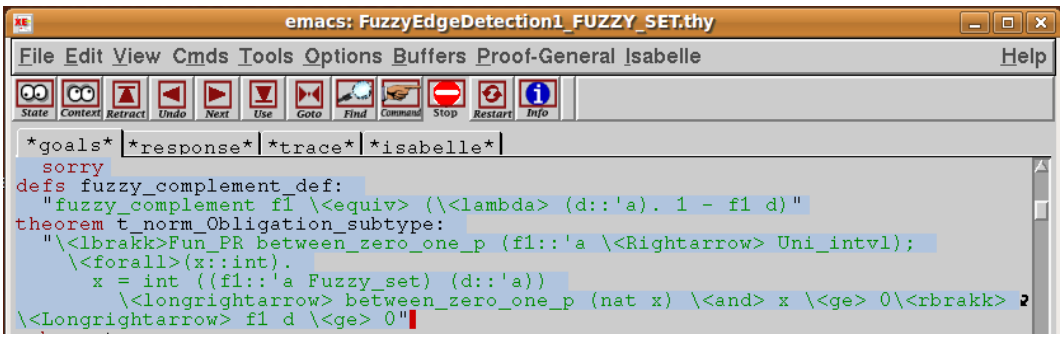



Figure 4.1: Screenshot of XEmacs.

Isabelle has two proof styles. The first is *Apply-style Proofs* and the other is *Structured Isar Proofs*.

## 4.2 Apply-style Proofs

### 4.2.1 apply(auto)

This command tells Isabelle to apply a proof strategy called *auto* to all sub-goals. Essentially, *auto* tries to simplify the subgoals [8]. Figure shows a sample proof that is proved with *apply(auto)*.

```
proof (prove): step 0

goal (1 subgoal):
 1. [| Fun_PR between_zero_one_p f1;
       ALL x. x = int (f1 d) --> between_zero_one_p (nat x) & 0 <= x |]
    ==> 0 <= f1 d
```

Figure 4.2: Example of a sub-goal.

```
theorem
  t_norm_Obligation_subtype:
    [| Fun_PR between_zero_one_p ?f1.0;
       ALL x. x = int (?f1.0 ?d) --> between_zero_one_p (nat x) & 0 <= x |]
    ==> 0 <= ?f1.0 ?d
```

Figure 4.3: The sub-goal in figure 4.2 is proved by *apply(auto)*.

### 4.2.2 apply(induct-tac x)

This command tells Isabelle to apply a proof strategy called induct_tac to perform induction to the variable x. Figure 4.4 shows an example of sub-goals in Isabelle before induction is applied, while Figure 4.5 shows an example of induct_tac being applied to the variable a.

### 4.2.3 apply(simp add: x1 x2)

This command tells Isabelle to apply a simplification proof strategy by adding x1 and x2, which are rules. Also there is another modifier: *del*. Modifier *add* can be changed to modifier *del* to delete simplification rules.

```
proof (prove): step 0

goal (1 subgoal):
 1. [| Fun_PR between_zero_one_p f; between_zero_one_p a; finite S |]
    ==> (alpha_cut (f, a) = S) =
        (ALL d. d : S --> (EX d1 d2. d1 <= d2 & d1 <= d & d <= d2))
```

Figure 4.4: The sub-goal before induction is applied

```
proof (prove): step 1

goal (2 subgoals):
 1. [| Fun_PR between_zero_one_p f; between_zero_one_p a; finite S |]
    ==> (alpha_cut (f, 0) = S) =
        (ALL d. d : S --> (EX d1 d2. d1 <= d2 & d1 <= d & d <= d2))
 2. !!n. [| Fun_PR between_zero_one_p f; between_zero_one_p a; finite S;
            (alpha_cut (f, n) = S) =
            (ALL d. d : S --> (EX d1 d2. d1 <= d2 & d1 <= d & d <= d2)) |]
         ==> (alpha_cut (f, Suc n) = S) =
             (ALL d. d : S --> (EX d1 d2. d1 <= d2 & d1 <= d & d <= d2))
```

Figure 4.5: The sub-goal after induction is applied

# Chapter 5

# SETUP OF SPECWARE, ISABELLE and XEMACS IN UBUNTU

To use Specware and Isabelle under same XEmacs, we need to work under a Linux operating system. In this work the choice was UBUNTU 9.04. Specware and Isabelle development environment requires the following software installed.

- Specware 4.2.5
- Isabelle 2008
- XEmacs 21.4.21

## 5.1 XEmacs 21.4.21

XEmacs can be installed by using the apt-get command in Ubuntu. It is installed automatically. But there must be an internet connection since XEmacs must be downloaded before the installing process. The figure below shows the command line.

```
keskemal@ubuntu:~$ sudo apt-get install xemacs21
```

Figure 5.1: Command to install XEmacs.

## 5.2 Specware 4.2.5

Specware installation is similar to install any software in Windows. Just execute setuplinux.bin and follow the instructions.

## 5.3 Isabelle

The following files are required for the Isabelle 2008 installation:

- Isabelle2008.tar.gz
- ProofGeneral.tar.gz
- Polyml_x86-linux.tar.gz
- HOL_x86-linux.tar.gz

Installation of Isabelle/HOL on Linux (x86 and x86_64) works by downloading and unpacking the relevant packages. In the subsequent example the installation location is /usr/local, but any other directory works as well [15]:

```
$ tar -C /usr/local -xzf Isabelle2009.tar.gz
$ tar -C /usr/local -xzf ProofGeneral.tar.gz
$ tar -C /usr/local -xzf polyml_x86-linux.tar.gz
$ tar -C /usr/local -xzf HOL_x86-linux.tar.gz
```

Figure 5.2: Commands to install Isabelle [15].

# Chapter 6

# FUZZY SETS

## 6.1 Introduction to Fuzzy Sets

In classical sets the transition for an element between membership and non-membership in the universe of a given set is abrupt. While the transition for an element that is in the universe of fuzzy sets can be gradual. This transition takes different degrees of membership that shows the boundaries of fuzzy sets are vague and ambiguous. Hence, the membership of an element is defined by describing the vagueness and ambiguity using the membership function. In general, membership functions take values in the unit interval [0,1].

We can define fuzzy sets in multiple ways. In the following we show two examples of fuzzy set definition.

*Definition 1*: Fuzzy set $A$ is a function

$$A : X \rightarrow [0, 1]$$

where $X$ is the universe of discourse.

*Definition 2*: When the universe of discourse, $X$, is discrete and finite, fuzzy set $A$ is:

$$A = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \cdots \right\} = \left\{ \sum_i \frac{\mu_A(x_i)}{x_i} \right\}$$

When the universe, $X$, is continuous and infinite the fuzzy set $A$ is:

$$A = \left\{ \int \frac{\mu_A(x)}{x} \right\}$$

The "+" sign in the first notation and the *integral* sign in the second notation represent the function-theoretic union.

## 6.2 Basic Concepts

One of the most important concepts of fuzzy sets is the concept of an $\alpha$ - cut. Definition of $\alpha$ - cut is

$$^{\alpha}A = \{x | \mu_A(x) \geq \alpha\}$$

where $0 \leq \alpha \leq 1$. In the definition, $\mu_A(x)$ denotes the membership function and the set $^{\alpha}A$ is a crisp set which derived from fuzzy set $A$. Any element $x$ from $^{\alpha}A$ (i.e. $x \in^{\alpha} A$) that is greater than or equal to the value $\alpha$ has a grade of membership in fuzzy set $A$ .

One other concept is strong $\alpha$ - cut. Definition of strong $\alpha$ - cut is

$$^{\alpha+}A = \{x | \mu_A(x) > \alpha\}$$

*Examples for $\alpha$ - cut and strong $\alpha$ - cut.*

Let us consider an universe $X = \{a, b, c, d\}$ and a fuzzy set $A$ on this universe.

$$A = \left\{ \frac{1}{a} + \frac{0.5}{b} + \frac{0.2}{c} + \frac{0.8}{d} \right\}$$

Let us define $\alpha$ - cut values: $\alpha = 0, 0.4, 0.5, 0.8$.

$$\begin{array}{ll} ^{0}A = \{a, b, c, d\} & ^{0+}A = \{a, b, c, d\} \\ ^{0.4}A = \{a, b, d\} & ^{0.4+}A = \{a, b, d\} \\ ^{0.5}A = \{a, b, d\} & ^{0.5+}A = \{a, d\} \\ ^{0.8}A = \{a, d\} & ^{0.8+}A = \{a\} \end{array}$$

The next concept *support* is the $\alpha$ - cut set of ${}^{0}A$. Symbolically,

$$supp(A) =^{0+} A = \{x \in X | A(x) > 0\}$$

where $\alpha = 0$. As seen from the definition, it contains all elements in $A$, except *zero*. Clearly, the support of A is exactly the same as the strong $\alpha$ - cut of A for $\alpha = 0$.

Another important concept is *core*. Core of $A$ contains all elements for which the degree of membership is 1 in $A$.

$$core(A) =^{1} A = \{x \in X | A(x) = 1\}$$

The largest value of $\alpha$ for which the $\alpha$ - cut is not empty is called the *height* of fuzzy set $A$.

$$h(A) = sup_{x \in X} A(x)$$

If the height of fuzzy set $A$ equals to one (i.e. $h(A) = 1$), fuzzy set A is called *normal*, otherwise it is called *subnormal*.

If for any elements $x, y, z$ in fuzzy set A, the relation $z < y < x$ implies that

$$\mu_A(y) \geq min\,[\mu_A(z), \mu_A(x)]$$

then A is a *convex* fuzzy set for $x, y, z \in \mathbb{R}$.

Lastly, three basic operations in crisp sets can be generalized as *standard fuzzy set operations*. These are *complement*, *intersection* and *union*.

The most natural way to express *standard complement* is

$$\bar{A}(x) = 1 - A(x)$$

for all $x \in X$.

The *standard intersection*, denoted $A \cap B$, can be defined by this equation

$$(A \cap B)(x) = min\,[A(x), B(x)]$$

Also the *standard union*, denoted $A \cup B$, can be defined in the same way,

$$(A \cup B)(x) = max\,[A(x), B(x)]$$

For the last two equation, min and max denote the minimum and maximum operator.

## 6.3 Fuzzy Set Operations

In this section standard fuzzy operations (fuzzy complement, intersection and union) are introduced.

### 6.3.1 Fuzzy Complement

Let $A$ is a fuzzy set defined on a universal set $X$, its complement $\bar{A}$ is another fuzzy set on $X$ such that for each $x \in X$ , $A(x)$ denotes the degree to which $x$ *belongs* to $A$ while $\bar{A}(x)$ denotes the degree to which $x$ *does not belong* to $A$.

$$\bar{A}(x) = 1 - A(x)$$

for all $x \in X$.

Fuzzy sets overlap their complements as seen from figure 6.1. When a member belongs to the fuzzy set A with the degree of 0.5, then it also belongs to the fuzzy set $\bar{A}(x)$ with a degree of 0.5. This is the main difference between fuzzy set theory and classical set theory since a set never overlaps its complement in classical set theory.

There are two characterization theorems of fuzzy complements.

*Theorem 1:* Let $c$ be a function from [0,1] to [0,1]. Then $c$ is a fuzzy complement (involutive) iff there exists a continuous function $g$ from [0,1] to $\mathbb{R}$ such that $g(0) = 0$, $g$ is strictly increasing, and

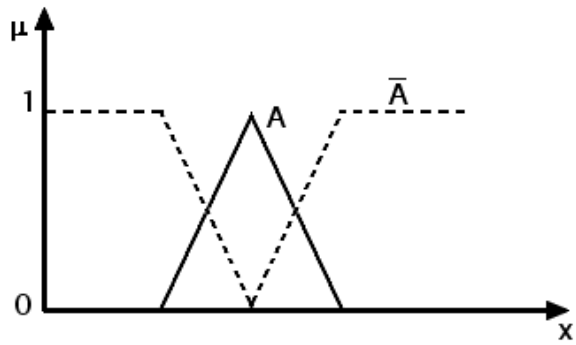



Figure 6.1: Complement of fuzzy set $A$.

$$c(a) = g^{-1}(g(1) - g(a))$$

for all a $\in$ [0,1].[1]

*Theorem 2:* Let $c$ be a function from [0,1] to [0,1]. Then $c$ is a fuzzy complement iff there exists a continuous function $f$ from [0,1] to $\mathbb{R}$ such that $f(1) = 0$, $g$ is strictly decreasing, and

$$c(a) = f^{-1}(f(0) - f(a))$$

for all a $\in$ [0,1].[1]

### 6.3.2 Fuzzy Union : t - CONORM

Let $A$ and $B$ are fuzzy sets defined on a universal set $X$. The standard fuzzy union of $A$ and $B$ , denoted by $A \cup B$, is defined by using their membership functions.

$$(A \cup B)(x) = max[A(x), B(x)]$$

for all $x \in X$.

In classical set theory, the equation

$$A \cup \bar{A} = X$$

is always true. However it does not hold for fuzzy sets under standard fuzzy union. It can be seen easily that this equation is invalid for all elements $x \in X$ such that $A(x) \notin 0, 1$. For example, $A(x) = 0.8$, then $\bar{A} = 1 - 0.8 = 0.2$. Hence,

$$(A \cup B)(x) = max[0.8, 0.2] = 0.8$$

If this was a classical set, $x$ was expected a member of $X$ with full membership ($x = 1$), but it is obvious that it does not hold for fuzzy sets under standard fuzzy union.

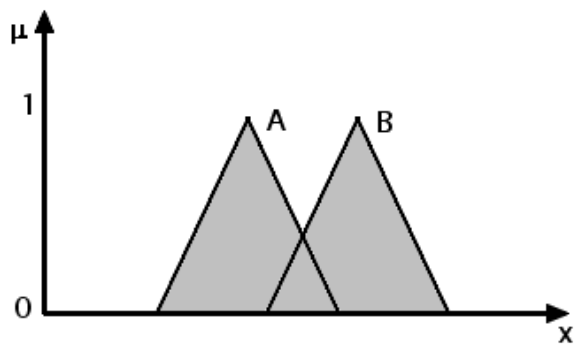



Figure 6.2: Union of fuzzy sets $A$ and $B$.

There is one characterization theorem of t-conorms.

*Theorem:* Let $u$ be a binary operation on the unit interval. Then, $u$ is an *Archimedean* t-conorm if and only if there exists an increasing generator $g$ such that

$$u(a, b) = g^{(-1)}(g(a) + g(b))$$

for all $a, b \in [0, 1]$. [1]

### 6.3.3 Fuzzy Intersection : t - NORM

Like fuzzy union, let $A$ and $B$ are fuzzy sets defined on a universal set $X$. The standard fuzzy intersection of $A$ and $B$ , denoted by $A \cap B$, is defined by using their membership functions.

$$(A \cap B)(x) = min[A(x), B(x)]$$

for all $x \in X$.

In classical set theory, the equation

$$A \cap \bar{A} = \emptyset$$

is always true. However it does not hold for fuzzy sets under standard fuzzy intersection. It can be seen easily that this equation is invalid for all elements $x \in X$ such that $A(x) \notin 0, 1$. For example, $A(x) = 0.8$, then $\bar{A} = 1 - 0.8 = 0.2$. Hence,

$$(A \cap B)(x) = min[0.8, 0.2] = 0.2$$

If this was a classical set, $x$ was expected a member of $X$ with the degree of 0 ($x = 0$), but it is obvious that it does not hold for fuzzy sets under standard fuzzy intersection.

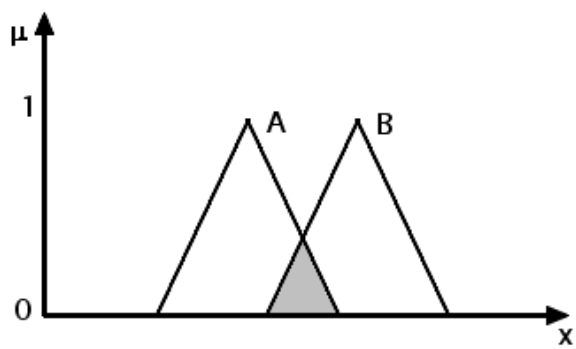



Figure 6.3: Intersection of fuzzy sets $A$ and $B$.

There is one characterization theorem of t-norms.

*Theorem:* Let $i$ be a binary operation on the unit interval. Then, $i$ is an *Archimedean* t-norm if and only if there exists a decreasing generator $f$ such that

$$i(a,b) = f^{(-1)}(f(a) + f(b))$$

for all $a, b \in [0,1]$. [1]

### 6.3.4 Properties of Fuzzy Sets

Fuzzy sets follow the same properties as classical sets. The following is a list of frequently used properties of fuzzy sets.

| | |
|---|---|
| *Commutativity* | $A \cup B = B \cup A$ |
| *Associativity* | $A \cup (B \cup C) = (A \cup B) \cup C$ |
| | $A \cap (B \cap C) = (A \cap B) \cap C$ |
| *Distributivity* | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ |
| | $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ |
| *Idempotency* | $A \cup A = A \quad and \quad A \cap A = A$ |
| *Identity* | $A \cup \emptyset = A \quad and \quad A \cap X = A$ |
| | $A \cap \emptyset = \emptyset \quad and \quad A \cup X = X$ |
| *Transitivity* | $If\ A \subseteq B \subseteq C, \quad then\ A \subseteq C$ |
| *Involution* | $\bar{\bar{A}} = A$ |

## 6.4 Fuzzy Arithmetic

### 6.4.1 Fuzzy Numbers

A fuzzy number is described in terms of a number and a linguistic modifier such as *approximately, around, or nearly.* For example *around seven* is fuzzy, since it contains some numbers that are on either side of the central value *seven.* The membership function decreases from 1 to 0 both sides of central value. This kind of fuzzy sets are called *fuzzy numbers.* Membership functions can be shown in the form:

$$A : \mathbb{R} \rightarrow [0, 1]$$

Not all membership functions in this form are fuzzy numbers. In order to qualify as a fuzzy number, a fuzzy set A on $\mathbb{R}$ must provide at least the following three properties:

1. Fuzzy set $A$ must be normal. The only condition for a fuzzy set to be normal is its height must equal to 1.

$$h(A) = \sup_{x \in X} A(x) = 1$$

2. Alpha-cuts of fuzzy set $A$ must be a closed interval for every $\alpha \in (0, 1]$. Since $\alpha$ - cuts of fuzzy number are required to be closed intervals, every fuzzy number is a convex fuzzy set. If for any elements $x, y, z$ in fuzzy set $A$, the relation $z < y < x$ implies that

$$\mu_A(y) \geq min\,[\mu_A(z), \mu_A(x)]$$

then $A$ is a *convex* fuzzy set for $x, y, z \in \mathbb{R}$.

3. The support of fuzzy set $A$, ${}^{0+}A$, must be bounded.

These properties are necessary for defining meaningful arithmetic operations on fuzzy numbers.

### 6.4.2 Arithmetic Operations On Intervals

Fuzzy set and fuzzy number can be described by the intervals associated with $\alpha$-cuts. As a result of that, we can define arithmetic operations on fuzzy numbers in terms of arithmetic operations on closed intervals.

Let's consider wo interval numbers $I_1$ and $I_2$t

$$I_1 = [a, b] \qquad where \qquad a \leq b$$

$$I_2 = [c, d] \qquad where \qquad c \leq d$$

Denote a general arithmetic operation for these two interval numbers:

$$I_1 * I_2 = [a, b] * [c, d]$$

We can represent the basic four arithmetic interval operations using this scheme:

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \cdot [c, d] = [min(ac, ad, bc, bd), max(ac, ad, bc, bd)]$$

$$[a, b] \div [c, d] = [a, b] \cdot [\frac{1}{d}, \frac{1}{c}] \qquad 0 \notin [c, d]$$

$$\alpha[a, b] = \begin{cases} (\alpha a, \alpha b) & \text{for } \alpha > 0 \\ (\alpha a, \alpha b) & \text{for } \alpha < 0 \end{cases}$$

where $ac, ad, bc$, and $bd$ are arithmetic products and $1/d$ and $1/c$ are quotients.

Arithmetic operations on closed intervals satisfy some useful properties. To overview them, let $A = [a1, a2]$, $B = [b1, b2]$, $C = [c1, c2]$, $0 = [0, 0]$, $1 = [1, 1]$. Using these symbols, the properties are formulated as follows [1]:

**1**. $A + B = B + A$

$A \cdot B = B \cdot A$ (commutativity)

**2**. $(A + B) + C = A + (B + C)$

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$ (associativity)

**3**. $A = 0 + A = A + 0$

$A = 1 \cdot A = A \cdot 1$ (identity)

**4**. $A \cdot (B + C) \subseteq A \cdot B + A \cdot C$ (subdistributivity)

**5**. If $b \cdot c \geq 0$ for every $b \in B$ and $c \in C$, then $A \cdot (B + C) = A \cdot B + A \cdot C$ (distributivity). Furthermore, if $A = [a, a]$, then $a \cdot (B + C) = a \cdot B + a \cdot C$.

**6**. $0 \in A - A$ and $1 \in A/A$.

**7**. If $A \subseteq E$ and $B \subseteq F$, then:

$$\begin{aligned} A + B &\subseteq E + F \\ A - B &\subseteq E - F \\ A \cdot B &\subseteq E \cdot F \\ A/B &\subseteq E/F \text{ (inclusion monotonicity)} \end{aligned}$$

### 6.4.3 Arithmetic Operations On Fuzzy Numbers

In order to use fuzzy numbers in practice, it is required to define usual operations on fuzzy numbers, such as addition, subtraction, multiplication, and division. From last section, it is known that fuzzy numbers can be described by the intervals associated with $\alpha$-cuts and these are all closed intervals of real numbers. Also it is known how to apply four basic operations to closed intervals. Hence, arithmetic operations on fuzzy numbers can be defined.

Let $A$ and $B$ denote two fuzzy numbers and $*$ denote any of the four basic arithmetic operations. Then, $A * B$, by defining its $\alpha - cut$, ${}^{\alpha}(A * B)$, as

$${}^{\alpha}(A * B) = {}^{\alpha} A *^{\alpha} B$$

for any $\alpha \in (0, 1]$. It can also be expressed as

$$A * B = \bigcup_{\alpha \in [0,1]} ({}^{\alpha} A *^{\alpha} B) \times \alpha$$

so the result will be fuzzy number.[1]

As an example of employing equations above, let's consider two triangular-shape fuzzy numbers A and B defined as follows:

$$A(x) = \begin{cases} 0, & \text{for } x \leq -1 \text{ and } x > 3 \\ \frac{x}{2}, & \text{for } -1 \leq x < 1 \\ \frac{(4-x)}{2}, & \text{for } 1 < x \leq 3 \end{cases}$$

$$B(x) = \begin{cases} 0, & \text{for } x \leq 1 \text{ and } x > 5 \\ \frac{(x+2)}{2}, & \text{for } 1 < x \leq 3 \\ \frac{(2-x)}{2}, & \text{for } 3 < x \leq 5. \end{cases}$$

From these formulas, $\alpha - cuts$:

$$^{\alpha}A = [^{\alpha}a_1, ^{\alpha}a_2]$$

$$^{\alpha}B = [^{\alpha}b_1, ^{\alpha}b_2]$$

Then,

$$A(^{\alpha}a_1) = (^{\alpha}a_1 + 1)/2 = \alpha$$

$$A(^{\alpha}a_2) = (3 -^{\alpha} a_2)/2 = \alpha$$

From these equations,

$$^{\alpha}a_1 = 2\alpha - 1$$

$$^{\alpha}a_2 = 3 - 2\alpha$$

Hence,

$$^{\alpha}A = [2\alpha - 1, 3 - 2\alpha]$$

When the same process is applied to B we obtain

$$^{\alpha}B = [2\alpha + 1, 5 - 2\alpha]$$

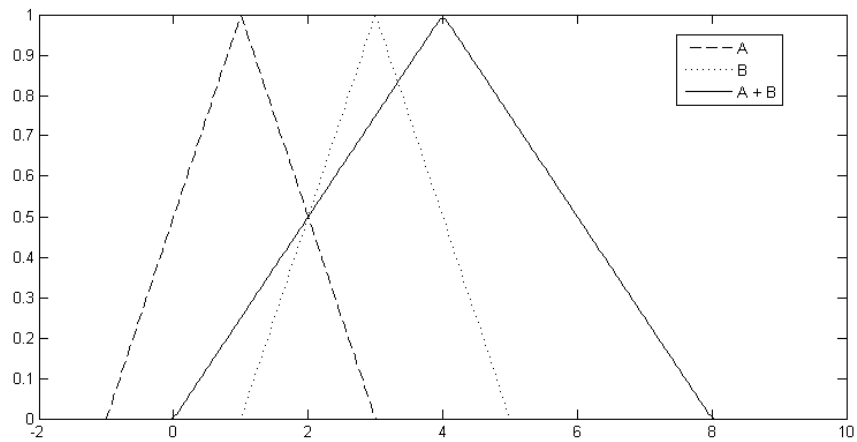



Figure 6.4: Addition of fuzzy sets $A$ and $B$.

Using these intervals,

$$(A+B)(x) = \begin{cases} 0, & \text{for } x \leq 0 \text{ and } x > 8 \\ \frac{x}{4}, & \text{for } 0 \leq x < 4 \\ \frac{(8-x)}{4}, & \text{for } 4 < x \leq 8 \end{cases}$$

$$(A-B)(x) = \begin{cases} 0, & \text{for } x \leq -6 \text{ and } x > 2 \\ \frac{(x+6)}{4}, & \text{for } -6 \leq x < -2 \\ \frac{(2-x)}{4}, & \text{for } -2 < x \leq 2 \end{cases}$$

$$(A \cdot B)(x) = \begin{cases} 0, & \text{for } x < -5 \text{ and } x \geq 15 \\ \frac{[3-(4-x)^{1/2}]}{2} & \text{for } -5 \leq x < 0 \\ \frac{(1+x)^{1/2}}{2}, & \text{for } 0 \leq x < 3 \\ \frac{[4-(1+x)^{1/2}]}{2}, & \text{for } 3 \leq x < 15 \end{cases}$$

$$(A/B)(x) = \begin{cases} 0, & \text{for } x \leq -1 \text{ and } x \geq 3 \\ \frac{(x+1)}{(2-2x)} & \text{for } -1 \leq x < 0 \\ \frac{(5x+1)}{(2x+2)}, & \text{for } 0 \leq x < 1/3 \\ \frac{(3-x)}{(2x+2)}, & \text{for } 1 < x \leq 3 \end{cases}$$

The extension principle can be used to define usual operations on fuzzy num-

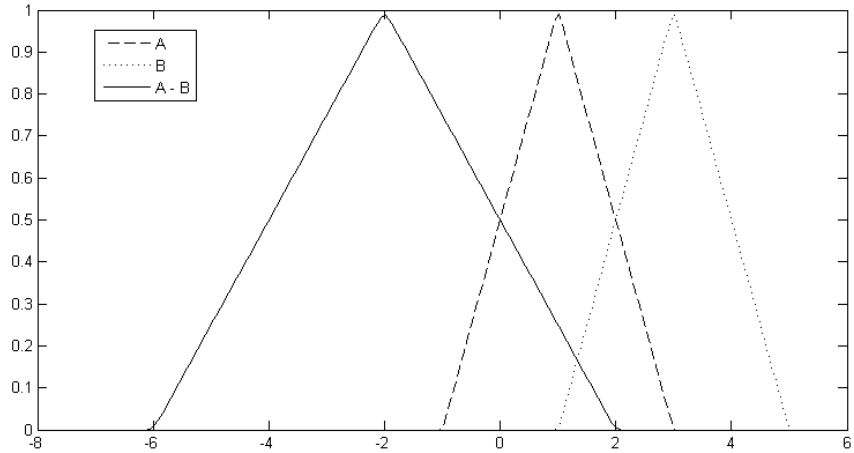



Figure 6.5: Substraction of fuzzy sets $A$ and $B$.

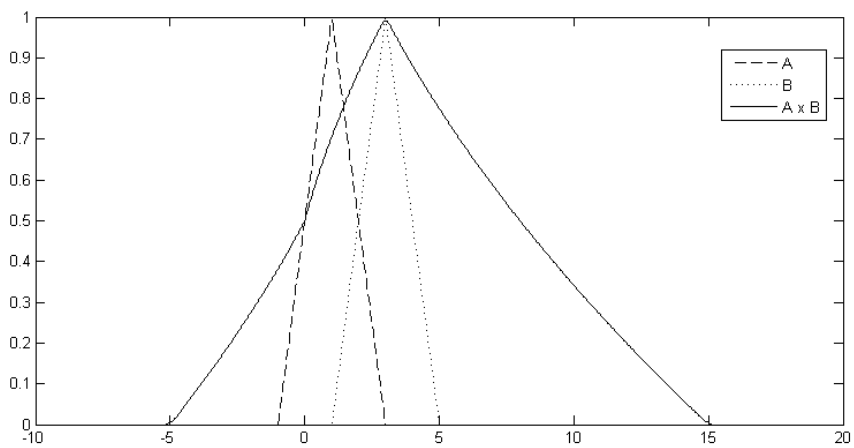



Figure 6.6: Multiplication of fuzzy sets $A$ and $B$.

bers. Let $A$ and $B$ denote two fuzzy numbers and $\star$ denote any of the four basic arithmetic operations. Then, $A \star B$ is defined by the equation,

$$(A \star B)(z) = \sup_{z=x\star y} \min[A(x), B(y)]$$

for all $z \in \mathbb{R}$. More specifically, the four arithmetic equations are defined as:

$$(A + B)(z) = \sup_{z=x+y} \min[A(x), B(y)]$$

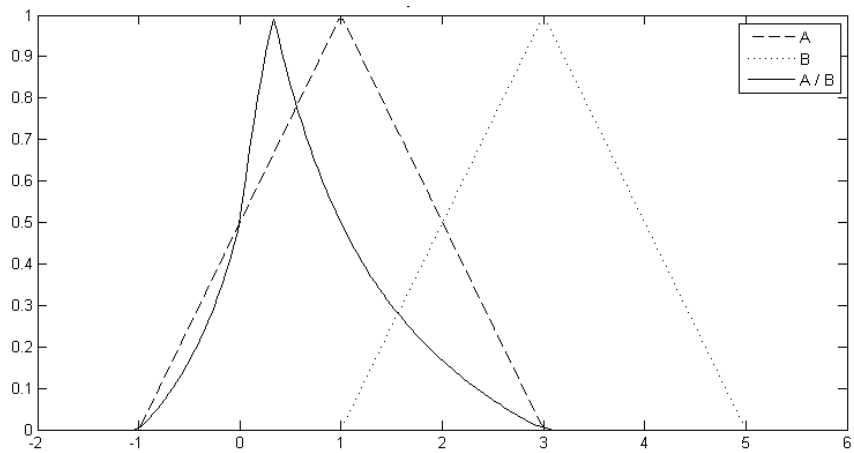



Figure 6.7: Division of fuzzy sets $A$ and $B$.

$$(A - B)(z) = \sup_{z=x-y} \min[A(x), B(y)]$$

$$(A \cdot B)(z) = \sup_{z=x \cdot y} \min[A(x), B(y)]$$

$$(A/B)(z) = \sup_{z=x/y} \min[A(x), B(y)]$$

### 6.4.4 Lattice Of Fuzzy Numbers

The set $\mathbb{R}$ of real numbers is linearly ordered. For every pair of real numbers, $x$ and $y$, either $x \leq y$ or $y \leq x$. The pair $(\mathbb{R}, \leq)$ is a lattice, which is also expressed in terms of two lattice operations,

$$\min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y \leq x \end{cases}$$

$$\max(x, y) = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y \leq x \end{cases}$$

for every pair $x, y \in \mathbb{R}$. The linear ordering of real numbers does not extend to fuzzy numbers, but fuzzy numbers can be ordered partially in a natural way and that this partial ordering forms a distributive lattice. So the lattice operations on fuzzy numbers can be shown as : for any two fuzzy numbers $A$ and $B$,

$$MIN(A, B)(z) = \sup_{z=\min(x,y)} \min[A(x), B(y)]$$

$$MAX(A, B)(z) = \sup_{z=\max(x,y)} \min[A(x), B(y)]$$

for all $z \in \mathbb{R}$. The operations MIN and MAX are totally different from

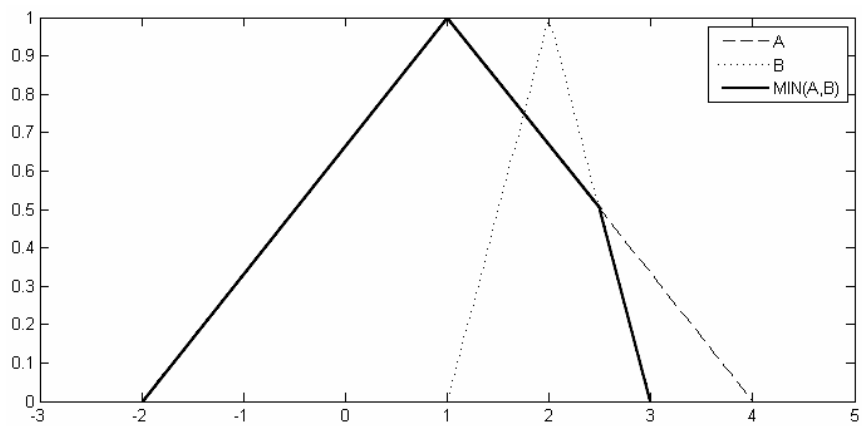



Figure 6.8: fuzzy-min of fuzzy sets $A$ and $B$.

the standard fuzzy intersection and union, min and max. This difference can be seen from the following equations :

$$A(x) = \begin{cases} 0, & \text{for } x < -2 \text{ and } x > 4 \\ \frac{(x+2)}{3}, & \text{for } -2 \leq x \leq 1 \\ \frac{(4-x)}{3}, & \text{for } 1 \leq x \leq 4 \end{cases}$$

$$B(x) = \begin{cases} 0, & \text{for } x < 1 \text{ and } x > 3 \\ x - 1, & \text{for } 1 \leq x < 2 \\ 3 - x, & \text{for } 2 \leq x \leq 3 \end{cases}$$

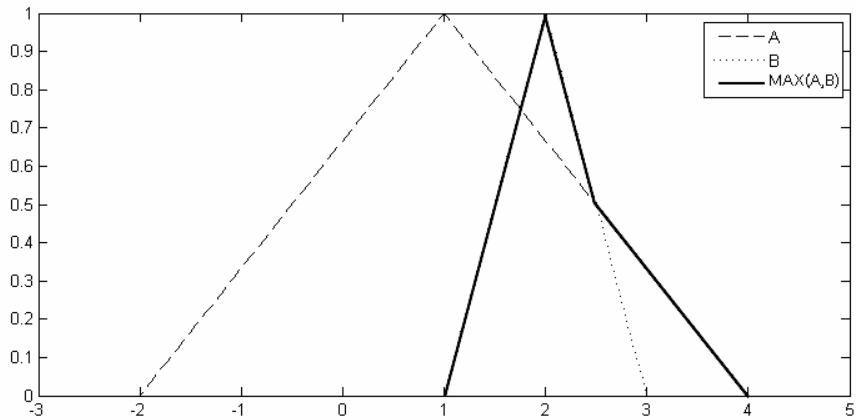



Figure 6.9: fuzzy-max of fuzzy sets $A$ and $B$.

$$MIN(A,B)(x) = \begin{cases} 0, & \text{for } x < -2 \text{ and } x > 3 \\ \frac{(x+2)}{3} & \text{for } -2 \leq x < 1 \\ \frac{(4-x)}{3}, & \text{for } 1 < x < 2.5 \\ 3-x, & \text{for } 2.5 < x \leq 3 \end{cases}$$

$$MAX(A,B)(x) = \begin{cases} 0, & \text{for } x < 1 \text{ and } x > 4 \\ x-1 & \text{for } 1 \leq x \leq 2 \\ 3-x, & \text{for } 2 < x \leq 2.5 \\ \frac{(4-x)}{3}, & \text{for } 2.5 < x \leq 4 \end{cases}$$

## 6.5 Fuzzy Reasoning Operators

In this section basic fuzzy reasoning operators are extended from the crisp logic. Fuzzy logic connectives such as *fuzzy implication, fuzzy biimplication, fuzzy and, fuzzy or* and *fuzzy negation* are specified. *Fuzzy equal, fuzzy less than and equal*, as well as *fuzzy greater than and equal* are three operators that have been specified.

Depending on partial ordering of fuzzy numbers, sometimes two fuzzy

numbers are comparable. In many fuzzy decision problems, the output alternatives are often represented in terms of fuzzy numbers. The requirement might be to express a crisp preference of alternatives, so it might need a method for construction a crisp total ordering of fuzzy numbers.

Numerous methods for total ordering of fuzzy numbers have been investigated and suggested, but the issue of choosing a proper ordering method in a certain circumstance is an open research. Here two methods suggested in [2] are listed.

The first method uses *Hamming Distance*. For any given fuzzy numbers $A$ and $B$, the Hamming Distance, $d(A, B)$ is defined by the formula:

$$d(A, B) = \int_R |A(x) - B(x)|dx$$

For any fuzzy numbers $A$ and $B$, first $fuzzy - min(A, B)$ is calculated, which is the least upper bound in the lattice. Then the Hamming distances $d(fuzzy - min(A, B), A)$ and $d(fuzzy - min(A, B), B)$ are calculated.

The compatibility of partial ordering of comparable fuzzy numbers with the ordering defined by the Hamming distance can be proven, because if $A \tilde{\leq} B$, then fuzzy-min(A,B)=B and hence, $A \leq B$.

Another method is based on $\alpha - cuts$. Given fuzzy numbers A and B, select a particular value of $\alpha \in [0, 1]$ and determine the $\alpha - cuts$ ${}^{\alpha}A = [a_1, a_2]$ and ${}^{\alpha}B = [b_1, b_2]$. Then the total ordering of two fuzzy numbers can be defined as:

$$A \leq B \text{ iff } a_2 \leq b_2$$

This definition is dependent upon the choice of a value $\alpha$. It is usually required that $\alpha > 0.5$.

However, in order to do fuzzy reasoning, the whole process has to be done in a fuzzified way, so that the fuzziness, or the level of uncertainty, can be maintained and could be reasoned about. In other words, fuzzy logic has to

be used. *Fuzzy equal, fuzzy less than and equal*, as well as *fuzzy greater than and equal* are three fuzzy logic operators are to be used in the approach. And *fuzzy implication, fuzzy biimplication, fuzzy and, fuzzy or* and *fuzzy negation* are fuzzy logical connectives that should be defined.

Each fuzzy logical operator or connective is associated with a mapping $[0,1]^2 \to [0,1]$. So the truth value of a proposition is admitted as the range of truth values. And the truth value of a combined composition can be determined from the truth values of the atomic propositions.

**Definition :** Let $T$ be a t-norm. Fuzzy implication $\tilde{\Rightarrow}$ is defined as

$$\tilde{\Rightarrow}: [0,1] \times [0,1] \to [0,1],$$

$$\beta \tilde{\Rightarrow} \gamma = \sup\{\alpha \in [0,1] | T(\alpha, \beta) \leq \gamma$$

Definition : Let $T$ be continuous t-norm.

$$\tilde{\Rightarrow}: [0,1] \times [0,1] \to [0,1],$$

and

$$\alpha \tilde{\Leftrightarrow} \beta = T(\alpha \tilde{\Rightarrow} \beta, \beta \tilde{\Rightarrow} \alpha)$$

are called the fuzzy implications and fuzzy biimplication, respectively, induced by $T$.

This definition is motivated by the equivalence

$$\alpha \Leftrightarrow \beta \equiv (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$$

The following theorem can make the calculation of fuzzy biimplication easier.

**Theorem :** Let T be a t-norm, $\alpha, \beta \in [0,1]$. Then

$$\alpha \tilde{\Leftrightarrow} \beta = \max(\alpha, \beta) \tilde{\Rightarrow} \min(\alpha, \beta)$$

holds.

Depending on which t-norm is used, fuzzy implication and fuzzy biimplication could be in different versions.

Assume t-norm $T = T_{Luka}$, then

$$\begin{aligned}
\alpha \tilde{\Rightarrow} \beta &= \sup\{\gamma \in [0,1] | \max(\alpha + \gamma - 1, 0) \leq \beta\} \\
&= \sup\{\gamma \in [0,1] | \alpha + \gamma - 1, 0 \leq \beta\} \\
&= \sup\{\gamma \in [0,1] | \gamma \leq 1 - \alpha + \beta\} \\
&= \min(1 - \alpha + \beta, 1)
\end{aligned}$$

which is called Lukasiewicz implication. The corresponding biimplication is :

$$\begin{aligned}
\alpha \tilde{\Leftrightarrow} \beta &= \min(1 - \max(\alpha, \beta) + \min(\alpha, \beta), 1) \\
&= 1 - |\alpha - \beta|
\end{aligned}$$

Assume t-norm $T = T_{min}$, then

$$\begin{aligned}
\alpha \tilde{\Rightarrow} \beta &= \sup\{\gamma \in [0,1] | \min(\alpha, \gamma) \leq \beta\} \\
&= \begin{cases} 1, & \text{if } \alpha \leq \beta \\ \beta, & \text{otherwise} \end{cases}
\end{aligned}$$

which is called Gödel implication. The corresponding biimplication is :

$$\begin{aligned}
\alpha \tilde{\Leftrightarrow} \beta &= \max(\alpha, \beta) \tilde{\Rightarrow} \min(\alpha, \beta) \\
&= \begin{cases} 1, & \text{if } \alpha = \beta \\ \min(\alpha, \beta), & \text{otherwise} \end{cases}
\end{aligned}$$

Assume t-norm $T = T_{prod}$, then

$$\begin{aligned}\alpha \tilde{\Rightarrow} \beta &= \sup\{\gamma \in [0,1] | \alpha \cdot \gamma \leq \beta\} \\ &= \begin{cases} \frac{\beta}{\alpha}, & \text{if } \beta < \alpha \\ 1, & \text{otherwise} \end{cases}\end{aligned}$$

which is called Goguen implication. The corresponding biimplication is :

$$\begin{aligned}\alpha \tilde{\Leftrightarrow} \beta &= \max(\alpha, \beta) \tilde{\Rightarrow} \min(\alpha, \beta) \\ &= \begin{cases} 1, & \text{if } \alpha = \beta \\ \frac{\min(\alpha,\beta)}{\max(\alpha,\beta)}, & \text{otherwise} \end{cases}\end{aligned}$$

The basic concept of the crisp logic (classical logic or two-valued logic) is that every proposition is either True or False. In fuzzy logic, the truth value could be extended to a value between 0 an 1, that is $[0, 1]$, instead of $\{0, 1\}$. This value represents the uncertainty level of the proposition. Let $\Upsilon(\phi) \in [0, 1]$ denotes the truth value of the proposition $\phi$. The truth values of statements that contain quantifiers are normally determined as :

$$\Upsilon(\forall x : P(x)) = \inf\{\Upsilon(P(x))|x\}$$

$$\Upsilon(\forall x : P(x)) = \sup\{\Upsilon(P(x))|x\}$$

With the previous support, now we can consider the truth value of *fuzzy equal, fuzzy less than and equal,* as well as *fuzzy greater than and equal.*

For two fuzzy sets, or fuzzy numbers in particular, $\mu$ and $\nu$, the truth value of the value of the statement '$\mu = \nu$', where $\tilde{=}$ represents fuzzy equal, can be defined as :

$$\begin{aligned}\Upsilon(\mu \tilde{=} \nu) &= \Upsilon(\forall x : (x \in \mu \tilde{\Leftrightarrow} x \in \nu)) \\ &= \Upsilon(\forall x : ((x \in \mu \tilde{\Rightarrow} x \in \nu) \tilde{\wedge} (x \in \nu \tilde{\Rightarrow} x \in \mu))\end{aligned}$$

So the truth value of $\mu \tilde{=} \nu$ depends on the truth functions chosen for the fuzzy implication $\tilde{\Rightarrow}$ and the fuzzy conjunction $\tilde{\wedge}$. Using Lukasiewicz implication yield :

$$\Upsilon(\mu \tilde{=} \nu) = \inf\{1 - |\mu(x) - \nu(x)| | x \in Real\}$$

whereas if Gödel implication is used, it will yield :

$$\Upsilon(\mu \tilde{=} \nu) = \inf\{g\ddot{o}d(\mu(x), \nu(x)) | x \in Real\}$$

where

$$g\ddot{o}d(\alpha, \beta) = \begin{cases} 1, & \text{if } \alpha = \beta \\ \min(\alpha, \beta), & \text{otherwise} \end{cases}$$

For two fuzzy numbers $\mu$ and $\nu$, the truth value of the statement '$\mu \tilde{\geq} \nu$', where $\tilde{\geq}$ represents fuzzy greater than and equal, can be defined as :

$$\Upsilon(\mu \tilde{\geq} \nu) = \Upsilon(\forall x : (x \in \nu \tilde{\Rightarrow} x \in \mu))$$

So the truth value of $\mu \tilde{\geq} \nu$ depends on the truth functions chosen for the fuzzy implication $\tilde{\Rightarrow}$ . Using Lukasiewicz implication yield :

$$\Upsilon(\mu \tilde{\geq} \nu) = \inf\{\min(1 - \nu(x) + \mu(x), 1) | x \in Real\}$$

For two fuzzy numbers $\mu$ and $\nu$, the truth value of the statement '$\mu \tilde{\leq} \nu$', where $\tilde{\leq}$ represents fuzzy greater than and equal, can be defined as :

$$\Upsilon(\mu \tilde{\leq} \nu) = \Upsilon(\forall x : (x \in \mu \tilde{\Rightarrow} x \in \nu))$$

So the truth value of $\mu \tilde{\leq} \nu$ depends on the truth functions chosen for the fuzzy implication $\tilde{\Rightarrow}$ . Using Lukasiewicz implication yield :

$$\Upsilon(\mu \tilde{\leq} \nu) = \inf\{\min(1 - \mu(x) + \nu(x), 1) | x \in Real\}$$

Logical Mapping Table for Reasoning Operators gives the mapping of crisp reasoning operators to fuzzy equivalent.

Where the truth value symbol $\Upsilon$ is eliminated. A complete mapping table of crisp to fuzzy is given at the end as Appendix A.

## 6.6 Fuzzy Set Specification

In this work, Specware is chosen as a tool for the handling of the fuzzy uncertainty. And since Specware requires that all functions be total, the first definition of fuzzy set (i.e., a fuzzy set is represented as a function) in Chapter 1 is chosen for building specifications. The diagram of the specification of fuzzy set is shown in figure 6.4.

| Logical Mapping Table | |
|---|---|
| CRISP | FUZZY |
| Number | Fuzzy Number (FN) |
| True, False | [0,1] |
| $\Rightarrow$ | $\Rightarrow: [0,1] \times [0,1] \rightarrow [0,1]$<br>1: $a \Rightarrow b = (1, 1 - a + b)$<br>2: $a \Rightarrow b = \begin{cases} 1, & \text{if } a \leq b \\ b & \text{other} \end{cases}$ |
| $\Leftrightarrow$ | $\Leftrightarrow: [0,1] \times [0,1] \rightarrow [0,1]$<br>1: $a \Leftrightarrow b = 1 - \lvert a - b \rvert$<br>2: $a \Leftrightarrow b = \begin{cases} 1, & \text{if } a = b \\ min(a,b) & \text{other} \end{cases}$ |
| $=$ | $=: FN \times FN \rightarrow [0,1]$<br>1: $\mu = v = inf\,\{1 - \lvert\mu(x) - v(x)\rvert \mid x \in \mathbb{R}\}$<br>1: $\mu = v = inf\,\{g(\mu(x), v(x)) \mid x \in \mathbb{R}\}$<br>where $g(a,b) = \begin{cases} 1, & \text{if } a = b \\ b & \text{other} \end{cases}$ |
| $\geq$ | $\geq: FN \times FN \rightarrow [0,1]$<br>$\mu \geq v = inf\,\{min(1 - v(x) + \mu(x), 1) \mid x \in \mathbb{R}\}$ |
| $\leq$ | $\leq: FN \times FN \rightarrow [0,1]$<br>$\mu \leq v = inf\,\{min(1 - \mu(x) + v(x), 1) \mid x \in \mathbb{R}\}$ |

Table 6.1: Logical Mapping Table

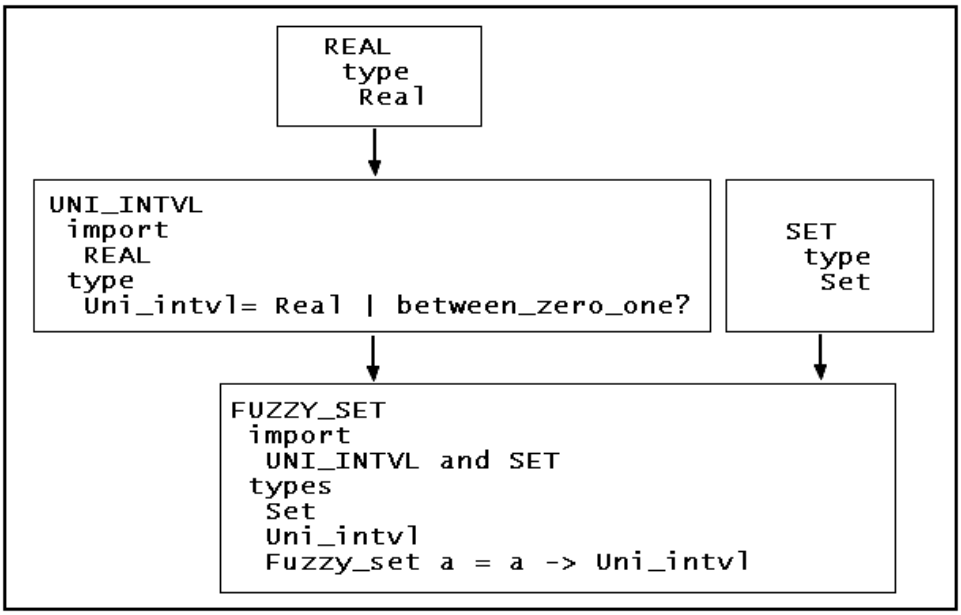



Figure 6.10: Diagram for fuzzy-set.

The spec UNI-INTVL imports REAL and introduces a new sort : Uni-intvl, which characterizes unit interval $[0, 1]$.

```
UNI_INTVL = spec
import Real
op between_zero_one?(x:Real):  Boolean =
x < one && zero <= x
type Uni_intvl = (Real |
between_zero_one?)
```

FUZZY-SET is a definitional extension of the colimit of UNI-INTVL and SET. It defines a function sort :

```
FUZZY_SET = spec
import /Library/General/Set, UNI_INTVL
type Fuzzy_set a = a -> Uni_intvl
```

where a is the type of elements in Set. In the FUZZY-SET spec, the basic fuzzy concepts such as fuzzy complement, union, and intersection are also specified which can also be found as Appendix B.

Fuzzy number is imported from the spec FUZZY_SET. In FUZZY_NUMBER spec new type Set_of_Real is defined for using real numbers set.

```
FUZZY_NUMBER = spec
import FUZZY_SET
type Fuzzy_number = Fuzzy_set Real
type Set_of_Real = Set Real
```

In chapter 6.4, the arithmetic operations on intervals and fuzzy numbers were discussed. In the next sections an example of fuzzy edge detection will be given. This example deals with fuzzy numbers, so the arithmetic operations on fuzzy numbers have to be specified. This is done in FUZZY_ARITHM spec. Fuzzy arithmetic operators are specified in this spec, which is a definitional extension of FUZZY_NUMBER, with fuzzy operations being of the following types :

```
op fuzzy_add :  Fuzzy_number * Fuzzy_number -> Fuzzy_number
op fuzzy_sub :  Fuzzy_number * Fuzzy_number -> Fuzzy_number
op fuzzy_mult:  Fuzzy_number * Fuzzy_number -> Fuzzy_number
op fuzzy_div :  Fuzzy_number * Fuzzy_number -> Fuzzy_number
op fuzzy_min :  Fuzzy_number * Fuzzy_number -> Fuzzy_number
op fuzzy_max :  Fuzzy_number * Fuzzy_number -> Fuzzy_number
```

The definition of each operation is given in the spec FUZZY_ARITHM, which is included in the Appendix B.

Let $A$ and $B$ denote two fuzzy numbers and $*$ denote any of the four basic arithmetic operations. Then, $A * B$ is defined by the equation,

$$(A * B)(z) = \sup_{z=x*y} \min[A(x), B(y)]$$

for all $z \in \mathbb{R}$. More specifically, the four arithmetic operations are defined as,

$$(A + B)(z) = \sup_{z=x+y} \min[A(x), B(y)]$$
$$(A - B)(z) = \sup_{z=x-y} \min[A(x), B(y)]$$
$$(A \cdot B)(z) = \sup_{z=x\cdot y} \min[A(x), B(y)]$$
$$(A/B)(z) = \sup_{z=x/y} \min[A(x), B(y)]$$

| Arithmetic Mapping Table | |
|---|---|
| CRISP | FUZZY |
| Number | Fuzzy Number (FN) |
| $+$ | $+ : FN \times FN \rightarrow FN$<br>$(A + B)(z) = \sup_{z=x+y} min[A(x), B(y)], \forall z \in \mathbb{R}$ |
| $-$ | $- : FN \times FN \rightarrow FN$<br>$(A - B)(z) = \sup_{z=x-y} min[A(x), B(y)], \forall z \in \mathbb{R}$ |
| $\cdot$ | $\cdot : FN \times FN \rightarrow FN$<br>$(A \cdot B)(z) = \sup_{z=x\cdot y} min[A(x), B(y)], \forall z \in \mathbb{R}$ |
| $\div$ | $\div : FN \times FN \rightarrow FN$<br>$(A \div B)(z) = \sup_{z=x\div y} min[A(x), B(y)], \forall z \in \mathbb{R}$ |

Table 6.2: Arithmetic Mapping Table

Arithmetic Mapping Table gives the mapping of crisp operators to fuzzy equivalent where $*$ represents a fuzzy operator.

## 6.7 Fuzzy Information Processing

Fuzzy information processing consists of three stages which are : fuzzification, fuzzy reasoning, and defuzzification. They are explained in the following

subsections.

### 6.7.1 Fuzzification

There are two kinds of approaches for fuzzification. The first approach is to fuzzify the input measurements by introducing one kind of fuzzification functions to each input variable to express the associated measurement uncertainty. The purpose of the fuzzification function is to interpret measurements of input variables, each expressed by a real number, for instance the intensity value of each pixel, as more realistic fuzzy approximation of the respective real numbers. This fuzzification function, which is also called membership function can be of the types *triangular, trapezoidal, gaussian* and so on.

Another way for fuzzification is to generate several fuzzy property sets out of the original image. For example, Dark, Normal, Bright fuzzy sets can be generated from an image, with each pixel corresponding to a value between 0 and 1, representing the likelihood the pixel belongs to each set.

Here the first approach is chosen since the fuzzy numbers can be used in this approach. For a given value $c$, the triangular fuzzy number $A$ is defined, such that for all $x \in Real$, $A(x)$ satisfies the equation :

$$A(x) = \begin{cases} 0, & \text{if } x < c - \delta \\ & \text{or } x > c + \delta \\ \frac{(x-c+\delta)}{\delta}, & \text{if } c - \delta \leq x \leq c \\ \frac{(c+\delta-x)}{\delta}, & \text{if } c \leq x \leq c + \delta \end{cases}$$

In this equation, $\delta$ represents the uncertainty level. The larger the $\delta$, the more uncertain the input data.

And also for a given value $c$, the trapezoidal fuzzy number $B$ is defined, such that for all $x \in \mathrm{Re}\,al$, $B(x)$ satisfies the equation :

$$\begin{cases} 0, & \text{if } x < c - \delta_1 - \delta_2 \\ & \text{or } x > c + \delta_1 + \delta_2 \\ 1, & \text{if } x < c - \delta_1 \\ & \text{or } x > c + \delta_1 \\ \frac{(x-c+\delta_1+\delta_2)}{\delta_2}, & \text{if } c - \delta_1 - \delta_2 \leq x \leq c - \delta_1 \\ \frac{(c+\delta_1+\delta_2-x)}{\delta_2}, & \text{if } c + \delta_1 \leq x \leq c + \delta_1 + \delta_2 \end{cases}$$

In this equation, $\delta_{1,2}$ represents the uncertainty level. The larger $\delta_{1,2}$ is, the more uncertain the input data is.

For a given value $c$, the Gaussian fuzzy number $C$ is defined, such that for all $x \in \mathrm{Re}\,al$, $C(x)$ satisfies the equation :

$$C(x) = \exp(-(\frac{x-\mu}{\sigma})^2)$$

One kind of typical input data for an information fusion system is image, which is generally sampled into a rectangular array of pixels. Each pixel has an x-y coordinate that corresponds to its location within the image, and an intensity value representing brightness of a pixel.

The spec IMAGE imports INTEGER and Real, and defines a type.

```
IMAGE = spec
import Real
type Image = Integer * Integer -> Real
endspec
```

The spec FUZZIFICATION is generated by taking the colimit of IMAGE and FUZZY_REASONING, which is going to be explained in the next subsection.

```
FUZZIFICATION = spec
import IMAGE, FUZZY_REASONING
type Nz = {i:Real | i ~=zero}
endspec
```

Using the FUZZIFICATION spec, three different versions of fuzzification are generated : TRI_FUZZIFY, TRA_FUZZIFY and GAUSS_FUZZIFY, representing triangular, trapezoidal and gaussian fuzzification methods. The detailed specs are included at the end in Appendix B.

In each spec, two *operations* are defined. For instance in TRI_FUZZIFY, the following two operators are defined :

```
op tri_fuzzify :  Real * Nz -> Fuzzy_number
op tri_fuzzify_2 :  Real -> Fuzzy_number
```

where operation *tri_fuzzify* takes a crisp number and some uncertainty level, and generates a fuzzy triangular number. The operation *tri_fuzzify_2* deals with the situation when the uncertainty level is zero, which means there is no fuzziness about the result. The latter operation is specified so that a crisp number can also be regarded as a fuzzy number.

### 6.7.2 Fuzzy Reasoning

In chapter 6.5, basic fuzzy reasoning operators were extended from the crisp logic. In order to do fuzzy reasoning, the whole reasoning process has to be done in fuzzy logic so that the fuzziness, or the level of uncertainty, can be maintained and could be reasoned. In another word, fuzzy logic have to be used. *Fuzzy equal, fuzzy less than and equal*, as well as *fuzzy greater than and equal* are three fuzzy logic operators going to be used in the approach. And *fuzzy implication, fuzzy biimplication, fuzzy and, fuzzy or* and *fuzzy negation* are fuzzy logical connectives that should be defined.

Each fuzzy logical operator or connective is associated with a mapping $[0,1]^2 \rightarrow [0,1]$. So the truth value of a proposition is admitted as the range of truth values. And the truth value of a combined composition can be determined from the truth values of the atomic propositions. These were all explained in details in chapter 6.5.

Fuzzy reasoning operators are specified in the spec FUZZY_REASONING, which is a *definitional extension* of the spec FUZZY_ARITHM, with the fuzzy operations being of the following type:

```
op fequal :  Fuzzy_number * Fuzzy_number -> Uni_intvl
op fgeq :  Fuzzy_number * Fuzzy_number -> Uni_intvl
op fleq :  Fuzzy_number * Fuzzy_number -> Uni_intvl
```

In the spec FUZZY_REASONING only Lukasiewicz implication was specified. There are also other applicable versions which were explained in details, but can be specified as a future work. The crisp to fuzzy logic table for these reasoning operators were shown in Table 6.1.

### 6.7.3 Defuzzification

This is the process of calculating single-output numerical value for a fuzzy output variable on the basis of the inferred resulting membership function for this variable. The input in the defuzzification process is a fuzzy number and the output is a crisp number.

There are several defuzzification methods proposed in the literature : *centroid calculation*, which returns the center of the area under the curve of the fuzzy number; *center of maximum*, which returns the average of the maximum value of the fuzzy number; *largest of maximum*, and *smallest of maximum*. As

an example for the specifications the center of maximum method is chosen to implement the defuzzification process.

In this method, the defuzzified value, *defuzzify(F)*, is defined as the average of the smallest value and the largest value, for which *height(F)* is 1:

Defuzzification is implemented in the spec DEFUZZIFICATION, which is a definitional extension of the spec FUZZY_NUMBER. The defuzzify operation is defined as :

```
op defuzzify :  Fuzzy_number -> Real
```

$$defuzzify(F) = \frac{(inf(M) + sup(M))}{2}$$

where $M$ = an interval [z] s.t. $F(z) = height(F) = 1$

Another version of this method is also specified by using $\alpha - cut$ :

$$defuzzify(F) = \frac{(inf(M) + sup(M))}{2}$$

where $M$ = an interval [z] s.t. $F(z) =^{\alpha} F$

All specifications about this spec is also included in the Appendix B.

# Chapter 7

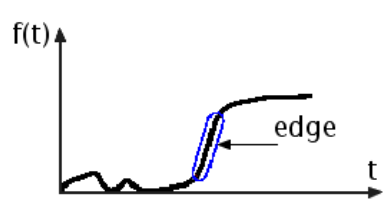
# AN EXAMPLE: EDGE DETECTION ALGORITHM

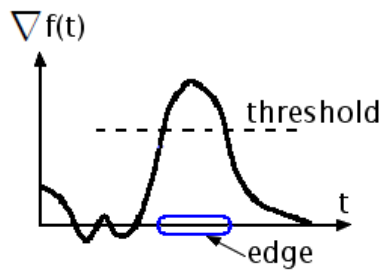
## 7.1 Edge Detection Algorithm

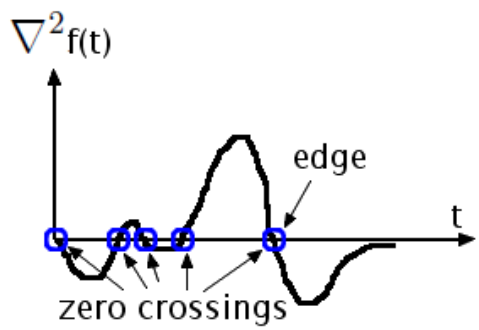
An edge in an image could be considered as a boundary at which a significant change of intensity, $I$, occurs.

Detecting an edge is very useful in object identification, because edges represent shapes of objects. There are many algorithms for edge detection. The objective of an edge detection algorithm is to locate the regions where the *intensity* is changing rapidly. So the whole process can be decomposed into two steps, the first is to derive edge points in an image, the second is to apply edge detection method only to these points.

For instance, if the gradient of this signal is taken (which, in one dimension, is just the first derivative with respect to t) the following occurs :






Clearly, the gradient has a large peak centered around the edge. By comparing the gradient to a threshold, an edge can be detected whenever the threshold is exceeded (as shown above). In this case, the edge is found, but the edge has become "thick" due to the thresholding. However, since it is known that the edge occurs at the peak, it can be localized by computing the Laplacian (in one dimension, the second derivative with respect to t) and finding the zero crossings.

The above figure shows the Laplacian of one-dimensional signal. As expected, the edge corresponds to a zero crossing, but other zero crossings are also seen which correspond to small ripples in the original signal.

Here, the Laplacian-based method is used to derive edge points. Edge points are where the second-order derivatives of the points are zero [7] . So edge points can be searched by looking for zero-crossing points of $\nabla^2 I(x, y)$, which can be calculated by the equation

$$\nabla^2 I(x,y) = I(x+1,y) + I(x-1,y) + I(x,y+1) + I(x,y-1) - 4I(x,y)$$

In order to avoid false edge points, local variance is estimated and compared with a threshold. The local variance can be estimated by

$$\sigma^2(x,y) = \frac{1}{(2M+1)^2} \sum_{k_1=x-M}^{x+M} \sum_{k_2=y-M}^{y+M} [(I(k_1,k_2) - m(k_1,k_2)]^2$$

where

$$m(x,y) = \frac{1}{(2M+1)^2} \sum_{k_1=x-M}^{x+M} \sum_{k_2=y-M}^{y+M} I(k_1,k_2)$$

with M typically chosen around 2. Since $\sigma^2(x,y)$ is compared with a threshold, the scaling factor $\frac{1}{(2M+1)^2}$ can be eliminated.

Zero-crossing detection means comparing a pixel (say x, y) with the adjacent ones on the next row and next column. Declaring zero-crossing points as edges results in a large number of points being declared to be edge points.

If there is an 'zero-crossing' action, that particular pixel is accepted as one of the

edge components, where

$$\text{if } pixel(x,y) < 0 \text{ and } pixel(x,y+1) > 0 \text{ or } pixel(x+1,y) > 0,$$

$$\text{then } pixel(x,y) = \text{ zero-crossing point}$$

$$\text{else if } pixel(x,y) > 0 \text{ and } pixel(x,y+1) < 0 \text{ or } pixel(x+1,y) < 0,$$

$$\text{then } pixel(x,y) = \text{ zero-crossing point}$$

A pixel at $(x,y)$ satisfies an *edge point* if and only if the second-order derivatives of the points are zero-crossing points and the local variance is greater than or equal to the threshold. The illustration of this definition is shown in figure below:

```
axiom edge_point?_def is
fa(I: Image, x:  Integer, y:  Integer)
(edge_point?(I, x, y) <=>
(zero_crossing_1(I,x,y) || zero_crossing_2(I,x,y)) &&
(thrd < var(I, x, y)))
```

## 7.2 Fuzzy Edge Detection Modeling

After setting up the Specware and Isabelle environment, the following Specware specifications were built that are needed to model Fuzzy Edge Detection.

- Fuzzy Set
- Fuzzy Number
- Fuzzy Arithmetic
- Fuzzy Reasoning
- Image
- Fuzzification
- Triangular Fuzzification
- Defuzzification
- Edge Point
- Edge and Fuzzy Edge

SNARK and Isabelle theorem provers were used to prove proof obligations. SNARK, proves theorems by using automatic tactics, however it could prove only simple theorems. However some options were changed. So the Isabelle

theorem prover was chosen to prove proof-obligations for all. Although most of proof-obligations could be proved by Isabelle, one of the theorems about FuzzyEdgeDetection still could not be proved.

Since Specware has not provided Real numbers library, it is needed to define Real numbers as Natural numbers in specifications.

$$\begin{array}{ll} \text{Real Number,} & [0,1] \\ \text{Natural Numbers,} & [0,100] \end{array}$$

### 7.2.1 Fuzzy Set

Fuzzy set constitutes the base of a fuzzy system. Unit interval is needed to define fuzzy set. So UNI_INTVL spec is as follows:

```
UNI_INTVL = spec
op between_zero_one?(x:Nat):  Boolean =
x < 100 && 0 <= x
type Uni_intvl = (Nat | between_zero_one?)
endspec
```

Figure 7.1: Unit Interval specification

Here, operation *between_zero_one?* is a boolean function; it returns *true* if number $x$ is between 0 and 100. Otherwise it returns $false$. Using this operation a new type *Uni_intvl* was built. It is a set that includes natural numbers only between 0 and 100.

Hence, fuzzy set is declared as a *type* by importing FiniteSet and UNI_INTVL. It consists of a finite set and creates a new set in *Uni_intvl* as shown in figure 7.2. Here $a$ denotes a finite set.

Fuzzy set operations - fuzzy complement, intersection, union, alpha-cut, and height - are declared as operations as shown in figure 7.3.

Fuzzy set operations are defined in figure 7.4. They are binary operations on the unit interval.

```
import /Library/General/FiniteSet, UNI_INTVL
%% Fuzzy set declaration as a type
type Fuzzy_set a = a -> Uni_intvl
```

Figure 7.2: Type *Fuzzy Set* definition

```
op [a] fuzzy_complement :  Fuzzy_set a -> Fuzzy_set a
op [a] t_norm :  Fuzzy_set a * Fuzzy_set a -> Fuzzy_set a
op [a] t_conorm :  Fuzzy_set a * Fuzzy_set a -> Fuzzy_set a
op [a] alpha_cut :  Fuzzy_set a * Uni_intvl -> FiniteSet a
```

Figure 7.3: Declaration of Fuzzy Set operations

### 7.2.2 Fuzzy Number (*FuzzyNumber.sw*)

Fuzzy number is declared as a *type* by importing FUZZY_SET as shown in figure 7.5. A fuzzy number is a function, thus the type of fuzzy numbers is a type of functions.

When the *height* of fuzzy number is equal to 1 it is *normal*. Since unit interval was defined as a set of numbers between 0 and 100, 100 is used instead of 1 to denote the full membership.

Also *convexity* can be defined as in figure 7.7.

```
def [a] fuzzy_complement(f1) = fn d -> 100 - f1(d)
def [a] t_norm(f1, f2) = fn d -> min(f1(d), f2(d))
def [a] t_conorm (f1, f2) = fn d -> max(f1(d), f2(d))
def [a] alpha_cut (f, a) = fn d -> a <= f(d)
def [a] height(f:  Fuzzy_set a):  Uni_intvl =
the(h) (fa(x:  a) f x <= h) && (ex(x:  a) f x = h)
```

Figure 7.4: Definition of Fuzzy Set operations

```
import FUZZY_SET
type Fuzzy_number = Fuzzy_set Nat
```

Figure 7.5: Type *Fuzzy Number* definition

```
axiom normality is
fa(f:Fuzzy_number)
height(f) = 100
```

Figure 7.6: Definition of *normality*

### 7.2.3 Fuzzy Arithmetic (*FuzzyArithm.sw*)

In this specification, arithmetic operations (addition, substraction, multiplication, division) on fuzzy numbers are declared and defined. Fuzzy number specification is required to define arithmetic operations, so first FUZZY_NUMBER spec is imported. Fuzzy addition is declared as an operation that consists of two fuzzy numbers and it yields a fuzzy number. Fuzzy addition declaration is shown in figure 7.8. Other operations are declared in a similar way.

Definitions of these operations are specified following the second method suggested by Klir. Here the fuzzy numbers are represented by continuous membership functions.

$\star$ in $z = x \star y$ equation denotes any of the four basic arithmetic operations.

```
axiom convexity is
fa(f:Fuzzy_number, x1:Nat, x2:Nat, lamd:  Uni_intvl)
f(lamd * x1) + ((100 - lamd) * x2) >= min(f(x1), f(x2))
```

Figure 7.7: Definition of *convexity*

```
%% FUZZY NUMBER spec is imported.
import FUZZY_NUMBER
%% Fuzzy addition declaration
op fuzzy_add :  Fuzzy_number * Fuzzy_number -> Fuzzy_number
%% Substraction, Multiplication, and Division are declared
by same way.
```

Figure 7.8: Op *fuzzy_add* declaration

```
axiom fuzzy_add_def is
fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat,
a:Uni_intvl)
fuzzy_add(f1, f2)(z) = a <=> z=x+y => (f1(x) <= a || f2(y) <=
a) &&
(ex(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat,
a:Uni_intvl)
z=x+y && ((f1(x) = a && a <= f2(y)) || (f2(y) = a && a <=
f1(x))))
```

Figure 7.9: Op *fuzzy_add* definition

So changing all $z = x + y$ to $z = x - y$, $z = x * y$, and $x = z * y$ in specification above yields other operations on fuzzy numbers. These specifications are available in the appendix.

Lattice operations on fuzzy numbers can be specified as figure 7.10.

### 7.2.4 Fuzzy Reasoning (*FuzzyReasoning.sw*)

Fuzzy equal, fuzzy less than and equal, as well as fuzzy greater than and equal are three operators that have been specified in this part. They are declared as operations such that each operation consists of two fuzzy number and yields a number in unit interval.

Definitions for these three operations are shown in figure 7.12.

```
axiom fuzzy_min_def is
fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat,
a:Uni_intvl)
fuzzy_min(f1, f2)(z) = a <=> z= min(x,y) => (f1(x) <= a ||
f2(y) <= a) &&
(ex(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat,
a:Uni_intvl)
z= min(x,y) && ((f1(x) = a && a <= f2(y)) || (f2(y) = a && a
<= f1(x))))
```

Figure 7.10: Op *fuzzy_min* definition

```
op fequal :  Fuzzy_number * Fuzzy_number -> Uni_intvl
op fgeq :  Fuzzy_number * Fuzzy_number -> Uni_intvl
op fleq :  Fuzzy_number * Fuzzy_number -> Uni_intvl
```

Figure 7.11: Declaration of fuzzy reasoning operators

### 7.2.5 Image (*Image.sw*)

A type named *image* is declared such that it consists of pixels, each including two integers plus a natural number. The two integer numbers such as $x$, and $y$ denote pixel of an image. The intensity is represented by a natural number. Type declaration for image can be seen in figure 7.13.

### 7.2.6 Fuzzification (*Fuzzification.sw*)

In this specification, IMAGE and FUZZY_REASONING are imported and a type named *Nz* is declared which is a set of natural numbers except zero. This spec is written to be used in the fuzzification method (triangular, trapezoidal, gaussian) specifications. The Fuzzification spec is shown in Figure 7.14.

```
axiom fequal_def is
fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat)
fequal(f1, f2) = inf (100 - (abs(f1(x) - f2(x))))
axiom fgeq_def is
fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat)
fgeq(f1, f2) = inf (min ((100 - f1(x) + f2(x)),100))
axiom fleq_def is
fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat)
fleq(f1, f2) = inf (min ((100 - f2(x) + f1(x)),100))
```

Figure 7.12: Definition of fuzzy reasoning operators

```
IMAGE = spec
type Image = Integer * Integer -> Nat
endspec
```

Figure 7.13: Type *Image* definition

### 7.2.7 Triangular Fuzzification (*TriFuzzify.sw*)

Triangular fuzzification is specified by importing FUZZIFICATION. A type named *Tri_fuzzy_image* is declared as shown in figure 7.15. This type denotes triangular fuzzified images such that two integers denote a pixel with a fuzzy number representing the fuzzy intensity of the pixel.

Minimum and maximum as lattice operations on unit interval are defined next.

```
FUZZIFICATION = spec
import IMAGE, FUZZY_REASONING
type Nz = {i:Nat | i~=0}
endspec
```

Figure 7.14: Type *Nz* definition

```
import FUZZIFICATION
type Tri_fuzzy_image = Integer * Integer -> Fuzzy_number
```

Figure 7.15: Specification of Fuzzy Set

```
op uni_min :  Uni_intvl * Uni_intvl -> Uni_intvl
op uni_max :  Uni_intvl * Uni_intvl -> Uni_intvl
def uni_min(u1,u2) = if u1 <= u2 then u1 else u2
def uni_max(u1,u2) = if u2 <= u1 then u2 else u1
```

Figure 7.16: Declaration and definition of *uni_min* and *uni_max*

To generate a triangular membership function for each crisp value, an operation named *tri_fuzzify* is defined as shown in figure 7.17. The uncertainty level(*delta*) is given as an operation (*constant*). Definition of *tri_fuzzify_2* deals with the situation when uncertainty level is zero.

```
op tri_fuzzify :  Nat * Nz -> Fuzzy_number
op tri_fuzzify_2 :  Nat -> Fuzzy_number
op tri_fuzzify_image :  Image * Nz -> Tri_fuzzy_image
op fuzzify :  Nat * Nz -> Fuzzy_number
op fuzzify_1 :  Nat * Nz -> Fuzzy_number
op delta :  Nz
```

Figure 7.17: Triangular fuzzification op declarations

For a given value $c$, the triangular fuzzy number $A$ is defined, such that for all $z \in \mathbb{R}$, $A(z)$ satisfies the equation;

$$A(z) = \begin{cases} 0, & \text{if } z < c - \delta \text{ or } z > c + \delta \\ \frac{(z-c+\delta)}{\delta}, & \text{if } c - \delta \leq z \leq c \\ \frac{(c+\delta-z)}{\delta}, & \text{if } c \leq z \leq c + \delta \end{cases}$$

Specifications for *tri_fuzzify*, and *tri_fuzzify_2* are shown in figure 7.18.

```
axiom tri_fuzzify_def is
fa(e:  Nat, f:  Fuzzy_number)
(tri_fuzzify(e, delta) = f <=>
(fa(x:  Nat)(x < (e - delta) || (e + delta) < x <=>
f(x) = 0 || (e - delta) <= x && x <= e <=>
f(x) = div((x - e + delta), (delta)) || e <= x && x <=
(e + delta) <=>
f(x) = div((e - x + delta), (delta)))))


axiom tri_fuzzify_2_def is
fa(e:  Nat, f:  Fuzzy_number)
(tri_fuzzify_2(e) = f <=>
(fa(x:  Nat)(x < e || e < x <=>
f(x) = 0 || x <= e && e <= x <=>
f(x) = 1)))
```

Figure 7.18: Definitions of *tri_fuzzify* and *tri_fuzzify_2*

Operation tri_fuzzify_image yields fuzzy images that have triangular membership degrees in type *tri_fuzzy_image*.

```
axiom tri_fuzzify_image_def is
fa(I: Image, F:Tri_fuzzy_image)
(tri_fuzzify_image(I, delta) = F <=> (fa(x:  Integer, y:  Integer)
(F(x,y) = tri_fuzzify(I(x, y), delta))))
```

Figure 7.19: Op *tri_fuzzify_image* definition

In order to use the triangular fuzzy numbers, the arithmetic equations on necessary operations were given subsection 7.2.3.

### 7.2.8 Defuzzification (*defuzzification.sw*)

The defuzzify operation is defined as :

$$defuzzify(F) = \frac{(inf(M) + sup(M))}{2}$$

where $M$ = an interval [z] s.t. $F(z) = height(F) = 1$

Hence, specification of defuzzification is:

```
op defuzzify_1 :  Fuzzy_number -> Nat
def defuzzify_1(F) = div((inf(height_intvl(F)) +
sup(height_intvl(F))),(1 + 1))
```

Figure 7.20: Op *defuzzify_1* declaration and definition

Here, *inf* and *sup* denote infimum and supremum. *height_intvl* is an operation takes a fuzzy number and yields a set of natural numbers. Specifications for *inf*, *sup* and *height_intvl* can be seen in figure 7.21.

```
op inf :  Intvl -> Nat
op sup :  Intvl -> Nat
op height_intvl :  Fuzzy_number -> Intvl
axiom inf_def is
fa(I: Intvl, a:  Nat, x:  Nat)
(inf(I) = a <=>
(x in?  I) => a <= x)
axiom sup_def is
fa(I: Intvl, a:  Nat, x:  Nat)
(sup(I) = a <=>
(x in?  I) => x <= a)
axiom height_intvl_def is
fa(F: Fuzzy_number, I: Intvl, x:  Nat)
(height_intvl(F) = I <=>
F(x) = 100 <=> (x in?  I))
```

Figure 7.21: Declaration and definition of operations used in *defuzzify_1*

Another version of defuzzification is also specified by using $\alpha - cut$ :

$$defuzzify(F) = \frac{(inf(M) + sup(M))}{2}$$

where $M$ = an interval [z] s.t. $F(z) =^{\alpha} F$

Hence,

```
op defuzzify_2 :  Fuzzy_number -> Nat
def defuzzify_2(F) = div((inf(alpha_intvl(F))+
sup(alpha_intvl(F))),(1 + 1))
```

Figure 7.22: Op *defuzzify_2* declaration and definition

Specifications for *inf* and *sup* are same as above. alpha_intvl definition is shown in figure 7.23.

```
op alpha_intvl :  Fuzzy_number -> Intvl
op alpha :  Uni_intvl %% alpha is constant
axiom alpha_intvl_def is
fa(F: Fuzzy_number, I: Intvl, x:  Nat)
(alpha_intvl(F) = I <=>
F(x) = alpha <=> (x in?  I))
```

Figure 7.23: Op *alpha_intvl* declaration and definition

### 7.2.9 Fuzzy Edge (*fuzzyEdge.sw*)

In this part of specifications, *Laplacian*, *zerocrossing* and *local variance* are formulated and defined according to the arithmetic operations of triangular fuzzy numbers. The first part which has to be defined in the definition of fuzzy_edge_point is the Laplacian part. Since Laplacian is;

$$\nabla^2 I(x,y) = I(x+1,y) + I(x-1,y) + I(x,y+1) + I(x,y-1) - 4I(x,y)$$

Hence, Laplacian is specified like in figure 7.24.

```
op flap :  Tri_fuzzy_image * Integer * Integer -> Fuzzy_number

axiom flap_def is
fa(F: Tri_fuzzy_image, x:  Integer, y:  Integer)
flap(F, x, y) =
fuzzy_sub (fuzzy_add (fuzzy_add (F((x+1), y),
F((x-1), y)),
fuzzy_add (F(x, (y+1)),
F(x, (y-1)))),
fuzzy_add (fuzzy_add (F(x,y), F(x,y)),
fuzzy_add (F(x,y), F(x,y))))
```

Figure 7.24: Laplacian declaration and definition as an operation

In order to explain the definition of *flap*, it is decomposed into parts which are named as follows, and then the arithmetic operation definitions are substituted into particular parts.

$$
\begin{aligned}
L_1 &= F((x+1), y) \\
L_2 &= F((x-1), y) \\
L_3 &= F(x, (y+1)) \\
L_4 &= F(x, (y-1)) \\
L_5 &= F(x, y) \\
L_{12} &= fuzzy\_add(F((x+1), y), F((x-1), y)) \\
L_{34} &= fuzzy\_add(F(x, (y+1)), F(x, (y-1))) \\
L_{1234} &= fuzzy\_add(L_{12}, L_{34}) \\
L_6 &= fuzzy\_mult(4, L_5)
\end{aligned}
$$

$$flap = fuzzy\_sub(L_{1234}, L_6)$$

After separating each part of the definition, the definition of fuzzy Laplacian operator can be substituted into the particular parts. Arithmetic equations of flap:

$$\mu_{L_{12}}(z) = \mu_{L_1}(z) \widetilde{+} \mu_{L_2}(z)$$

$$\mu_{L_{12}}(z) = \begin{cases} 0, & \text{if } z < L_1 + L_2 - 2\delta \text{ or } z > L_1 + L_2 + 2\delta \\ \frac{z-(L_1+L_2)+2\delta}{2\delta}, & \text{if } L_1 + L_2 - 2\delta \leq z \leq L_1 + L_2 \\ \frac{(L_1+L_2+2\delta-z)}{2\delta}, & \text{if } L_1 + L_2 \leq z \leq L_1 + L_2 + 2\delta \end{cases}$$

$$\mu_{L_{34}}(z) = \mu_{L_3}(z) \widetilde{+} \mu_{L_4}(z)$$

$$\mu_{L_{34}}(z) = \begin{cases} 0, & \text{if } z < L_3 + L_4 - 2\delta \text{ or } z > L_3 + L_4 + 2\delta \\ \frac{z-(L_3+L_4)+2\delta}{2\delta}, & \text{if } L_3 + L_4 - 2\delta \leq z \leq L_3 + L_4 \\ \frac{(L_3+L_4+2\delta-z)}{2\delta}, & \text{if } L_3 + L_4 \leq z \leq L_3 + L_4 + 2\delta \end{cases}$$

$$\mu_{L_{1234}}(z) = \mu_{L_{12}}(z) \widetilde{+} \mu_{L_{34}}(z)$$

$$\mu_{L_{1234}}(z) = \begin{cases} 0, & \text{if } z < L_1 + L_2 + L_3 + L_4 - 4\delta \\ & \text{or } z > L_1 + L_2 + L_3 + L_4 + 4\delta \\ \frac{z-(L_1+L_2+L_3+L_4)+4\delta}{4\delta}, & \text{if } L_1 + L_2 + L_3 + L_4 - 4\delta \leq z \leq \\ & L_1 + L_2 + L_3 + L_4 \\ \frac{(L_1+L_2+L_3+L_4+4\delta-z)}{4\delta}, & \text{if } L_1 + L_2 + L_3 + L_4 \leq z \leq \\ & L_1 + L_2 + L_3 + L_4 + 4\delta \end{cases}$$

$$\mu_{L_6}(z) = 4 \widetilde{*} \mu_{L_5}(z)$$

$$\mu_{L_6}(z) = \begin{cases} 0, & \text{if } z < 4(L_5 - \delta) \\ & \text{or } z > 4(L_5 + \delta) \\ \frac{z-4(L_5-\delta)}{4(\delta)}, & \text{if } 4(L_5 - \delta) \leq z \leq 4(L_5) \\ \frac{4(L_5+\delta)-z}{4(\delta)}, & \text{if } 4(L_5) \leq z \leq 4(L_5 + \delta) \end{cases}$$

$$\mu_{flap}(z) = \mu_{L_{1234}}(z) \widetilde{-} \mu_{L_6}(z)$$

$$\mu_{flap}(z) = \begin{cases} 0, & \text{if } z < L_1 + L_2 + L_3 + L_4 - 4L_5 - 8\delta \\ & \text{or } z > L_1 + L_2 + L_3 + L_4 - 4L_5 + 8\delta \\ \frac{z-(L_1+L_2+L_3+L_4-4L_5-8\delta)}{8\delta}, & \text{if } L_1 + L_2 + L_3 + L_4 - 4L_5 - 8\delta \\ & \leq z \leq L_1 + L_2 + L_3 + L_4 - 4L_5 \\ \frac{L_1+L_2+L_3+L_4-4L_5+8\delta-z}{8\delta}, & \text{if } L_1 + L_2 + L_3 + L_4 - 4L_5 \\ & \leq z \leq L_1 + L_2 + L_3 + L_4 + 4L_5 + 8\delta \end{cases}$$

Zero-crossing detection means comparing a pixel (say (x, y)) with the adjacent ones on the next row and next column. Declaring zero-crossing points as edge results in a large number of points being declared to be edge points. In the specs, zero-crossing is divided into two parts; *zero_crossing_1* and *zero_crossing_2*, which are fuzzified as shown in figure 7.25.

```
op fuzzy_zero_crossing_1 :
Tri_fuzzy_image * Integer * Integer -> Uni_intvl
axiom fuzzy_zero_crossing_1_def is
fa(F: Tri_fuzzy_image, x:  Integer, y:  Integer)
(fuzzy_zero_crossing_1(F, x, y) =
(uni_min (fleq (flap(F, x, y), tri_fuzzify(0, delta)),
(uni_max (fgeq (flap(F, x, (y+1)), tri_fuzzify(0,
delta)),
fgeq (flap(F, (x+1), y), tri_fuzzify(0, delta)))))))

op fuzzy_zero_crossing_2 :
Tri_fuzzy_image * Integer * Integer -> Uni_intvl
axiom fuzzy_zero_crossing_2_def is
fa(F: Tri_fuzzy_image, x:  Integer, y:  Integer)
(fuzzy_zero_crossing_2(F, x, y) =
(uni_min (fgeq (flap(F, x, y), tri_fuzzify(0, delta)),
(uni_max (fleq (flap(F, x, (y+1)), tri_fuzzify(0,
delta)),
fleq (flap(F, (x+1), y), tri_fuzzify(0, delta)))))))
```

Figure 7.25: Ops *fuzzy_zero_crossing_1* and *fuzzy_zero_crossing_2* declarations and definitions

For fuzzy_zero_crossing_1:

$\mu_{flap}(z)\widetilde{\leq}\mu_{zero}(z) = \inf\{\min(1 - \mu_{flap}(z) + \mu_{zero}(z), 1)|z \in Real\}$ which has Tri_fuzzy_image as $(F, x, y)$.

Since $\mu_{flap}(z), \mu_{zero}(z)$ are fuzzy_numbers, it is obvious that the result will be

$(1 - \mu_{flap}(z) + \mu_{zero}(z))$, if $(1 - \mu_{flap}(z) + \mu_{zero}(z))$ is named as $l_1$, then for Tri_fuzzy_image $(F, x, y)$,

$$\mu_{flap}(z)\widetilde{\leq}\mu_{zero}(z) \equiv l_1$$

$\mu_{flap}(z)\widetilde{\geq}\mu_{zero}(z) = \inf\{\min(1-\mu_{zero}(z)+\mu_{flap}(z),1)|z \in Real\}$ which has Tri_fuzzy_image as $(F, x, y+1)$.

Since $\mu_{flap}(z), \mu_{zero}(z)$ are fuzzy_numbers, it is obvious that the result will be

$(1-\mu_{zero}(z)+\mu_{flap}(z))$, if $(1-\mu_{zero}(z)+\mu_{flap}(z))$ is named as $l_2$, then for Tri_fuzzy_image $(F, x, y+1)$,

$$\mu_{flap}(z)\widetilde{\geq}\mu_{zero}(z) \equiv l_2$$

$\mu_{flap}(z)\widetilde{\geq}\mu_{zero}(z) = \inf\{\min(1-\mu_{zero}(z)+\mu_{flap}(z),1)|z \in Real\}$ which has Tri_fuzzy_image as $(F, x+1, y)$.

Since $\mu_{flap}(z), \mu_{zero}(z)$ are fuzzy_numbers, it is obvious that the result will be

$(1-\mu_{zero}(z)+\mu_{flap}(z))$, if $(1-\mu_{zero}(z)+\mu_{flap}(z))$ is named as $l_3$, then for Tri_fuzzy_image $(F, x+1, y)$,

$$\mu_{flap}(z)\widetilde{\geq}\mu_{zero}(z) \equiv l_3$$

and it is known from the specs that

op uni_min : Uni_intvl, Uni_intvl $\rightarrow$ Uni_intvl

op uni_max: Uni_intvl, Uni_intvl $\rightarrow$ Uni_intvl

The notation $\widetilde{\wedge} \equiv$ uni_min and $\widetilde{\vee} \equiv$ uni_max are used for the operators above.

$\widetilde{\wedge}: \ [0,1] \times [0,1] \rightarrow [0,1]$

$a\widetilde{\wedge}b = \min(a,b)$

$\widetilde{\vee}: \ [0,1] \times [0,1] \rightarrow [0,1]$

$a\widetilde{\vee}b = \max(a,b)$

So, fuzzy_zero_crossing_1 can be written as

$fuzzy\_zero\_crossing\_1(F,x,y) = l_1\widetilde{\wedge}(l_2\widetilde{\vee}l_3)$, if $(l_2\widetilde{\vee}l_3) \equiv l_4$ then

$fuzzy\_zero\_crossing\_1(F,x,y) = \min(l_1, (\max(l_2,l_3) = \min(l_1,l_4) = t_1$

And for fuzzy_zero_crossing_2

$\mu_{flap}(z)\widetilde{\geq}\mu_{zero}(z) = \inf\{\min(1-\mu_{zero}(z)+\mu_{flap}(z),1)|z \in Real\}$ which has Tri_fuzzy_image as $(F,x,y)$.

Since $\mu_{flap}(z), \mu_{zero}(z)$ are fuzzy_numbers, it is obvious that the result will be

$(1-\mu_{zero}(z)+\mu_{flap}(z))$, if $(1-\mu_{zero}(z)+\mu_{flap}(z))$ is named as $g_1$, then for Tri_fuzzy_image $(F,x,y)$,

$$\mu_{flap}(z)\widetilde{\geq}\mu_{zero}(z) \equiv g_1$$

$\mu_{flap}(z)\widetilde{\leq}\mu_{zero}(z) = \inf\{\min(1-\mu_{flap}(z)+\mu_{zero}(z),1)|z \in Real\}$ which has Tri_fuzzy_image as $(F,x,y+1)$.

Since $\mu_{flap}(z), \mu_{zero}(z)$ are fuzzy_numbers, it is obvious that the result will be

$(1-\mu_{flap}(z)+\mu_{zero}(z))$, if $(1-\mu_{flap}(z)+\mu_{zero}(z))$ is named as $g_2$, then for Tri_fuzzy_image $(F,x,y+1)$,

$$\mu_{flap}(z)\widetilde{\leq}\mu_{zero}(z) \equiv g_2$$

$\mu_{flap}(z)\widetilde{\leq}\mu_{zero}(z) = \inf\{\min(1-\mu_{flap}(z)+\mu_{zero}(z),1)|z \in Real\}$ which has Tri_fuzzy_image as $(F,x+1,y)$.

Since $\mu_{flap}(z), \mu_{zero}(z)$ are fuzzy_numbers, it is obvious that the result will be

$(1-\mu_{flap}(z)+\mu_{zero}(z))$, if $(1-\mu_{flap}(z)+\mu_{zero}(z))$ is named as $g_3$, then for Tri_fuzzy_image $(F,x+1,y)$,

$$\mu_{flap}(z)\widetilde{\leq}\mu_{zero}(z) \equiv g_3$$

So, fuzzy_zero_crossing_2 can be written as

$fuzzy_zero_crossing_2(F,x,y) = g_1\widetilde{\wedge}(g_2\widetilde{\vee}g_3)$, if $(g_2\widetilde{\vee}g_3) \equiv g_4$ then

$fuzzy\_zero\_crossing\_2(F,x,y) = \min(g_1,(\max(g_2,g_3) = \min(g_1,g_4) = t_2$

Fuzzy variance part consists of five operations. They are declared in figure 7.26.

```
op fvar :  Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
op fysum :  Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
op felmt :  Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
op fm :  Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
op fsub :  Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
```

Figure 7.26: Specification of Fuzzy Variance

The local variance is

$$\sigma^2(x,y) = \frac{1}{(2M+1)^2} \sum_{k_1=x-M}^{x+M} \sum_{k_2=y-M}^{y+M} [(I(k_1,k_2) - m(k_1,k_2)]^2$$

where

$$m(x,y) = \frac{1}{(2M+1)^2} \sum_{k_1=x-M}^{x+M} \sum_{k_2=y-M}^{y+M} I(k_1,k_2)$$

with M typically chosen around 2. Since $\sigma^2(x,y)$ is compared with a threshold, the scaling factor can be eliminated. In the m formula, $I(k_1,k_2)$ is the intensity value. Within the intervals given in the formula, there will be different values for each intensity. When the formula m is fuzzified, all these different intensity values will have the same uncertainty level , which is $\delta$.

The result of the addition for two of these fuzzified intensity values will be

$$\mu_{I_1 \tilde{+} I_2}(z) = \begin{cases} 0, & \text{if } z < I_1 + I_2 - 2\delta \text{ or } z > I_1 + I_2 + 2\delta \\ \frac{z-(I_1+I_2)+2\delta}{2\delta}, & \text{if } I_1 + I_2 - 2\delta \leq z \leq I_1 + I_2 \\ \frac{(I_1+I_2+2\delta-z)}{2\delta}, & \text{if } I_1 + I_2 \leq z \leq I_1 + I_2 + 2\delta \end{cases}$$

where $I_1, I_2$ are two different intensity values with same uncertainty level $\delta$ ($\delta \in Nz$).

Since variance is going to be compared with a threshold, *threshold* is also fuzzified:

$$\mu_{threshold}(z) = \begin{cases} 0, & \text{if } z < I_{threshold} \\ & \text{or } z > I_{threshold} \\ 1, & \text{if } z = I_{threshold} \end{cases}$$

So, the comparison between fuzzy_variance and fuzzy_threshold can be written as follows:

$\mu_{var}(z)\widetilde{\geq}\mu_{threshold}(z) = \inf\{\min(1 - \mu_{threshold}(z) + \mu_{var}(z), 1)\}$

Since $\mu_{var}(z), \mu_{threshold}(z)$ are fuzzy-numbers, it is obvious that the result will be

$(1 - \mu_{threshold}(z) + \mu_{var}(z))$, if $(1 - \mu_{threshold}(z) + \mu_{var}(z))$ is named as $a$, which is a Uni-intvl,

$$\mu_{var}(z)\widetilde{\geq}\mu_{threshold}(z) \equiv a$$

$Fuzzy\_edge\_point(F, x, y) = uni\_\min(uni\_\max(t_1, t_2)), a)$ can be written.

If $t_1\widetilde{\vee}t_2 \equiv \max(t_1, t_2) = b$, then

$Fuzzy\_edge\_point(F, x, y) = a\widetilde{\wedge}b \equiv \min(a, b)$ will be a number in $[0, 1]$, where F:Tri_fuzzy_image, x,y:Integer and $\delta \in Nz$. After all the parts of the definition of fuzzy_edge_point is defined as shown in figure 7.27.

```
op Fuzzy_edge_point :  Tri_fuzzy_image * Integer *
Integer -> Uni_intvl
axiom Fuzzy_edge_point_def is
fa(F: Tri_fuzzy_image, x:  Integer, y:  Integer)
Fuzzy_edge_point(F, x, y) =
(uni_min (uni_max (fuzzy_zero_crossing_1(F, x, y),
fuzzy_zero_crossing_2(F, x, y)),
fequal (flap(F, x, y), tri_fuzzify(0, delta))))
```

Figure 7.27: Op *Fuzzy_edge_point* declaration and definition

### Theorem

In order to show that reasoning about the impact of the uncertainty of input information can be specified formally in every aspect of the fusion process, a theorem is implemented on the edge detection algorithm, which is :

$$delta1 \geq delta2 => Fuzzy\_edge\_point\_1 \geq Fuzzy\_edge\_point\_2$$

all $x, y \in Integer$, $I \in Image$, and $delta1, delta2 \in Nz$. Where delta1 and delta2 are two different values chosen to fuzzify the input data and represent the uncertainty levels of the input information, and $Fuzzy\_edge\_point\_1$ and $Fuzzy\_edge\_point\_2$ are the generated uncertainty values for deriving the results of $Fuzzy\_edge\_point(x, y)$ for two different fuzzified images.

```
theorem fuzzyEdge is
(fa(x:  Integer, y:  Integer, I: Image, delta1:  Nz,
delta2:  Nz)
(delta2 <= delta1) =>
(Fuzzy_edge_point(tri_fuzzify_image(I, delta2), x, y))
<=
(Fuzzy_edge_point(tri_fuzzify_image(I, delta1), x, y)))
```

Figure 7.28: Theorem *fuzzyEdge*

### 7.2.10 Proving in SNARK and Isabelle

As mentioned section 3.4, SNARK is an automated theorem prover. In SNARK, the user can make changes in execution process only by using options. With these capabilities, Theorem *fuzzyEdge* could not be proved by SNARK, although, some options were active. Theorem proving experiments with the fuzzyEdge theorem can be seen in Figure 7.29.

However, Isabelle is a powerful interactive theorem prover, the proof requires extensive knowledge and experience in logical calculus and Isabelle. In order to get Isabelle to prove theorem *fuzzyEdge*, all specifications are translated to Isabelle's language. Some basic theorems were proved. Isabelle proof terms embedded specifications can be seen in figures between 7.31-7.37. In order to prove these theorems, apply (sim add: ) method is used.

```
     theorem fuzzyEdge is
     (fa(x: Integer, y: Integer, I: Image, delta1: Nz, delta2: Nz)
        (delta2 <= delta1) =>
        (Fuzzy_edge_point(tri_fuzzify_image(I, delta2), x, y)) <=
        (Fuzzy_edge_point(tri_fuzzify_image(I, delta1), x, y)))

   endspec

   p1 = prove fuzzyEdge in FUZZY_EDGE

ISO8-----XEmacs: FuzzyEdgeDetection2.sw      (Specware Font hs)----L477--C37--Bot--------------------
creating directory: /home/keskemal/Specware-4-2-5/Examples/Kodlar/Snark/FuzzyEdgeDetection2/
    Expanded spec file: /home/keskemal/Specware-4-2-5/Examples/Kodlar/Snark/FuzzyEdgeDetection2/p1.sw
    Snark Log file: /home/keskemal/Specware-4-2-5/Examples/Kodlar/Snark/FuzzyEdgeDetection2/p1.log
p1: Theorem fuzzyEdge in FUZZY_EDGE is NOT proved using Snark in 0.3 seconds.
*
```

Figure 7.29: SNARK with no options.

```
   p1 = prove fuzzyEdge in FUZZY_EDGE
        options  "(use-resolution nil)
                  (use-hyperresolution t)
                  (use-negative-hyperresolution nil)
                  (use-paramodulation)
                  (use-factoring)
                  (use-literal-ordering-with-hyperresolution 'literal-ordering-p)
                  (use-literal-ordering-with-negative-hyperresolution 'literal-ordering-p)
                  (use-literal-ordering-with-resolution 'literal-ordering-a)
                  (use-literal-ordering-with-paramodulation 'literal-ordering-p)
                  (use-ac-connectives)
                  (run-time-limit 50)
                  (assert-supported nil)
                  (use-code-for-numbers nil)
                  (print-symbol-ordering)
                  (print-final-rows)"

ISO8-----XEmacs: FuzzyEdgeDetection2.sw      (Specware Font hs)----L492--C37--Bot--------------------
;;; Elaborating proof-term at /home/keskemal/Specware-4-2-5/Examples/Kodlar/FuzzyEdgeDetection2#p1
    Expanded spec file: /home/keskemal/Specware-4-2-5/Examples/Kodlar/Snark/FuzzyEdgeDetection2/p1.sw
    Snark Log file: /home/keskemal/Specware-4-2-5/Examples/Kodlar/Snark/FuzzyEdgeDetection2/p1.log
p1: Theorem fuzzyEdge in FUZZY_EDGE is NOT proved using Snark in 0.3 seconds.
*
```

Figure 7.30: SNARK with options.

As an example, Isabelle theory files for monotocity are shown in figures 7.37, 7.38, and 7.39:

Theorem *fuzzyEdge* could not be proved. Figure 7.37 shows first subgoal of theorem *fuzzyEdge* . In this subgoal, Fuzzy_edge_point and tri_fuzzify_image definitions are known, so it can be simplified by using these definitions.

Isabelle *apply(simp add: axiom1 axiom2)* rule are applied to first subgoal. Axioms: *Fuzzy_edge_point_def*, *tri_fuzzify_image_subtype_constr*, *tri_fuzzify_image_def*, *fuzzy_zero_crossing_1_def*, *fuzzy_zero_crossing_2_def*, and *tri_fuzzify_def* are used as shown in figure 7.37

After simplification *apply induction* rule is applied to *delta1* and *delta2*. Then two more simplications are applied. Lastly, *apply auto* rule is applied.

```
theorem complement is
fa(f: Fuzzy_set Nat, x: Nat)
fuzzy_complement(fuzzy_complement(f)) x = f x

proof Isa
by (auto simp add: between_zero_one_p_def
fuzzy_complement_def abc)
end-proof
```

Figure 7.31: Proof of theorem *Complement*

```
theorem commutativity is
fa(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat, x: Nat)
t_norm(f1, f2)(x) =t_norm(f2, f1)(x)

proof Isa
by (simp add: between_zero_one_p_def t_norm_def min_def)
end-proof
```

Figure 7.32: Proof of theorem *Commutativity*

After these processes, there existed one more subgoal. Since it is about a hundred lines long, it was not put here but can be found in the appendix.

```
theorem assosiativity is
fa(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat, f3: Fuzzy_set
Nat, x: Nat)
t_norm(f1, t_norm(f2, f3))(x) = t_norm(t_norm(f1, f2),
f3)(x)

proof Isa
by (simp add: between_zero_one_p_def t_norm_def min_def)
end-proof
```

Figure 7.33: Proof of theorem *Assosiativity*

```
theorem monotonicity is
fa(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat, f3:
Fuzzy_set Nat, x: Nat)
f2(x)<= f3(x) => t_norm(f1, f2)(x) <= t_norm(f1, f3)(x)

proof Isa
by (simp add: between_zero_one_p_def t_norm_def min_def)
end-proof
```

Figure 7.34: Proof of theorem *Monotonicity*

```
theorem theo1 is
fa(e: Nat, f: Fuzzy_number, x: Nat)
fuzzify(10, delta)(100) = 100

proof Isa:
by (simp add: fuzzify_def)
end-proof
```

Figure 7.35: Proof of theorem *Fuzzify*

```
theorem theo2 is
fa(u1:Uni_intvl, u2:  Uni_intvl)
uni_min(10, 2)= 2

proof Isa theo2_Obligation_subtype:
by (simp add:  between_zero_one_p_def)
end-proof

proof Isa theo2_Obligation_subtype0:
by (simp add:  between_zero_one_p_def)
end-proof

proof Isa theo2:
by (simp add:  between_zero_one_p_def uni_min_def)
end-proof
```

Figure 7.36: Proof of theorem *Uni_min*

```
theorem monotonicity:
  "\<lbrakk>Fun_PR between_zero_one_p (f1::nat \<Rightarrow> Uni_intvl);
    Fun_PR between_zero_one_p (f2::nat \<Rightarrow> Uni_intvl);
    Fun_PR between_zero_one_p (f3::nat \<Rightarrow> Uni_intvl);
    f2 x \<le> f3 x\<rbrakk> \<Longrightarrow>
   t_norm(f1, f2) x \<le> t_norm(f1, f3) x"
  apply (simp add: between_zero_one_p_def t_norm_def min_def)
done
ISO8--**-XEmacs: FuzzyEdgeDetection2_FUZZY_SET.thy      (Isar script Font

proof (prove): step 0

goal (1 subgoal):
 1. [| Fun_PR between_zero_one_p f1; Fun_PR between_zero_one_p f2;
       Fun_PR between_zero_one_p f3; f2 x <= f3 x |]
    ==> t_norm (f1, f2) x <= t_norm (f1, f3) x
```

Figure 7.37: Subgoal for *Monotonicity*.

```
theorem monotonicity:
  "\<lbrakk>Fun_PR between_zero_one_p (f1::nat \<Rightarrow> Uni_intvl);
    Fun_PR between_zero_one_p (f2::nat \<Rightarrow> Uni_intvl);
    Fun_PR between_zero_one_p (f3::nat \<Rightarrow> Uni_intvl);
    f2 x \<le> f3 x\<rbrakk> \<Longrightarrow>
   t_norm(f1, f2) x \<le> t_norm(f1, f3) x"
  apply (simp add: between_zero_one_p_def t_norm_def min_def)
done
ISO8--**-XEmacs: FuzzyEdgeDetection2_FUZZY_SET.thy      (Isar script Font

proof (prove): step 1

goal:
No subgoals!
```

Figure 7.38: Theorem *Monotonicity* proved in step 1 in Isabelle.

```
theorem monotonicity:
  "\<lbrakk>Fun_PR between_zero_one_p (f1::nat \<Rightarrow> Uni_intvl);
    Fun_PR between_zero_one_p (f2::nat \<Rightarrow> Uni_intvl);
    Fun_PR between_zero_one_p (f3::nat \<Rightarrow> Uni_intvl);
    f2 x \<le> f3 x\<rbrakk> \<Longrightarrow>
   t_norm(f1, f2) x \<le> t_norm(f1, f3) x"
  apply (simp add: between_zero_one_p_def t_norm_def min_def)
done
ISO8--**-XEmacs: FuzzyEdgeDetection2_FUZZY_SET.thy      (Isar script Font
theorem
  monotonicity:
    [| Fun_PR between_zero_one_p ?f1.0; Fun_PR between_zero_one_p ?f2.0;
       Fun_PR between_zero_one_p ?f3.0; ?f2.0 ?x <= ?f3.0 ?x |]
    ==> t_norm (?f1.0, ?f2.0) ?x <= t_norm (?f1.0, ?f3.0) ?x
```

Figure 7.39: Theorem *Monotonicity*.

```
theorem fuzzyEdge:
  "\<lbrakk>(delta1::nat) \<noteq> 0; (delta2::nat) \<noteq> 0; delta2 \<le> delta1\<rbrakk> \<Longrightarro
w>
   Fuzzy_edge_point(tri_fuzzify_image(I, delta2), x, y)
     \<le> Fuzzy_edge_point(tri_fuzzify_image(I, delta1), x, y)"
  apply (simp add: Fuzzy_edge_point_def  tri_fuzzify_image_subtype_constr tri_fuzzify_image_def  fuzzy_zero_
crossing_1_def fuzzy_zero_crossing_2_def tri_fuzzify_def)
  apply (induct_tac delta1)
  apply (induct_tac delta2)
  apply (simp)
  apply (simp)
  apply (auto
end

ISO8--**-XEmacs: FuzzyEdgeDetection2_FUZZY_EDGE.thy      (Isar script Font Scripting )----L123--C64--Bot-----

proof (prove): step 0

goal (1 subgoal):
 1. [| delta1 ~= 0; delta2 ~= 0; delta2 <= delta1 |]
    ==> Fuzzy_edge_point (tri_fuzzify_image (I, delta2), x, y)
        <= Fuzzy_edge_point (tri_fuzzify_image (I, delta1), x, y)
```

Figure 7.40: Theorem *fuzzyEdge* proving step 1 in Isabelle.

```
FuzzyEdgeDetection2_FU... | FuzzyEdgeDetection2_FU... | Scratch.thy
   Fuzzy_edge_point(tri_fuzzify_image(I, delta2), x, y)
    \<le> Fuzzy_edge_point(tri_fuzzify_image(I, delta1), x, y)"
  apply (simp add: Fuzzy_edge_point_def  tri_fuzzify_image_subtype_constr tri_fuzzify_image_def  fuzzy_zero_crossing_1_def f
uzzy_zero_crossing_2_def tri_fuzzify_def)
  apply (induct_tac delta1)
  apply (induct_tac delta2)
  apply (simp)
ISO8--**-XEmacs: FuzzyEdgeDetection2_FUZZY_EDGE.thy      (Isar script Font Scripting )----L123--C64--94%----------------------
proof (prove): step 1

goal (1 subgoal):
 1. [| 0 < delta2; delta2 <= delta1 |]
    ==> uni_min
          (uni_max
            (uni_min
              (fleq (flap (tri_fuzzify_image (I, delta2), x, y), tri_fuzzify (0, delta)),
               uni_max
                (fgeq (flap (tri_fuzzify_image (I, delta2), x, y + 1), tri_fuzzify (0, delta)),
                 fgeq (flap (tri_fuzzify_image (I, delta2), x + 1, y), tri_fuzzify (0, delta)))),
             uni_min
              (fgeq (flap (tri_fuzzify_image (I, delta2), x, y), tri_fuzzify (0, delta)),
               uni_max
                (fleq (flap (tri_fuzzify_image (I, delta2), x, y + 1), tri_fuzzify (0, delta)),
                 fleq (flap (tri_fuzzify_image (I, delta2), x + 1, y), tri_fuzzify (0, delta))))),
          FuzzyEdgeDetection2_FUZZY_REASONING.fequal
            (flap (tri_fuzzify_image (I, delta2), x, y), tri_fuzzify (0, delta)))
        <= uni_min
            (uni_max
              (uni_min
                (fleq (flap (tri_fuzzify_image (I, delta1), x, y), tri_fuzzify (0, delta)),
                 uni_max
                  (fgeq (flap (tri_fuzzify_image (I, delta1), x, y + 1), tri_fuzzify (0, delta)),
                   fgeq (flap (tri_fuzzify_image (I, delta1), x + 1, y), tri_fuzzify (0, delta)))),
               uni_min
                (fgeq (flap (tri_fuzzify_image (I, delta1), x, y), tri_fuzzify (0, delta)),
                 uni_max
                  (fleq (flap (tri_fuzzify_image (I, delta1), x, y + 1), tri_fuzzify (0, delta)),
                   fleq (flap (tri_fuzzify_image (I, delta1), x + 1, y), tri_fuzzify (0, delta))))),
             FuzzyEdgeDetection2_FUZZY_REASONING.fequal
              (flap (tri_fuzzify_image (I, delta1), x, y), tri_fuzzify (0, delta)))
```

Figure 7.41: Theorem *fuzzyEdge* proving step 2 in Isabelle.

```
apply (induct_tac delta1)
apply (induct_tac delta2)
apply (simp)
apply (simp)
apply (auto
```

Figure 7.42: Other steps in Isabelle.

# Chapter 8

# CONCLUSION

## 8.1 Conclusion

In the course of this work, we attempted to learn Formal Method tools starting from minimal knowledge about these tools. Since formal method tools are complex and not very well documented, it took a long time to learn how to use them. In addition to Specware, the SNARK theorem prover (that comes bundled with Specware) and the Isabelle theorem prover, were the formal method tools used in this thesis.

A verification process is consists of two main parts: specification and theorem proving. Specware is used to build specifications and SNARK and Isabelle are used to prove theorems. In this work, we found Specware is a powerful specification tool. Using Specware we were able to represent very complex computational processes and the logic behind their processing algorithms. Specware language (Metaslang) supports higher order logic and thus we were able to represent operations that are performed not only on various data types, but also on algorithms.

However, an insufficiency of examples and documentation is the biggest problem with Specware. For a long time, it was attempted to get SNARK to prove theorems, but it could prove nothing more than simple theorems. In

this process we found some bugs in SNARK. We came to this conclusion after a number of interactions with the developers of Specware and SNARK.

As a result, we changed the theorem prover to Isabelle. In contrast to SNARK, Isabelle is an interactive theorem prover. But Isabelle comes with an overwhelming load of documentation. Therefore, it is hard to find a logical starting point. Although Isabelle can prove some theorems automatically, for some more complex theorems it is possible to use tactics and other options by means of adding them into specifications. This feature is very useful in proving such theorems.

XEmacs is the software development environment for Specware. It integrates Specware and Isabelle in the same environment allowing the user to work with both tools within the same environment. XEmacs proved to be a very handy tool in this kind of work.

Metaslang is the specification language of Specware. It is not not an imperative programming language. It is a declarative logic based language. Metaslang can be viewed as a language to express mathematics. Thus coding in Metaslang is actually the same as deriving formulas in mathematics.

In this project, first, an information system was specified abstractly then more complex specifications were built incrementatally. Initially, the specifications were non-constructive, i.e., they were based on *existential specifications.* In other words, a logical formula was stating that a solution exists, without actually pointing to a specific solution. Such specifications cannot be easily translated into code. So the use of such specifications is somewhat limited. These kinds of specifications can be used for verifying that a specification is consistent, but not that it is correct.

To address this limitation, Specware provides constructs that allow for representing constructive specifications. One of the constructs that have to be used to develop such specifications is the construct of *lambda form.* It is part of

Lambda Calculus. In essence, Lambda Calculus allows to express and manipulate functions. Since a large class of algorithms are just functions, Lambda Calculus provides good basis for specifying such algorithms. Thus lambda forms were used extensively in developing the constructive specifications in this work.

The main goal of this work was to investigate the use of formal method tools in the software development process. It is a well known fact in software engineering that the cost of modification of software grows exponentially with the stage in the software development process. The cost of removing an error in a software system after its deployment is thousands of times more expensive than the cost of eliminating the cause of such an error in the specification phase. Thus from the economical point of view, it pays to invest in the initial phases, in particular in the specification phase, in order to avoid much higher cost in the final phases. The goal of using formal method tools in software development is just this.

Toward this goal, we have developed an library of specifications for capturing the processing of uncertainty (fuzzy information). Then we developed a specification of a fuzzy information processing system. We formulated a number of theorems to test the specification of this system. In the course of this work we were able to verify some of the features of the system. Some of the theorems were not proved. This might be due to two reasons - either our specification was not refined to a sufficient level of detail, or our analysis tools were not powerful enough. Or simply we did devote a sufficient amount of effort to this verification process. Whatever happens to be the case, it is clear that we were able to stress test the specification. Although we we did not pursue this task in this work, Specware can also generate code from a specification.

In conclusion, through our work in this thesis fuzzy set theory which represents uncertainty of categorization was specified formally with Metaslang in Specware. As an example, fuzzy information processing specifications were applied to an edge detection algorithm and to a theorem based on this specification. Although they are theoretically possible and some fundamental theorems were proved, Isabelle and SNARK failed to prove some theorems. Overall, we saw that it is possible to verify systems before they are built and it was found that Specware and Isabelle can be great formal method tools to build specifications and to verify them. We believe that our work will be useful for future Specware and Isabelle users and can be considered as a useful approach for information fusiom system formalization and verification.

## 8.2 Future Works

- Proving of the Model Using Isabelle

  Some theorems of model, such as*theorem fuzzyEdge*, can be proved by using Isabelle after getting extensive knowledge and experience in logical calculus and Isabelle. Isabelle has lots of capabilities that have not been used in this thesis.

- Refinement and Code Generation

  We spent all of our effort to prove theorems and to write specifications constructively. However, we could build some concrete specifications; it is needed to refine the specifications further by considering entire specifications. Specware has the capability of code generation from specifications. So it is possible to generate executable Lisp code from refined specifications.

- Different Fuzzy Rules

Even though we implemented three fuzzification methods - *triangular, trapozaidal, gaussian* - the work was focused only on the triangular fuzzification method. As a future work, other fuzzification methods can be applied to edge detection algorithms. Furthermore, fuzzy reasoning specifications can be defined by using different methods other than Lukasiewicz.

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# Appendix A

# Crisp To Fuzzy Mapping Table

| Crisp to Fuzzy Mapping Table | |
|---|---|
| CRISP | FUZZY |
| Number | Fuzzy Number (FN) |
| True, False | [0,1] |
| $+$ | $\tilde{+} : FN \times FN \rightarrow FN$<br>$(A\tilde{+}B)(z) = \sup_{z=x+y} min[A(x), B(y)], \forall z \in \mathbb{R}$ |
| $-$ | $\tilde{-} : FN \times FN \rightarrow FN$<br>$(A\tilde{-}B)(z) = \sup_{z=x-y} min[A(x), B(y)], \forall z \in \mathbb{R}$ |
| $\cdot$ | $\tilde{\cdot} : FN \times FN \rightarrow FN$<br>$(A\tilde{\cdot}B)(z) = \sup_{z=x\cdot y} min[A(x), B(y)], \forall z \in \mathbb{R}$ |
| $\div$ | $\tilde{\div} : FN \times FN \rightarrow FN$<br>$(A\tilde{\div}B)(z) = \sup_{z=x\div y} min[A(x), B(y)], \forall z \in \mathbb{R}$ |
| $min$ | $fuzzy - min : FN \times FN \rightarrow FN$<br>$fuzzy - min(A, B)(z) = \sup_{z=min(x,y)} min[A(x), B(y)], \forall z \in \mathbb{R}$ |

| | |
|---|---|
| $max$ | $fuzzy - max : FN \times FN \rightarrow FN$<br>$fuzzy{-}max(A,B)(z) = \sup_{z=max(x,y)} min[A(x), B(y)], \forall z \in \mathbb{R}$ |
| $\Rightarrow$ | $\tilde{\Rightarrow} : [0,1] \times [0,1] \rightarrow [0,1]$<br>1: $a\tilde{\Rightarrow}b = (1, 1-a+b)$<br>2: $a\tilde{\Rightarrow}b = \begin{cases} 1, & \text{if } a \leq b \\ b & \text{other} \end{cases}$ |
| $\Leftrightarrow$ | $\tilde{\Leftrightarrow} : [0,1] \times [0,1] \rightarrow [0,1]$<br>1: $a\tilde{\Leftrightarrow}b = 1 - \lvert a-b \rvert$<br>2: $a\tilde{\Leftrightarrow}b = \begin{cases} 1, & \text{if } a = b \\ min(a,b) & \text{other} \end{cases}$ |
| $=$ | $\tilde{=} : FN \times FN \rightarrow [0,1]$<br>1: $\mu\tilde{=}v = inf\,\{1 - \lvert\mu(x) - v(x)\rvert \mid x \in \mathbb{R}\}$<br>1: $\mu\tilde{=}v = inf\,\{g(\mu(x), v(x)) \mid x \in \mathbb{R}\}$<br>where $g(a,b) = \begin{cases} 1, & \text{if } a = b \\ b & \text{other} \end{cases}$ |
| $\geq$ | $\tilde{\geq} : FN \times FN \rightarrow [0,1]$<br>$\mu\tilde{\geq}v = inf\,\{min(1 - v(x) + \mu(x), 1) \mid x \in \mathbb{R}\}$ |
| $\leq$ | $\tilde{\leq} : FN \times FN \rightarrow [0,1]$<br>$\mu\tilde{\leq}v = inf\,\{min(1 - \mu(x) + v(x), 1) \mid x \in \mathbb{R}\}$ |

# Appendix B

# Formal Information Fusion Library Specs 1

```
%% Formal Information Fusion Library Specs
%% Fuzzy Set specification
%% Fuzzy Set is type: Fuzzy_set a = a -> Uni_intvl

%% UNI_INTVL specification
   UNI_INTVL = spec
%     import Nat

     op between_zero_one?(x:Nat): Boolean =
     x < 100 && 0 <= x

%% Uni_intvl is a type defined between 0 and 1.
     type Uni_intvl = (Nat | between_zero_one?)

   endspec

%% FUZZY_SET specification
   FUZZY_SET = spec
     import /Library/General/FiniteSet, UNI_INTVL

%% Definition of Fuzzy_set type
     type Fuzzy_set a = a -> Uni_intvl

%% Declarations of Fuzzy Complement, t_norm, and t_conorm operations
     op [a] fuzzy_complement : Fuzzy_set a -> Fuzzy_set a
     op [a] t_norm : Fuzzy_set a * Fuzzy_set a -> Fuzzy_set a
     op [a] t_conorm : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a

%% Definitions of Fuzzy Complement, t_norm, and t_conorm.
     def [a] fuzzy_complement(f1) = fn d -> 100 - f1(d)
     def [a] t_norm(f1, f2) = fn d -> min(f1(d), f2(d))
     def [a] t_conorm (f1, f2) = fn d -> max(f1(d), f2(d))
```

```
%% Definitions of min and max
     op min (x1: Nat, x2: Nat): Nat =
       if x1 <= x2 then x1 else x2
     op max (x1: Nat, x2: Nat): Nat =
       if x1 <= x2 then x2 else x1

%% Declarations of t_norm_min,  t_conorm_min, t_norm_Luka
%% t_conorm_Luka, t_norm_prod, t_conorm_prod, alpha_cut operations
     op [a] t_norm_min : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_conorm_min : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_norm_Luka : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_conorm_Luka : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_norm_prod : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_conorm_prod : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] alpha_cut : Fuzzy_set a * Uni_intvl -> FiniteSet a

%% Definitions of t_norm_min,  t_conorm_min, t_norm_Luka
%% t_conorm_Luka, t_norm_prod, t_conorm_prod, alpha_cut
     def [a] t_norm_min (f1, f2) = fn d -> min(f1(d), f2(d))
     def [a] t_conorm_min (f1, f2) = fn d -> max(f1(d), f2(d))
     def [a] t_norm_Luka(f1, f2) = fn d -> max(0, f1(d) + f2(d) - 100)
     def [a] t_conorm_Luka (f1, f2) = fn d -> min(f1(d) + f2(d), 100)
     def [a] t_norm_prod (f1, f2) = fn d -> f1(d)*f2(d)
     def [a] t_conorm_prod (f1, f2) = fn d -> (f1(d) + f2(d)) - (f1(d) * f2(d))
     def [a] alpha_cut (f, a) = fn d  -> a <= f(d)
     def [a] height(f: Fuzzy_set a): Uni_intvl =
     the(h) (fa(x: a) f x <= h) && (ex(x: a) f x = h)

   endspec



%% Formal Information Fusion Library Specs
%% Fuzzy number specification
%% Fuzzy_number is imported from Fuzzy_set

   FUZZY_NUMBER = spec
     import FUZZY_SET

%% Definitions of Fuzzy_number and Set_of_Nat
     type Fuzzy_number = Fuzzy_set Nat
     type Set_of_Nat = FiniteSet Nat

%% Fuzzy number is normal when height(f) has full membership degree
     axiom normality is
     fa(f:Fuzzy_number)
       height(f) = 100

%% Fuzzy number is convex
     axiom convexity is
     fa(f:Fuzzy_number, x1:Nat, x2:Nat, lamd: Uni_intvl)
       f(lamd * x1) + ((100 - lamd) * x2) >= min(f(x1), f(x2))

   endspec
```

```
%% Formal Information Fusion Library Specs
%% FUZZY_ARITHM
%% In this part fuzzy arithmetic operations are specified

   FUZZY_ARITHM = spec
     import FUZZY_NUMBER

     op fuzzy_add : Fuzzy_number * Fuzzy_number -> Fuzzy_number
     op fuzzy_sub : Fuzzy_number * Fuzzy_number -> Fuzzy_number
     op fuzzy_mult: Fuzzy_number * Fuzzy_number -> Fuzzy_number
     op fuzzy_div : Fuzzy_number * Fuzzy_number -> Fuzzy_number
     op fuzzy_min : Fuzzy_number * Fuzzy_number -> Fuzzy_number
     op fuzzy_max : Fuzzy_number * Fuzzy_number -> Fuzzy_number

%% Definition of Fuzzy Addition
     axiom fuzzy_add_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
     fuzzy_add(f1, f2)(z) = a <=> z=x+y => (f1(x) <= a || f2(y) <= a) &&
     (ex(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
        z=x+y && ((f1(x) = a && a <= f2(y)) || (f2(y) = a && a <= f1(x))))

%% Definition of Fuzzy Substraction
     axiom fuzzy_sub_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
     fuzzy_sub(f1, f2)(z) = a <=> z=x-y => (f1(x) <= a || f2(y) <= a) &&
     (ex(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
        z=x-y && ((f1(x) = a && a <= f2(y)) || (f2(y) = a && a <= f1(x))))

%% Definition of Fuzzy Multiplication
     axiom fuzzy_mult_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
     fuzzy_mult(f1, f2)(z) = a <=> z=x*y => (f1(x) <= a || f2(y) <= a) &&
     (ex(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
        z=x*y && ((f1(x) = a && a <= f2(y)) || (f2(y) = a && a <= f1(x))))

%% Definition of Fuzzy Division
     axiom fuzzy_div_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
     fuzzy_div(f1, f2)(z) = a <=> x=z*y => (f1(x) <= a || f2(y) <= a) &&
     (ex(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
        x=z*y && ((f1(x) = a && a <= f2(y)) || (f2(y) = a && a <= f1(x))))

%% Definition of Fuzzy Minimum
     axiom fuzzy_min_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
     fuzzy_min(f1, f2)(z) = a <=> z= min(x,y) => (f1(x) <= a || f2(y) <= a) &&
     (ex(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
        z= min(x,y) && ((f1(x) = a && a <= f2(y)) || (f2(y) = a && a <= f1(x))))

%% Definition of Fuzzy Maximum
     axiom fuzzy_max_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
     fuzzy_max(f1, f2)(z) = a <=> z= max(x,y) => (f1(x) <= a || f2(y) <= a) &&
     (ex(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat, y:Nat, z:Nat, a:Uni_intvl)
        z= max(x,y) && ((f1(x) = a && a <= f2(y)) || (f2(y) = a && a <= f1(x))))
```

```
   endspec


%% Formal Information Fusion Library Specs
%% FUZZY_REASONING
%% Fuzzy reasoning operations are defined.

   FUZZY_REASONING = spec
     import FUZZY_ARITHM

     op fequal : Fuzzy_number * Fuzzy_number -> Uni_intvl
     op fgeq   : Fuzzy_number * Fuzzy_number -> Uni_intvl
     op fleq   : Fuzzy_number * Fuzzy_number -> Uni_intvl
     op inf    : Nat -> Uni_intvl

%% Definition of Fuzzy equal
     axiom fequal_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat)
     fequal(f1, f2) = inf (100 - (abs(f1(x) - f2(x))))

%% Definition of Fuzzy greater than and equal
     axiom fgeq_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat)
     fgeq(f1, f2) = inf (min ((100 - f1(x) + f2(x)),100))

%% Definition of Fuzzy less than and equal
     axiom fleq_def is
     fa(f1:Fuzzy_number, f2:Fuzzy_number, x:Nat)
     fleq(f1, f2) = inf (min ((100 - f2(x) + f1(x)),100))

   endspec


%% Formal Information Fusion Library Specs
%% IMAGE
%% In this part image is defined.

   IMAGE = spec
     type Image = Integer * Integer -> Nat
   endspec


%% Formal Information Fusion Library Specs
%% FUZZIFICATION
%% Fuzzification spec is defined.

   FUZZIFICATION = spec
     import IMAGE, FUZZY_REASONING
     type Nz = {i:Nat | i~=0}
   endspec
```

```
%% Formal Information Fusion Library Specs
%% TRI_FUZZIFY
%% Triangular fuzzification is defined.

   TRI_FUZZIFY = spec
     import FUZZIFICATION

     type Tri_fuzzy_image = Integer * Integer -> Fuzzy_number

     op uni_min : Uni_intvl * Uni_intvl -> Uni_intvl
     op uni_max : Uni_intvl * Uni_intvl -> Uni_intvl
     op tri_fuzzify : Nat * Nz -> Fuzzy_number
     op tri_fuzzify_2 : Nat -> Fuzzy_number
     op tri_fuzzify_image : Image * Nz -> Tri_fuzzy_image
     op fuzzify : Nat * Nz -> Fuzzy_number
     op fuzzify_1 : Nat * Nz -> Fuzzy_number

%% Uncertainty level is given as a const
%% delta is the uncertainty level of the fuzzification
     op delta : Nz

     def uni_min(u1,u2) = if u1 <= u2 then u1 else u2
     def uni_max(u1,u2) = if u2 <= u1 then u2 else u1

%% op tri_fuzzify is to generate a triangular membership
%% function for each crisp value, define an Around label.
     axiom tri_fuzzify_def is
     fa(e: Nat, f: Fuzzy_number)
     (tri_fuzzify(e, delta) = f <=> (fa(x: Nat)
       (x < (e - delta) || (e + delta) < x <=>
          f(x) = 0 || (e - delta) <= x && x <= e <=>
          f(x) = div((x - e + delta), (delta)) || e <= x && x <= (e + delta) <=>
          f(x) = div((e - x + delta), (delta)))))

%% op tri_fuzzify_2 deals with the situation when
%% uncertainty level is zero
     axiom tri_fuzzify_2_def is
     fa(e: Nat, f: Fuzzy_number)
     (tri_fuzzify_2(e) = f <=> (fa(x: Nat)
       (x < e || e < x <=>
          f(x) = 0 || x <= e && e <= x <=>
          f(x) = 1)))

%% Definition of image fuzzification
     axiom tri_fuzzify_image_def is
     fa(I: Image, F:Tri_fuzzy_image)
     (tri_fuzzify_image(I, delta) = F <=>
        (fa(x: Integer, y: Integer)
           (F(x,y) = tri_fuzzify(I(x, y), delta))))

%% Definitions of fuzzify and fuzzify_l
     axiom fuzzify_def is
     fa(e: Nat, f: Fuzzy_number, x: Nat)
     (fuzzify(e, delta)(x) = tri_fuzzify(e, delta)(x) <=>
        x < e || fuzzify(e, delta)(x) = 100 <=>
```

```
        e <= x)

     axiom fuzzify_1_def is
     fa(e: Nat, f: Fuzzy_number, x: Nat)
     (fuzzify(e, delta)(x) = tri_fuzzify(e, delta)(x) <=>
        e < x || fuzzify(e, delta)(x) = 100 <=>
        x <= e)

   endspec



%% Formal Information Fusion Library Specs
%% TRA_FUZZIFY
%% Trapezoidal fuzzification is defined.

   TRA_FUZZIFY = spec
     import FUZZIFICATION

     type Tra_fuzzy_image = Integer * Integer -> Fuzzy_number

     op tra_fuzzify : Nat * Nz * Nz -> Fuzzy_number
     op tra_fuzzify_2 : Nat -> Fuzzy_number
     op tra_fuzzify_image : Image * Nz * Nz -> Tra_fuzzy_image

%% Uncertainty level is given as two consts
%% delta1 + delta2 is the uncertainty level of the fuzzification
     op delta1: Nz
     op delta2: Nz

%% op tra_fuzzify is to generate a trapezoidal membership
%% function for each crisp value, define an Around label.
     axiom tri_fuzzify_def is
     fa(e: Nat, f: Fuzzy_number)
     (tra_fuzzify(e, delta1, delta2) = f <=>
       (fa(x: Nat)
          (x<(e - (delta1+delta2)) || (e + (delta1+delta2))<x <=>
             f(x)=0 || (e - delta1)<=x && x<=(e + delta1) <=>
             f(x)=100  || (e - (delta1+delta2))<=x && x<=(e - delta1) <=>
             f(x)=(x-(e - (delta1+delta2)))* inv(delta2) || (e+delta1)<=x &&
             x<=(e+(delta1+delta2)) <=>
             f(x)=(e+(delta1+delta2)-x)* inv(delta2))))

%% op tra_fuzzify_2 deals with the situation when
%% uncertainty level is zero
     axiom tra_fuzzify_2_def is
     fa(e: Nat, f: Fuzzy_number)
     (tra_fuzzify_2(e) = f <=>
        (fa(x: Nat)
           (x<e || e<x <=>
              f(x)=0 || x<=e && e<=x <=>
              f(x)=100)))

%% definition of fuzzification of Image
     axiom tra_fuzzify_image_def is
     fa(I: Image, F:Tra_fuzzy_image)
```

```
     (tra_fuzzify_image(I, delta1, delta2) = F <=>
        (fa(x: Integer, y: Integer)
           (F(x,y) = tra_fuzzify(I(x, y), delta1, delta2))))
    endspec



%% Formal Information Fusion Library Specs
%% GAUSS_FUZZIFY
%% Gaussian fuzzification is defined.

   GAUSS_FUZZIFY = spec
     import FUZZIFICATION

     type Gauss_fuzzy_image = Integer * Integer -> Fuzzy_number

     op gauss_fuzzify : Nat * Nz * Nz -> Fuzzy_number
     op gauss_fuzzify_2 : Nat -> Fuzzy_number
     op gauss_fuzzify_image : Image * Nz * Nz -> Gauss_fuzzy_image
     op exp : Nat -> Nat
     op square : Nat -> Nat

%% Uncertainty level is given as two consts
     op sigma : Nz
     op mu : Nz

%% op gauss_fuzzify is to generate a Gaussian membership
%% function for each crisp value, define an Around label.
     axiom gauss_fuzzify_def is
     fa(e: Nat, f: Fuzzy_number)
     (gauss_fuzzify(e, sigma, mu) = f <=>
        (fa(x:Nat)
           f(x) = exp(-square((e-mu)*inv(sigma)))))

%% op gauss_fuzzify_2 deals with the situation when
%% uncertainty level is zero
     axiom gauss_fuzzify_2_def is
     fa(e: Nat, f: Fuzzy_number)
     (gauss_fuzzify_2(e) = f <=> (fa(x: Nat)
                                     (x<e || e<x <=>
                                        f(x)=zero || x<=e && e<=x <=>
                                        f(x)=one)))

%% definition of fuzzification of Image
     axiom gauss_fuzzify_image_def is
     fa(I: Image, F: Gauss_fuzzy_image)
     (gauss_fuzzify_image(I, sigma, mu) = F <=>
        (fa(x:Integer, y:Integer)
           (F(x,y) = gauss_fuzzify(I(x,y), sigma, mu))))

   endspec



%% Formal Information Fusion Library Specs
%% DEFUZZIFICATION 1
```

```
%% Defuzzification is defined.

   DEFUZZIFICATION = spec
     import FUZZY_NUMBER

%% type Intvl is a set of natural numbers
     type Intvl = Set_of_Nat

     op defuzzify_1 : Fuzzy_number -> Nat
     op inf : Intvl -> Nat
     op sup : Intvl -> Nat
     op height_intvl : Fuzzy_number -> Intvl
     op div : Nat * Nat -> Nat

%% defuzzify(F) = (inf(M) + sup(M))/2
%% M=an interval [z] s.t. F(z) = height(F) = one
     def defuzzify_1(F) =
       div((inf(height_intvl(F))+ sup(height_intvl(F))),(1 + 1))

     axiom inf_def is
     fa(I: Intvl, a: Nat, x: Nat)
     (inf(I) = a <=>
        (x in? I) => a <= x)

     axiom sup_def is
     fa(I: Intvl, a: Nat, x: Nat)
     (sup(I) = a <=>
        (x in? I) => x <= a)

     axiom height_intvl_def is
     fa(F: Fuzzy_number, I: Intvl, x: Nat)
     (height_intvl(F) = I <=>
        F(x) = 100 <=> (x in? I))

   endspec



%% Formal Information Fusion Library Specs
%% DEFUZZIFICATION 2
%% Defuzzification is defined.

   DEFUZZIFICATION_2 = spec
     import FUZZY_NUMBER

     type Intvl = Set_of_Nat

     op defuzzify_2 : Fuzzy_number -> Nat
     op inf : Intvl -> Nat
     op sup : Intvl -> Nat
     op alpha_intvl : Fuzzy_number -> Intvl
     op div : Nat * Nat -> Nat

     op alpha : Uni_intvl

%% defuzzify_2(F) = (inf(M) + sup(M))/2
```

```
%% M=an interval [z] s.t. F(z) = alpha_cut of F
     def defuzzify_2(F) =
       div((inf(alpha_intvl(F))+ sup(alpha_intvl(F))),(1 + 1))

     axiom inf_def is
     fa(I: Intvl, a: Nat, x: Nat)
     (inf(I) = a <=>
        (x in? I) => a <= x)

     axiom sup_def is
     fa(I: Intvl, a: Nat, x: Nat)
     (sup(I) = a <=>
        (x in? I) => x <= a)

     axiom alpha_intvl_def is
     fa(F: Fuzzy_number, I: Intvl, x: Nat)
     (alpha_intvl(F) = I <=>
        F(x) = alpha <=> (x in? I))

   endspec




%% Formal Information Fusion Library Specs
%% EDGE_POINT
%% Edge_point is defined.

   EDGE_POINT = spec
     import IMAGE

%% Laplacian_based algorithm is used to derive edge point
     op edge_point? : Image * Integer * Integer -> Boolean
     op lap : Image * Integer * Integer -> Nat
     op var : Image * Integer * Integer -> Nat
     op ysum: Image * Integer * Integer -> Nat
     op elmt: Image * Integer * Integer -> Nat
     op m   : Image * Integer * Integer -> Nat
     op sub : Image * Integer * Integer -> Nat
     op zero_crossing_1 : Image * Integer * Integer -> Boolean
     op zero_crossing_2 : Image * Integer * Integer -> Boolean

     op thrd : Nat

     type Edge_point = (Image * Integer * Integer | edge_point?)

     def lap(I, x, y) =
       (((I((x+1),y)+I((x-1),y)) + (I((y+1),x)+I((y-1),x))) -
          ((I(x,y)+I(x,y))+(I(x,y)+I(x,y))))

     def sub(I, x, y) =
       (((I(x, (y-(1+1))) + I(x, (y-1))) + (I(x, y) + I(x, (y+1)))) +
          I(x, (y+(1+1))))

     def m(I, x, y) =
       (((sub(I, (x-(1+1)), y) + sub(I, (x-1), y)) +
```

```
          (sub(I, x, y) + sub(I, (x+1), y))) + sub(I, (x+(1+1)), y))

     def elmt(I, x, y) =
       ((I(x, y) - m(I, x, y))*(I(x, y) - m(I, x, y)))

     def ysum(I, x, y) =
       (((elmt(I, x, (y-(1+1))) + elmt(I, x, (y-1))) +
          (elmt(I, x, y) + elmt(I, x, (y+1)))) + elmt(I, x, (y+(1+1))))

%% definition of local variance
     def var(I, x, y) =
       (((ysum(I, (x-(1+1)), y) + ysum(I, (x-1), y)) +
          (ysum(I, x, y) + ysum(I, (x+1), y))) + ysum(I, (x+(1+1)), y))

%% definition of zero_crossing
     axiom zero_crossing_1_def is
     fa(I: Image, x: Integer, y: Integer)
     (zero_crossing_1(I, x, y) = ((lap(I,x,y) < 0) &&
                                    (0 < lap(I,x,(y+1))) ||
                                  (0 < lap(I,(x+1), y))))

     axiom zero_crossing_2_def is
     fa(I: Image, x: Integer, y: Integer)
     (zero_crossing_2(I, x, y) = ((0 < lap(I, x, y)) &&
                                    (lap(I, x, (y+1)) < 0) ||
                                  (lap(I, (x+1), y) < 0)))
%% definition of edge_point
     axiom edge_point_def is
     fa(I: Image, x: Integer, y: Integer)
     (edge_point?(I, x, y) <=>
     (zero_crossing_1(I,x,y) || zero_crossing_2(I,x,y))
        && (thrd < var(I, x, y)))

   endspec



%% Formal Information Fusion Library Specs
%% EDGE
%% Edge is defined.
   EDGE = spec
     import EDGE_POINT

     op edge? : Image * Integer * Integer * Integer * Integer -> Boolean

     type Edge = (Image*Integer*Integer*Integer*Integer | edge?)

     axiom edge?_def is
     fa(I: Image, x1: Integer, y1: Integer, x2: Integer, y2: Integer)
     (edge?(I, x1, y1, x2, y2) <=>
       (edge_point?(I, x1, y1) && edge_point?(I, x2, y2) &&
          ((y2 = y1 && (fa(x: Integer) (x <= x2 && x1 < x =>
                                        edge_point?(I, x, y1)))) ||
             (x2 = x1 && (fa(y: Integer) (y <= y2 && y1 < y =>
                                          edge_point?(I, x1, y)))) ||
             (fa(x: Integer, y: Integer)
```

```
                (y < y2 && y1 <= y && x < x2 && x1 <= x =>
                   edge_point?(I, (x+1), (y+1)))))))
   endspec



%% Formal Information Fusion Library Specs
%% FUZZY_EDGE
%% how to derive fuzzy edge points
%% If (zero_crossing_1(I, x, y),zero_crossing_2(I,x,y)==1
%% and var(I, x, y) >= thrd
%% then (x, y) is an Edge_point.(crisp algorithm)
%% Fuzzy logic:
%% If zero_crossing_1(I, x, y)and zero_crossing_2(I, x, y) are true and
%% fvar(I, x, y) is fuzzy greater than thrd
%% then (x, y) is a Fuzzy_edge_point.

   FUZZY_EDGE = spec
     import TRI_FUZZIFY

     type ONE_SORT

     op flap  : Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
     op fvar  : Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
     op fysum : Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
     op felmt : Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
     op fm    : Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
     op fsub  : Tri_fuzzy_image * Integer * Integer -> Fuzzy_number
     op Fuzzy_edge_point : Tri_fuzzy_image * Integer * Integer -> Uni_intvl
     op Tri_fuzzy_image : Integer * Integer -> Fuzzy_number
     op fuzzy_zero_crossing_1 : Tri_fuzzy_image * Integer * Integer -> Uni_intvl
     op fuzzy_zero_crossing_2 : Tri_fuzzy_image * Integer * Integer -> Uni_intvl

     op thrd: Nat

     type Fuzzy_edge_point = Tri_fuzzy_image * Integer * Integer -> Uni_intvl

%% definition of fuzzy Laplacian
     def flap(F, x, y) =
       fuzzy_sub (fuzzy_add (fuzzy_add (F((x+1), y),
                                        F((x-1), y)),
                             fuzzy_add (F(x, (y+1)),
                                        F(x, (y-1)))),
                  fuzzy_add (fuzzy_add (F(x,y), F(x,y)),
                             fuzzy_add (F(x,y), F(x,y))))

     def fsub(F, x, y) =
       fuzzy_add (fuzzy_add (fuzzy_add (F(x, (y-(1+1))),
                                        F(x, (y-1))),
                             fuzzy_add (F(x, y),
                                        F(x, (y+1)))),
                  F(x, (y+(1+1))))

     def fm(F, x, y) =
       fuzzy_add (fuzzy_add (fuzzy_add (fsub(F, (x-(1+1)), y),
                                        fsub(F, (x-1), y)),
```

```
                              fuzzy_add (fsub(F, x, y),
                                         fsub(F, (x+1), y))),
                   fsub(F, (x+(1+1)), y))

     def felmt(F, x, y) =
       fuzzy_mult (fuzzy_sub (F(x, y), F(x, y)),
                   fuzzy_sub (F(x, y), F(x, y)))

     def fysum(F, x, y) =
       fuzzy_add (fuzzy_add (fuzzy_add (felmt(F, x, (y-(1+1))),
                                        felmt(F, x, (y-1))),
                              fuzzy_add (felmt(F, x, y),
                                         felmt(F, x, (y+1)))),
                   felmt(F, x, (y+(1+1))))

%% definition of fuzzy local variance
     def fvar(F, x, y) =
       fuzzy_add (fuzzy_add (fuzzy_add (fysum(F, (x-(1+1)), y),
                                        fysum(F, (x-1), y)),
                              fuzzy_add (fysum(F, x, y),
                                         fysum(F, (x+1), y))),
                   fysum(F, (x+(1+1)), y))

%% fuzzy_zero_crossing
     axiom fuzzy_zero_crossing_1_def is
     fa(F: Tri_fuzzy_image, x: Integer, y: Integer)
     (fuzzy_zero_crossing_1(F, x, y) =
        (uni_min (fleq (flap(F, x, y), tri_fuzzify(0, delta)),
                  (uni_max (fgeq (flap(F, x, (y+1)), tri_fuzzify(0, delta)),
                            fgeq (flap(F, (x+1), y), tri_fuzzify(0, delta)))))))

     axiom fuzzy_zero_crossing_2_def is
     fa(F: Tri_fuzzy_image, x: Integer, y: Integer)
     (fuzzy_zero_crossing_2(F, x, y) =
        (uni_min (fgeq (flap(F, x, y), tri_fuzzify(0, delta)),
                  (uni_max (fleq (flap(F, x, (y+1)), tri_fuzzify(0, delta)),
                            fleq (flap(F, (x+1), y), tri_fuzzify(0, delta)))))))

%% definition of Fuzzy edge point
     def Fuzzy_edge_point(F, x, y) =
       (uni_min (uni_max (fuzzy_zero_crossing_1(F, x, y),
                          fuzzy_zero_crossing_2(F, x, y)),
                 fequal (flap(F, x, y), tri_fuzzify(0, delta))))

%% Theorem fuzzyEdge
     theorem fuzzyEdge is
     (fa(x: Integer, y: Integer, I: Image, delta1: Nz, delta2: Nz)
        (delta2 <= delta1) =>
        (Fuzzy_edge_point(tri_fuzzify_image(I, delta2), x, y)) <=
        (Fuzzy_edge_point(tri_fuzzify_image(I, delta1), x, y)))

   endspec

   p1 = prove fuzzyEdge in FUZZY_EDGE
}
```

# Appendix C

# Formal Information Fusion Library Specs 2

Specifications are transformed to concrete specifications and added theorems and Isabelle proof terms. While some parts stayed same some parts changed. This part shows changed parts of specifications.

```
%% Formal Information Fusion Library Specs
%% Fuzzy Set specification
%% Fuzzy Set is type: Fuzzy_set a = a -> Uni_intvl

%% UNI_INTVL specification
   UNI_INTVL = spec
%     import Nat

     op between_zero_one?(x:Nat): Boolean =
     x < 100 && 0 <= x

%% Uni_intvl is a type defined between 0 and 1.
     type Uni_intvl = (Nat | between_zero_one?)

   endspec

%% FUZZY_SET specification
   FUZZY_SET = spec
     import /Library/General/FiniteSet, UNI_INTVL

%% Definition of Fuzzy_set type
     type Fuzzy_set a = a -> Uni_intvl

%% Declarations of Basic Fuzzy Operations
     op [a] fuzzy_complement : Fuzzy_set a -> Fuzzy_set a
     op [a] t_norm : Fuzzy_set a * Fuzzy_set a -> Fuzzy_set a
     op [a] t_conorm : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_norm_min : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_conorm_min : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_norm_Luka : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_conorm_Luka : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_norm_prod : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
     op [a] t_conorm_prod : Fuzzy_set a * Fuzzy_set a ->  Fuzzy_set a
```

```
     op [a] alpha_cut : Fuzzy_set a * Uni_intvl -> FiniteSet a

%% Definitions of Basic Fuzzy Operations
     def [a] fuzzy_complement(f1) d = 100 - f1(d)
     def [a] t_norm(f1, f2) d = min(f1(d),f2(d))
     def [a] t_conorm (f1, f2) d = max(f1(d), f2(d))
     def [a] t_norm_min (f1, f2) d = min(f1(d), f2(d))
     def [a] t_conorm_min (f1, f2) d = max(f1(d), f2(d))
     def [a] t_norm_Luka(f1, f2) d = max(0, f1(d) + f2(d) - 100)
     def [a] t_conorm_Luka (f1, f2) d = min(f1(d) + f2(d), 100)
     def [a] t_norm_prod (f1, f2) d = f1(d)*f2(d)
     def [a] t_conorm_prod (f1, f2) d = (f1(d) + f2(d)) - (f1(d) * f2(d))
     def [a] alpha_cut (f, a) d = (a <= f(d))
     def [a] height(f: Fuzzy_set a): Uni_intvl =
     the(h) (fa(x: a) f x <= h) && (ex(x: a) f x = h)

%% Definitions of min and max
     op min (x1: Nat, x2: Nat): Nat = if x1 <= x2 then x1 else x2
     op max (x1: Nat, x2: Nat): Nat = if x1 <= x2 then x2 else x1

%% Theorems and  Isabelle proof terms
     axiom abc is
     fa(a: Nat, b: Nat)
     a - (a - b) = b

%% Fuzzy complement of fuzzy complement of a fuzzy set equals to itself.
     theorem complement is
     fa(f: Fuzzy_set Nat, x: Nat)
     fuzzy_complement(fuzzy_complement(f)) x = f x

     proof Isa
     by (auto simp add: between_zero_one_p_def fuzzy_complement_def abc)
     end-proof

%% Fuzzy Set intersection/union  provides commutativity.
     theorem commutativity is
     fa(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat, x: Nat)
     t_norm(f1, f2)(x) =t_norm(f2, f1)(x)

     proof Isa
     by (simp add: between_zero_one_p_def t_norm_def min_def)
     end-proof

%% Fuzzy Set intersection/union  provides assosiativity.
     theorem assosiativity is
     fa(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat, f3: Fuzzy_set Nat, x: Nat)
     t_norm(f1, t_norm(f2, f3))(x) =  t_norm(t_norm(f1, f2), f3)(x)

     proof Isa
     by (simp add: between_zero_one_p_def t_norm_def min_def)
     end-proof

%% Fuzzy Set intersection/union  provides monotonicity.
     theorem monotonicity is
     fa(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat, f3: Fuzzy_set Nat, x: Nat)
     f2(x)<= f3(x) =>  t_norm(f1, f2)(x) <=  t_norm(f1, f3)(x)
```

```
    proof Isa
    by (simp add: between_zero_one_p_def t_norm_def min_def)
    end-proof
  endspec



%% Formal Information Fusion Library Specs
%% Fuzzy number specification
%% Fuzzy-number is imported from Fuzzy-set

  FUZZY_NUMBER = spec
    import FUZZY_SET

%% Definitions of Fuzzy_number and Set_of_Real
    type Fuzzy_number = Fuzzy_set Nat
    type Set_of_Nat = FiniteSet Nat

%% Fuzzy number is normal when height(f) has full membership degree
    axiom normality is
    fa(f:Fuzzy_number)
      height(f) = 100

%% Fuzzy number is convex
    axiom convexity is
    fa(f:Fuzzy_number, x1:Nat, x2:Nat, lamd: Uni_intvl)
      f(lamd * x1) + ((100 - lamd) * x2) >= min(f(x1), f(x2))

  endspec



%% Formal Information Fusion Library Specs
%% FUZZY-ARITHM
%% In this part fuzzy arithmetic operations are specified

  FUZZY_ARITHM = spec
    import FUZZY_NUMBER

    op isMinIn (r:Nat, sr: Set Nat) infixl 20 : Bool =
    r in? sr && (fa(r1) r1 in? sr => r <= r1)

    op hasMin? (sr: Set Nat) : Bool = (ex(r) r isMinIn sr)

    op minIn (sr: Set Nat | hasMin? sr) : Nat = the(r) r isMinIn sr

    op rangeCC (lo:Nat, hi:Nat| lo <= hi) : Set Nat=
    fn r -> lo <= r && r <= hi

    op rangeOO (lo:Nat, hi:Nat| lo <  hi) : Set Nat=
    fn r -> lo <  r && r <  hi

    op rangeCO (lo:Nat, hi:Nat| lo <  hi) : Set Nat=
    fn r -> lo <= r && r <  hi
```

```
    op rangeOC (lo:Nat, hi:Nat| lo <  hi) : Set Nat=
    fn r -> lo <  r && r <= hi

    type Range = {rs : Set Nat | ex(lo:Nat,hi:Nat)
             lo <= hi && rs = rangeCC (lo, hi) ||
             lo <  hi && rs = rangeOO (lo, hi) ||
             lo <  hi && rs = rangeCO (lo, hi) ||
             lo <  hi && rs = rangeOC (lo, hi)}

    op sup (rng:Range) : Nat=
    minIn (fn hi -> (fa(r) r in? rng => hi >= r))

    op fuzzy_add(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat) (z: Nat): Uni_intvl =
    sup(map (fn (x,y) -> min(f1 x, f2 y)) (fn (x,y) -> x + y = z))

    op fuzzy_sub(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat) (z: Nat): Uni_intvl =
    sup(map (fn (x,y) -> min(f1 x, f2 y)) (fn (x,y) -> x - y = z))

    op fuzzy_mult(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat) (z: Nat): Uni_intvl =
    sup(map (fn (x,y) -> min(f1 x, f2 y)) (fn (x,y) -> x * y = z))

    op fuzzy_div(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat) (z: Nat): Uni_intvl =
    sup(map (fn (x,y) -> min(f1 x, f2 y)) (fn (x,y) -> z * y = x))

    op fuzzy_min(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat) (z: Nat): Uni_intvl =
    sup(map (fn (x,y) -> min(f1 x, f2 y)) (fn (x,y) -> min(x, y) = z))

    op fuzzy_max(f1: Fuzzy_set Nat, f2: Fuzzy_set Nat) (z: Nat): Uni_intvl =
    sup(map (fn (x,y) -> min(f1 x, f2 y)) (fn (x,y) -> max(x, y) = z))

  endspec


%% Formal Information Fusion Library Specs
%% FUZZY-REASONING
%% Fuzzy reasoning operations are defined.

  FUZZY_REASONING = spec
    import FUZZY_ARITHM

%% Definition of Fuzzy equal
    op fequal (f1: Fuzzy_number, f2: Fuzzy_number): Uni_intvl =
    (fa(x:Nat) (inf (100 - (abs(f1(x) - f2(x))))))

%% Definition of Fuzzy greater than and equal
    op fgeq (f1: Fuzzy_number, f2: Fuzzy_number): Uni_intvl =
    (fa(x:Nat) inf (min ((100 + f1(x) - f2(x)),100))

%% Definition of Fuzzy less than and equal
    op fleq (f1: Fuzzy_number, f2: Fuzzy_number): Uni_intvl =
    (fa(x:Nat) inf (min ((100 + f2(x) - f1(x)),100))

  endspec
```

```
%% Formal Information Fusion Library Specs
%% IMAGE
%% In this part image is defined.

   IMAGE = spec
     type Image = Integer * Integer -> Nat
   endspec



%% Formal Information Fusion Library Specs
%% FUZZIFICATION
%% Fuzzification spec is defined.

   FUZZIFICATION = spec
     import IMAGE, FUZZY_REASONING
     type Nz = {i:Nat | i~=0}
   endspec



%% Formal Information Fusion Library Specs
%% TRI-FUZZIFY
%% Triangular fuzzification is defined.

   TRI_FUZZIFY = spec
     import FUZZIFICATION

     type Tri_fuzzy_image = Integer * Integer -> Fuzzy_number

     op uni_min : Uni_intvl * Uni_intvl -> Uni_intvl
     op uni_max : Uni_intvl * Uni_intvl -> Uni_intvl
     op tri_fuzzify : Nat * Nz -> Fuzzy_number
     op tri_fuzzify_2 : Nat -> Fuzzy_number
     op tri_fuzzify_image : Image * Nz -> Tri_fuzzy_image
     op fuzzify : Nat * Nz -> Fuzzy_number
     op fuzzify_1 : Nat * Nz -> Fuzzy_number

%% Uncertainty level is given as a const
%% delta is the uncertainty level of the fuzzification
     op delta : Nz

     def uni_min(u1,u2) = if u1 <= u2 then u1 else u2
     def uni_max(u1,u2) = if u2 <= u1 then u2 else u1

%% op tri-fuzzify is to generate a triangular membership
%% function for each crisp value, define an Around label.
     axiom tri_fuzzify_def is
     fa(e: Nat, f: Fuzzy_number)
     (tri_fuzzify(e, delta) = f =>
        (fa(x: Nat)
           if ((e - delta) <= x && x <= e)
             then f(x) = div((x - e + delta), (delta))
           else if (e <= x && x <= (e + delta))
                  then f(x) = div((e - x + delta), (delta))
```

```
                else f(x) = 0))

%% op tri-fuzzify-2 deals with the situation when
%% uncertainty level is zero
     axiom tri_fuzzify_2_def is
     fa(e: Nat, f: Fuzzy_number)
     (tri_fuzzify_2(e) = f =>
        (fa(x: Nat)
           if (e <= x && x <= e)
             then f(x) = 100
           else f(x) = 0))

%% Definition of image fuzzification
    axiom tri_fuzzify_image_def is
     fa(I: Image, F:Tri_fuzzy_image)
     (tri_fuzzify_image(I, delta) = F =>
        (fa(x: Integer, y: Integer)
           (F(x,y) = tri_fuzzify(I(x, y), delta))))

%% Definitions of fuzzify and fuzzify-l
    axiom fuzzify_def is
     fa(e: Nat, f: Fuzzy_number, x: Nat)
     (fuzzify(e, delta)(x) =
        (if (x < e)
           then tri_fuzzify(e, delta)(x)
         else 100))

    axiom fuzzify_1_def is
     fa(e: Nat, f: Fuzzy_number, x: Nat)
     (fuzzify(e, delta)(x) =
        (if (e < x)
           then tri_fuzzify(e, delta)(x)
         else 100))

%% Theorems and  Isabelle proof terms
     theorem theo1 is
     fa(e: Nat, f: Fuzzy_number, x: Nat)
     fuzzify(10, delta)(100) = 100

     proof Isa theo2_Obligation_subtype:
     by (simp add: fuzzify_def)
     end-proof

     theorem theo2 is
     fa(u1:Uni_intvl, u2: Uni_intvl)
     uni_min(10, 2)= 2

%% Obligation subtypes are created by Isabelle and needed to prove too.
     proof Isa theo2_Obligation_subtype:
     by (simp add: between_zero_one_p_def)
     end-proof

     proof Isa theo2_Obligation_subtype0:
     by (simp add: between_zero_one_p_def)
     end-proof
```

```
     proof Isa theo2:
     by (simp add: between_zero_one_p_def uni_min_def)
     end-proof

   endspec




%% Formal Information Fusion Library Specs
%% DEFUZZIFICATION 1
%% Defuzzification is defined.

   DEFUZZIFICATION = spec
     import FUZZY_NUMBER, FUZZY_ARITHM

%% type Intvl is a set of natural numbers
     type Intvl = Set_of_Nat

     op defuzzify_1 : Fuzzy_number -> Nat
     op inf : Intvl -> Nat
     op height_intvl : Fuzzy_number -> Intvl
     op div : Nat * Nat -> Nat

%% defuzzify(F) = (inf(M) + sup(M))/2
%% M=an interval [z] s.t. F(z) = height(F) = one
     def defuzzify_1(F) =
       div((inf(height_intvl(F))+ sup(height_intvl(F))),(1 + 1))

     axiom inf_def is
     fa(I: Intvl, a: Nat, x: Nat)
     (inf(I) = a <=>
        (x in? I) => a <= x)

%% We do not need to define sup again here.
%% sup is called from FUZZY_ARITHM

     axiom height_intvl_def is
     fa(F: Fuzzy_number, I: Intvl, x: Nat)
     (height_intvl(F) = I <=>
        F(x) = 100 <=> (x in? I))

   endspec



%% Formal Information Fusion Library Specs
%% DEFUZZIFICATION 2
%% Defuzzification is defined.

   DEFUZZIFICATION_2 = spec
     import FUZZY_NUMBER, FUZZY_ARITHM

     type Intvl = Set_of_Nat

     op defuzzify_2 : Fuzzy_number -> Nat
```

```
     op inf : Intvl -> Nat
     op alpha_intvl : Fuzzy_number -> Intvl
     op div : Nat * Nat -> Nat

     op alpha : Uni_intvl

%% defuzzify-2(F) = (inf(M) + sup(M))/2
%% M=an interval [z] s.t. F(z) = alpha-cut of F
     def defuzzify_2(F) =
       div((inf(alpha_intvl(F))+ sup(alpha_intvl(F))),(1 + 1))

     axiom inf_def is
     fa(I: Intvl, a: Nat, x: Nat)
     (inf(I) = a <=>
        (x in? I) => a <= x)

%% sup is called from FUZZY_ARITHM

     axiom alpha_intvl_def is
     fa(F: Fuzzy_number, I: Intvl, x: Nat)
     (alpha_intvl(F) = I <=>
        F(x) = alpha <=> (x in? I))

   endspec
```

# Appendix D

# Subgoal for Theorem FuzzyEdge Which Generated by Isabelle

```
proof (prove): step 1

goal (1 subgoal):
 1. [| 0 < delta2; delta2 <= delta1 |]
    ==> uni_min
          (uni_max
            (uni_min
              (fleq (%z. sup__c (Set__map
                (%(xa, ya).
                  FuzzyEdgeDetection2_FUZZY_SET.min
                    (sup__c (Set__map
                      (%(xa, ya).
                        FuzzyEdgeDetection2_FUZZY_SET.min
                          (sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (x + 1, y) xa,
        tri_fuzzify_image (I, delta2) (x - 1, y) ya))
   (%(x, y). x + y = xa)),
     sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (x, y + 1) xa,
        tri_fuzzify_image (I, delta2) (x, y - 1) ya))
   (%(x, y). x + y = ya))))
   (%(x, y). x + y = xa)),
         sup__c (Set__map
              %(xa, ya).
         FuzzyEdgeDetection2_FUZZY_SET.min
          (sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (x, y) xa,
        tri_fuzzify_image (I, delta2) (x, y) ya))
   (%(x, y). x + y = xa)),
       sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (x, y) xa,
```

```
     tri_fuzzify_image (I, delta2) (x, y) ya))
(%(x, y). x + y = ya))))
      (%(x, y). x + y = ya))))
       (%(x, y). int x - int y = int z)),
              tri_fuzzify (0, delta)),
           uni_max
            (fgeq (%z. sup__c (Set__map
               (%(xa, ya).
              FuzzyEdgeDetection2_FUZZY_SET.min
                  (sup__c (Set__map
                     (%(xa, ya).
                        FuzzyEdgeDetection2_FUZZY_SET.min
                         (sup__c (Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta2) (x + 1, y + 1) xa,
       tri_fuzzify_image (I, delta2) (x - 1, y + 1) ya))
  (%(x, y). x + y = xa)),
     sup__c (Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta2) (x, 2 + y) xa,
       tri_fuzzify_image (I, delta2) (x, y) ya))
  (%(x, y). x + y = ya))))
     (%(x, y). x + y = xa)),
 sup__c (Set__map
         (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
       (sup__c (Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta2) (x, y + 1) xa,
       tri_fuzzify_image (I, delta2) (x, y + 1) ya))
  (%(x, y). x + y = xa)),
         sup__c (Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta2) (x, y + 1) xa,
      tri_fuzzify_image (I, delta2) (x, y + 1) ya))
  (%(x, y). x + y = ya))))
     (%(x, y). x + y = ya))))
          (%(x, y). int x - int y = int z)),
                   tri_fuzzify (0, delta)),
             fgeq (%z. sup__c (Set__map
                  (%(xa, ya).
                FuzzyEdgeDetection2_FUZZY_SET.min
           (sup__c (Set__map
               (%(xa, ya).
            FuzzyEdgeDetection2_FUZZY_SET.min
                         (sup__c (Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta2) (2 + x, y) xa,
       tri_fuzzify_image (I, delta2) (x, y) ya))
  (%(x, y). x + y = xa)),
     sup__c (Set__map
```

```
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta2) (x + 1, y + 1) xa,
       tri_fuzzify_image (I, delta2) (x + 1, y - 1) ya))
  (%(x, y). x + y = ya))))
     (%(x, y). x + y = xa)),
            sup__c (Set__map
       (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (sup__c (Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta2) (x + 1, y) xa,
      tri_fuzzify_image (I, delta2) (x + 1, y) ya))
  (%(x, y). x + y = xa)),
  sup__c (Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta2) (x + 1, y) xa,
      tri_fuzzify_image (I, delta2) (x + 1, y) ya))
  (%(x, y). x + y = ya))))
           (%(x, y). x + y = ya))))
               (%(x, y). int x - int y = int z)),
                   tri_fuzzify (0, delta)))),
        uni_min
         (fgeq (%z. sup__c (Set__map
             (%(xa, ya).
              FuzzyEdgeDetection2_FUZZY_SET.min
                   (sup__c (Set__map
                     (%(xa, ya).
                   FuzzyEdgeDetection2_FUZZY_SET.min
                          (sup__c (Set__map
(%(xa, ya).
    FuzzyEdgeDetection2_FUZZY_SET.min
     (tri_fuzzify_image (I, delta2) (x + 1, y) xa,
      tri_fuzzify_image (I, delta2) (x - 1, y) ya))
(%(x, y). x + y = xa)),
        sup__c (Set__map
(%(xa, ya).
    FuzzyEdgeDetection2_FUZZY_SET.min
     (tri_fuzzify_image (I, delta2) (x, y + 1) xa,
      tri_fuzzify_image (I, delta2) (x, y - 1) ya))
(%(x, y). x + y = ya))))
                (%(x, y). x + y = xa)),
     sup__c (Set__map
      (%(xa, ya).
    FuzzyEdgeDetection2_FUZZY_SET.min
      (sup__c (Set__map
(%(xa, ya).
    FuzzyEdgeDetection2_FUZZY_SET.min
     (tri_fuzzify_image (I, delta2) (x, y) xa,
      tri_fuzzify_image (I, delta2) (x, y) ya))
(%(x, y). x + y = xa)),
          sup__c (Set__map
(%(xa, ya).
    FuzzyEdgeDetection2_FUZZY_SET.min
```

```
      (tri_fuzzify_image (I, delta2) (x, y) xa,
      tri_fuzzify_image (I, delta2) (x, y) ya))
 (%(x, y). x + y = ya))))
         (%(x, y). x + y = ya))))
     (%(x, y). int x - int y = int z)),
                 tri_fuzzify (0, delta)),
            uni_max
             (fleq (%z. sup__c (Set__map
      (%(xa, ya).
                  FuzzyEdgeDetection2_FUZZY_SET.min
                sup__c (Set__map
                      (xa, ya).
             FuzzyEdgeDetection2_FUZZY_SET.min
                (sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (x + 1, y + 1) xa,
        tri_fuzzify_image (I, delta2) (x - 1, y + 1) ya))
   (%(x, y). x + y = xa)),
         sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (x, 2 + y) xa,
        tri_fuzzify_image (I, delta2) (x, y) ya))
   (%(x, y). x + y = ya))))
               (%(x, y). x + y = xa)),
                 sup__c (Set__map
                   (%(xa, ya).
               FuzzyEdgeDetection2_FUZZY_SET.min
      (sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (x, y + 1) xa,
        tri_fuzzify_image (I, delta2) (x, y + 1) ya))
   (%(x, y). x + y = xa)),
          sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (x, y + 1) xa,
        tri_fuzzify_image (I, delta2) (x, y + 1) ya))
   (%(x, y). x + y = ya))))
                (%(x, y). x + y = ya))))
             (%(x, y). int x - int y = int z)),
                    tri_fuzzify (0, delta)),
              fleq (%z. sup__c (Set__map
           (%(xa, ya).
              FuzzyEdgeDetection2_FUZZY_SET.min
              (sup__c (Set__map

               FuzzyEdgeDetection2_FUZZY_SET.min
      (sup__c (Set__map
   (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (tri_fuzzify_image (I, delta2) (2 + x, y) xa,
        tri_fuzzify_image (I, delta2) (x, y) ya))
   (%(x, y). x + y = xa)),
```

```
        sup__c (Set__map
     (%(xa, ya).
         FuzzyEdgeDetection2_FUZZY_SET.min
          (tri_fuzzify_image (I, delta2) (x + 1, y + 1) xa,
          tri_fuzzify_image (I, delta2) (x + 1, y - 1) ya))
     (%(x, y). x + y = ya))))
               (%(x, y). x + y = xa)),
            sup__c (Set__map
                (%(xa, ya).
             FuzzyEdgeDetection2_FUZZY_SET.min
    (sup__c (Set__map
     (%(xa, ya)
         FuzzyEdgeDetection2_FUZZY_SET.min
          (tri_fuzzify_image (I, delta2) (x + 1, y) xa,
          tri_fuzzify_image (I, delta2) (x + 1, y) ya))
     (%(x, y). x + y = xa)),
           sup__c (Set__map
     (%(xa, ya).
         FuzzyEdgeDetection2_FUZZY_SET.min
          (tri_fuzzify_image (I, delta2) (x + 1, y) xa,
          tri_fuzzify_image (I, delta2) (x + 1, y) ya))
     (%(x, y). x + y = ya))))
         (%(x, y). x + y = ya))))
     (%(x, y). int x - int y = int z)),
                      tri_fuzzify (0, delta))))),
          FuzzyEdgeDetection2_FUZZY_REASONING.fequal
           (%z. sup__c (Set__map
           (%(xa, ya).
             FuzzyEdgeDetection2_FUZZY_SET.min
            (sup__c (Set__map
                (%(xa, ya).
        FuzzyEdgeDetection2_FUZZY_SET.min
            (sup__c (Set__map
                 (%(xa, ya).
               FuzzyEdgeDetection2_FUZZY_SET.min
(tri_fuzzify_image (I, delta2) (x + 1, y) xa,
tri_fuzzify_image (I, delta2) (x - 1, y) ya))
              (%(x, y). x + y = xa)),
              sup__c (Set__map
                            (%(xa, ya).
            FuzzyEdgeDetection2_FUZZY_SET.min
(tri_fuzzify_image (I, delta2) (x, y + 1) xa,
 tri_fuzzify_image (I, delta2) (x, y - 1) ya))
            (%(x, y). x + y = ya))))
           (%(x, y). x + y = xa)),
         sup__c (Set__map
                 (%(xa, ya).
        FuzzyEdgeDetection2_FUZZY_SET.min
               (sup__c (Set__map
                     (%(xa, ya).
                   FuzzyEdgeDetection2_FUZZY_SET.min
(tri_fuzzify_image (I, delta2) (x, y) xa,
tri_fuzzify_image (I, delta2) (x, y) ya))
                    %(x, y). x + y = xa)),
           sup__c (Set__map
                           (%(xa, ya).
```

```
             FuzzyEdgeDetection2_FUZZY_SET.min
(tri_fuzzify_image (I, delta2) (x, y) xa,
tri_fuzzify_image (I, delta2) (x, y) ya))
            (%(x, y). x + y = ya))))
              (%(x, y). x + y = ya))))
              (%(x, y). int x - int y = int z)),
            tri_fuzzify (0, delta)))
        <= uni_min
            (uni_max
              (uni_min
                (fleq (%z. sup__c (Set__map
                 (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
           (sup__c (Set__map
                          (%(xa, ya).
       FuzzyEdgeDetection2_FUZZY_SET.min
        (sup__c (Set__map
      (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
           (tri_fuzzify_image (I, delta1) (x + 1, y) xa,
           tri_fuzzify_image (I, delta1) (x - 1, y) ya))
      (%(x, y). x + y = xa)),
           sup__c (Set__map
      (%(xa, ya
          FuzzyEdgeDetection2_FUZZY_SET.min
           (tri_fuzzify_image (I, delta1) (x, y + 1) xa,
           tri_fuzzify_image (I, delta1) (x, y - 1) ya))
      (%(x, y). x + y = ya))))
         (%(x, y). x + y = xa)),
             sup__c (Set__map
                       (xa, ya).
            FuzzyEdgeDetection2_FUZZY_SET.min
                    (sup__c (Set__map
      (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
           (tri_fuzzify_image (I, delta1) (x, y) xa,
           tri_fuzzify_image (I, delta1) (x, y) ya))
      (%(x, y). x + y = xa)),
                 sup__c (Set__map
      (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
           (tri_fuzzify_image (I, delta1) (x, y) xa,
           tri_fuzzify_image (I, delta1) (x, y) ya))
      (%(x, y). x + y = ya))))
             (%(x, y). x + y = ya))))
           (%(x, y). int x - int y = int z)),
            tri_fuzzify (0, delta)),
          ni_max
   (fgeq (%z. sup__c (Set__map
             (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
            (sup__c (Set__map
                (%(xa, ya).
        FuzzyEdgeDetection2_FUZZY_SET.min
               (sup__c
(Set__map
```

```
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x + 1, y + 1) xa,
       tri_fuzzify_image (I, delta1) (x - 1, y + 1) ya))
  (%(x, y). x + y = xa)),
                        sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x, 2 + y) xa,
       tri_fuzzify_image (I, delta1) (x, y) ya))
  (%(x, y). x + y = ya))))
              (%(x, y). x + y = xa)),
                sup__c (Set__map
                        (%(xa, ya).
           FuzzyEdgeDetection2_FUZZY_SET.min (sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x, y + 1) xa,
       tri_fuzzify_image (I, delta1) (x, y + 1) ya))
  (%(x, y). x + y = xa)),
        sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x, y + 1) xa,
       tri_fuzzify_image (I, delta1) (x, y + 1) ya))
  (%(x, y). x + y = ya))))
             (%(x, y). x + y = ya))))
           (%(x, y). int x - int y = int z)),
           tri_fuzzify (0, delta)),
            fgeq (%z. sup__c (Set__map
              (%(xa, ya).
         FuzzyEdgeDetection2_FUZZY_SET.min
    (sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (2 + x, y) xa,
       tri_fuzzify_image (I, delta1) (x, y) ya))
  (%(x, y). x + y = xa)),
         sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x + 1, y + 1) xa,
       tri_fuzzify_image (I, delta1) (x + 1, y - 1) ya))
  (%(x, y). x + y = ya))))
       (%(x, y). x + y = xa)),
          sup__c (Set__map
       (%(xa, ya).
         FuzzyEdgeDetection2_FUZZY_SET.min
                (sup__c
(Set__map
  (%(xa, ya).
```

```
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x + 1, y) xa,
       tri_fuzzify_image (I, delta1) (x + 1, y) ya))
  (%(x, y). x + y = xa)),
                  sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x + 1, y) xa,
       tri_fuzzify_image (I, delta1) (x + 1, y) ya))
  (%(x, y). x + y = ya))))
      (%(x, y). x + y = ya))))
  (%(x, y). int x - int y = int z)),
            tri_fuzzify (0, delta)))),
              uni_min
                (fgeq (%z. sup__c (Set__map
         (%(xa, ya).
         FuzzyEdgeDetection2_FUZZY_SET.min
        sup__c (Set__map
              (%(xa, ya).
             FuzzyEdgeDetection2_FUZZY_SET.min
            (sup__c (Set__map
      (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
           (tri_fuzzify_image (I, delta1) (x + 1, y) xa,
           tri_fuzzify_image (I, delta1) (x - 1, y) ya))
      (%(x, y). x + y = xa)),
          sup__c (Set__map
      (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
           (tri_fuzzify_image (I, delta1) (x, y + 1) xa,
            tri_fuzzify_image (I, delta1) (x, y - 1) ya))
      (%(x, y). x + y = ya))))
           %(x, y). x + y = xa)),
      up__c (Set__map
                     (%(xa, ya).
     FuzzyEdgeDetection2_FUZZY_SET.min
   (sup__c (Set__map
      (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
           (tri_fuzzify_image (I, delta1) (x, y) xa,
          tri_fuzzify_image (I, delta1) (x, y) ya))
      (%(x, y). x + y = xa)),
            sup__c (Set__map
      (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
           (tri_fuzzify_image (I, delta1) (x, y) xa,
          tri_fuzzify_image (I, delta1) (x, y) ya))
      (%(x, y). x + y = ya))))
            (%(x, y). x + y = ya))))
              (%(x, y). int x - int y = int z)),
        tri_fuzzify (0, delta)),
   uni_max
    (fleq (%z. sup__c (Set__map
           (%(xa, ya).
        FuzzyEdgeDetection2_FUZZY_SET.min
```

```
          (sup__c (Set__map
           (%(xa, ya).
          FuzzyEdgeDetection2_FUZZY_SET.min
                        (sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x + 1, y + 1) xa,
        tri_fuzzify_image (I, delta1) (x - 1, y + 1) ya))
  (%(x, y). x + y = xa)),
               sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x, 2 + y) xa,
       tri_fuzzify_image (I, delta1) (x, y) ya))
  (%(x, y). x + y = ya))))
                (%(x, y). x + y = xa)),
                          sup__c (Set__map
                        (%(xa, ya).
                FuzzyEdgeDetection2_FUZZY_SET.min
                        (sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x, y + 1) xa,
      tri_fuzzify_image (I, delta1) (x, y + 1) ya))
  (%(x, y). x + y = xa)),
      sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x, y + 1) xa,
      tri_fuzzify_image (I, delta1) (x, y + 1) ya))
  (%(x, y). x + y = ya))))
                      (%(x, y). x + y = ya))))
     (%(x, y). int x - int y = int z)),
    tri_fuzzify (0, delta)),
         leq (%z. sup__c (Set__map
     (%(xa, ya).
              FuzzyEdgeDetection2_FUZZY_SET.min
     sup__c (Set__map
            %(xa, ya).
    FuzzyEdgeDetection2_FUZZY_SET.min
         (sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (2 + x, y) xa,
       tri_fuzzify_image (I, delta1) (x, y) ya))
  (%(x, y). x + y = xa)),
          sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x + 1, y + 1) xa,
```

```
      tri_fuzzify_image (I, delta1) (x + 1, y - 1) ya))
  (%(x, y). x + y = ya))))
      (%(x, y). x + y = xa)),
   sup__c (Set__map
      (%(xa, ya).
  FuzzyEdgeDetection2_FUZZY_SET.min
     (sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x + 1, y) xa,
      tri_fuzzify_image (I, delta1) (x + 1, y) ya))
  (%(x, y). x + y = xa)),
         sup__c
(Set__map
  (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
       (tri_fuzzify_image (I, delta1) (x + 1, y) xa,
       tri_fuzzify_image (I, delta1) (x + 1, y) ya))
  (%(x, y). x + y = ya))))
            (%(x, y). x + y = ya))))
        (%(x, y). int x - int y = int z)),
     tri_fuzzify (0, delta))))),
   FuzzyEdgeDetection2_FUZZY_REASONING.fequal
  (%z. sup__c (Set__map
          (%(xa, ya).
   FuzzyEdgeDetection2_FUZZY_SET.min
        (sup__c (Set__map
          (%(xa, ya).
  FuzzyEdgeDetection2_FUZZY_SET.min
          (sup__c (Set__map
               (%(xa, ya).
  FuzzyEdgeDetection2_FUZZY_SET.min
   (tri_fuzzify_image (I, delta1) (x + 1, y) xa,
   tri_fuzzify_image (I, delta1) (x - 1, y) ya))
      (%(x, y). x + y = xa)),
         sup__c (Set__map
             (%(xa, ya).
  FuzzyEdgeDetection2_FUZZY_SET.min
   (tri_fuzzify_image (I, delta1) (x, y + 1) xa,
   tri_fuzzify_image (I, delta1) (x, y - 1) ya))
             (%(x, y). x + y = ya))))
      (%(x, y). x + y = xa)),
         sup__c (Set__map
            (%(xa, ya).
      FuzzyEdgeDetection2_FUZZY_SET.min
               (sup__c (Set__map
                  (%(xa, ya).
  FuzzyEdgeDetection2_FUZZY_SET.min (tri_fuzzify_image
  (I, delta1) (x, y) xa,
  tri_fuzzify_image (I, delta1) (x, y) ya))
                (%(x, y). x + y = xa)),
                   sup__c (Set__map
                        (%(xa, ya).
  FuzzyEdgeDetection2_FUZZY_SET.min (tri_fuzzify_image
  (I, delta1) (x, y) xa,
```

```
tri_fuzzify_image (I, delta1) (x, y) ya))
                          (%(x, y). x + y = ya))))
                     (%(x, y). x + y = ya))))
          (%(x, y). int x - int y = int z)),
             tri_fuzzify (0, delta)))
```