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# Orbit closures of quiver representations

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### ORBIT CLOSURES OF QUIVER REPRESENTATIONS

A dissertation presented by

Kavita Sutar

 $\mathrm{to}$ 

Department of Mathematics

In partial fulfillment of the requirements for the degree of

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### ORBIT CLOSURES OF QUIVER REPRESENTATIONS

by

Kavita Sutar

#### ABSTRACT OF DISSERTATION

Submitted in the partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics in the graduate school of Arts and Sciences of Northeastern University, February 2012

### Abstract

Let Q be a Dynkin quiver. We study orbit closures in  $\operatorname{Rep}(Q, \underline{d})$ , the affine space of quiver representations of a fixed dimension vector. The orbits arise from the action of  $\operatorname{Gl}(\underline{d})$  on  $\operatorname{Rep}(Q, \underline{d})$  and we consider their closure in the Zariski topology.

We investigate the properties of coordinate rings of orbit closures for quivers of type  $A_3$  by considering the desingularization given by Reineke [Rei03]. We construct explicit minimal free resolutions of the defining ideals of the orbit closures thus giving us a minimal set of generators for the defining ideal. The resolution allows us to read off some geometric properties of the orbit closure. In addition, we give a characterization for the orbit closure to be Gorenstein.

Next, we investigate orbit closures of Dynkin quivers with every vertex being source or sink. We use this resolution to derive the normality of such orbit closures. As a consequence we obtain the normality of certain orbit closures of type E.

Finally we consider orbit closures of type equioriented  $A_n$ . In this context we consider varieties  $Z(\beta, \gamma)$  defined by Schofield [Sch92] and obtain conditions for these varieties to be orbit closures. We also obtain resolutions for a class of orbit closures and recover normality for this class. This is a special case of a more general result of Abeasis, Del Fra and Kraft [ADFK81].

**Keywords** : Quiver representations, orbit closures, desingularization, minimal free resolution, normal, Cohen-Macaulay, Gorenstein, Lascoux-Kempf-Weyman, geometric technique, Bott's theorem, vector bundles, Dynkin quiver.

To my mother

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### Introduction

The object of the thesis is to study closures of orbits of quiver representations. The space  $\operatorname{Rep}(Q,\underline{d})$  of representations of a quiver Q of a fixed dimension vector  $\underline{d}$  can be viewed as an affine space. The algebraic group  $\operatorname{Gl}(\underline{d})$  acts on  $\operatorname{Rep}(Q,\underline{d})$  by simultaneous change of basis at each vertex of Q. The orbits of this action are isomorphism classes of quiver representations. The closures of these orbits in the Zariski topology are the objects of our interest. For a representation V of quiver Q, we denote the closure of the orbit of V by  $\overline{O}_V$ . The above situation generalizes two classical problems from linear algebra: the classification of endomorphisms of a vector space and the classification of linear maps between two vector spaces. In the latter case, the corresponding orbit closures are the well-known determinantal varieties.

There are two aspects of our work. One is demonstrating the use of the geometric technique (also referred to as the Kempf-Lascoux-Weyman technique in literature) in calculating resolutions of varieties. It is based on Kempf's collapsing for homogeneous vector bundles [Kem76] and is a powerful technique for syzygy calculations. The orbit closures described above admit a desingularization which is ideal for the application of the technique.

The other aspect is the study of the geometry of these orbit closures. Like determinantal varieties, these orbit closures provide good examples of varieties and are of intrinsic interest to geometers. They have been objects of considerable interest and research in the recent years. A good survey of the current state of research in orbit closures can be found in [Zwa11].

Using the geometric technique we can calculate a minimal free complex  $\mathbf{F}_{\bullet}$  supported in the variety  $\overline{O}_V$  for any  $V \in \operatorname{Rep}(Q,\underline{d})$  where Q is a Dynkin quiver. If  $\mathbf{F}_{\bullet}$  has no terms in negative degrees then it provides a resolution of the coordinate ring of the normalization of  $\overline{O}_V$ . Furthermore, if  $F_0$  is the coordinate ring of the affine space  $\operatorname{Rep}(Q,\underline{d})$  then  $\overline{O}_V$ itself is normal so that in this case  $\mathbf{F}_{\bullet}$  gives a resolution of the coordinate ring of  $\overline{O}_V$ . It should be pointed out that the resolution obtained by the geometric technique in the case of quiver  $1 \to 2$  is the same as Lascoux's resolution; in this sense our resolutions generalize Lascoux's resolution.

An important ingredient in the calculation is the desingularization Z of  $\overline{O}_V$ . The construction of such a desingularization was given by Reineke [Rei03]. His construction holds for orbit closures arising from representations of any Dynkin quiver Q. The desingularization Z obtained by this construction is an incidence variety in the product space  $\operatorname{Rep}(Q,\underline{d}) \times \prod_{x \in Q_0} \operatorname{Flag}(d_*(x), V_x)$ . Here  $\operatorname{Flag}(d_*(x), V_x)$  denotes the variety of flags of dimensions specified by  $d_*(x)$  contained in the vector space  $V_x$  at vertex x of quiver Q. The number of subspaces in the flag and the dimensions  $d_*(x)$  depend on the choice of a directed partition of the Auslander-Reiten quiver of Q. For calculation reasons explained in Section 2.2, it is convenient to work with desingularizations for which the flags are of length one that is, when  $Z \subset \operatorname{Rep}(Q,\underline{d}) \times \prod_{x \in Q_0} \operatorname{Gr}(d(x), V_x)$ . We call such Z a 1-step desingularization. A representation V of Q admits a 1-step desingularization when all its indecomposable summands lie in exactly 2 parts of the partitioned Auslander-Reiten quiver. In theory, this restriction is not required for calculating a resolution but in order to simplify calculations we impose this condition on our orbit closures.

One of the strengths of the geometric technique is in the use of Bott's theorem which makes the calculation of the resolution algorithmic in some sense. In Section 4.1, we introduce the difference estimate  $D(\underline{\lambda})$  which denotes the degree of the term in  $\mathbf{F}_{\bullet}$  that a certain module (corresponding to  $\underline{\lambda}$ ) appears in. The key to showing that  $\mathbf{F}_{\bullet}$  is a resolution lies in proving that  $D(\underline{\lambda})$  is non-negative for every  $\underline{\lambda}$ . In fact we show something stronger, that  $D(\underline{\lambda}) \geq E_Q$  where  $E_Q$  is the Euler form of the quiver Q (this serves our purpose since  $E_Q > 0$  for Dynkin quiver Q). This we prove in the case of non-equioriented  $A_3$  (Proposition 4.1.4), for source-sink quivers (Theorem 5.1.3) and for equioriented  $A_n$ (Proposition 6.0.8). The proofs of these results involve combinatorics of Young tableaux and Bott's algorithm.

As a consequence, we effectively have an algorithm for calculating resolutions of orbit closures of representations of Dynkin quivers. These resolutions encode a lot of information about the geometry of the corresponding variety. In all the cases we consider, we are able to prove that the orbit closures are normal, Cohen-Macaulay with rational singularities. From the first term of the resolution one can read whether the defining ideal is determinantal (generated by some minors) or not. In any case one can tell what the generators of the defining ideal are. By analyzing the last term of the resolution once can draw inferences about the Gorenstein property. The existence of a resolution opens up many directions for exploring the geometric and algebraic properties of orbit closures.

The thesis is organized as follows: Chapters 1, 2 and 3 consist of background material. In Chapter 1 we discuss quivers and their representations and give a brief overview of Auslander-Reiten theory relevant to our context. Chapter 2 contains a review of orbit closures and a description of Reineke's desingularization for the same. In Chapter 3 we describe the geometric technique which is central to the calculation of resolutions.

Chapters 4, 5 and 6 form the crux of the thesis work. In Chapter 4 we consider the non-equioriented  $A_3$  quiver. This quiver is special because its Auslander-Reiten quiver admits a partition into 2 parts. This enables every orbit closure  $\overline{O}_V$  to admit a 1-step desingularization, a very desirable property. As a consequence the results obtained in this chapter hold for all orbit closures corresponding to non-equioriented  $A_3$ . First we contruct a minimal free resolution of the coordinate ring of  $\overline{O}_V$  (Section 4.1). We use this resolution to draw conclusions about geometric properties of  $\overline{O}_V$ ; in particular we show that  $\overline{O}_V$  is normal, Cohen-Macaulay and has rational singularities. This result is not new: Bobiński and Zwara show in [BZ01] that the orbit closures corresponding to  $A_n$ with arbitrary orientation are normal, Cohen-Macaulay with rational singularities. What is new is the resolution of the defining ideal of  $\overline{O}_V$  and consequently the method in which the conclusion is drawn. Next, we find a closed form for the first term  $F_1$  of the resolution; this enables us to describe in detail the nature of generators for the defining ideal of  $O_V$ (Section 4.2). We also analyze the last term of the resolution to investigate the Gorenstein property. We derive a sufficient condition for any normal orbit closure arising from a Dynkin quiver to be Gorenstein (Theorem 4.3.4). In case of non-equivarianted  $A_3$  quiver we prove a characterization of the Gorenstein property for  $\overline{O}_V$ . This characterization is based on the occurrence of indecomposables as summands of V with certain multiplicities so our proof proceeds case-by-case. Investigating the Gorenstein property in this manner for larger quivers appears to be quite complicated.

In Chapter 5 we consider the case of Dynkin quivers with orientation such that every vertex is either a source or a sink. We call these *source-sink* quivers. The resolution of orbit closures admitting a 1-step desingularization is calculated. The fact that the terms of these resolutions lie in positive degrees implies that the corresponding orbit closures are normal, Cohen-Macaulay with rational singularities. This is a hitherto unknown result for Dynkin quivers of type  $E_6$ ,  $E_7$  and  $E_8$ .

The case of equioriented  $A_n$  is considered in Chapter 6. The heart of the matter is again to show that the difference estimate  $D(\underline{\lambda})$  is non-negative. Once this is done, the geometric technique enables us to calculate resolutions of orbit closures admitting a 1-step desingularization. The result that orbit closures for equioriented  $A_n$  are normal, Cohen-Macaulay with rational singularities was first proven by Abeasis, Del Fra and Kraft in [ADFK81]. Their work initiated the study of orbit closures of representations of quivers. We recover this result for orbit closures in our case using the resolution of their defining ideals. An interesting generalization of these orbit closures are the varieties  $Y(\beta, \gamma)$  obtained as images of Schofield's incidence varieties  $Z(\beta, \gamma)$  [Sch92, DSW07]. We obtain conditions on dimension vectors  $\beta$  and  $\gamma$  for the variety  $Y(\beta, \gamma)$  to be an orbit closure. The positivity of the difference estimate  $D(\underline{\lambda})$  is true for the general varieties  $Y(\beta, \gamma)$ , so that we can construct a resolution of  $Y(\beta, \gamma)$  whenever  $Z(\beta, \gamma)$  is its desingularization. This is an interesting generalization and there are many possibilities to be explored here.

A word about the actual computations: these involve calculating the exterior powers of a certain vector bundle  $\xi$  using Cauchy's formula. This calculation makes use of the Littlewood-Richardson rule for tensoring Schur functors. Then we associate weights to the resulting vector bundles and use Bott's theorem to calculate the contribution of the vector bundles to the complex  $\mathbf{F}_{\bullet}$ . This is a long computation to do by hand. Fortunately, there exists a computer program that can do this calculation. This program developed by Jerzy Weyman and Jason Ribeiro calculates the contributions of the exterior powers  $\bigwedge^{t} \xi$ . The program is in its last stages of development and will be available online soon.

## Chapter 1

## Quivers

A quiver  $Q = (Q_0, Q_1)$  is a directed graph with  $Q_0$  being its set of vertices and  $Q_1$  being its set of arrows. A representation of a quiver Q is an assignment of finite dimensional vector spaces to the vertices and linear maps to the arrows of Q. Quiver representations were originally introduced to study problems in linear algebra. Later it was found that quiver representations play an important role in studying representations of finitedimensional algebras and also appear in the study of Kac-Moody Lie algebras, quantum groups, geometric invariant theory etc.

In Section 1.1 we introduce some preliminary notions related to quivers and their representations which we use in our work. In Section 1.2 we review parts of Auslander-Reiten theory relevant to this thesis. Most of our notation and terminology is adopted from the book 'Elements of the representation theory of Associative Algebras' by Assem, Simson and Skowroński [ASS06].

#### **1.1 Quiver representations**

**Definition 1.1.1.** A **quiver** is a pair  $Q = (Q_0, Q_1)$  where  $Q_0$  is the set of vertices and  $Q_1$  is a set of arrows.

For any arrow a in  $Q_1$ , we let ta denote the tail (starting point) and ha denote the head (ending point) of a; thus any arrow a can be denoted as  $ta \xrightarrow{a} ha$ .

Thus a quiver is simply an oriented graph without any restriction on the number of vertices and arrow and on the orientation of arrows. The term 'quiver' is used instead of 'graph' mainly to differentiate the subject material from graph theory and uses of graphs in other areas of mathematics.

A quiver Q is said to be **finite** if both  $Q_0$  and  $Q_1$  are finite sets. The underlying graph  $\hat{Q}$  is the undirected graph obtained by forgetting the orientation of arrows of Q. A **connected** quiver is one whose underlying graph is connected.

A **Dynkin quiver** is a quiver whose underlying unoriented graph is one of the following Dynkin diagrams:



A **path** of length k in a quiver is a sequence of k arrows  $a_1a_2...a_k$  such that  $ha_i =$ 

 $ta_{i+1}$ . The composition of two paths is defined by concatenation:

$$a_1 \dots a_k \circ b_1 \dots b_l = \begin{cases} a_1 \dots a_k b_1 \dots b_l & \text{if } ha_k = tb_1 \\ 0 & \text{otherwise} \end{cases}$$

The constant path at vertex i will be denoted by  $\epsilon_i$ .

**Definition 1.1.2.** The **path algebra** KQ of a quiver Q is a K-algebra whose underlying K-vector space has basis all paths in Q and product of two basis vectors is the composition of paths.

KQ is finite dimensional if and only if Q is finite and has no oriented cycles.

**Definition 1.1.3.** A representation  $V = ((V_i)_{i \in Q_0}, (V(a))_{a \in Q_1})$  of Q is given by assigning a finite dimensional K-vector space  $V_i$  to every vertex  $i \in Q_0$  and K-linear maps  $V_{ta} \xrightarrow{V(a)} V_{ha}$  to every arrow  $a \in Q_1$ . The **dimension vector** of a representation  $((V_x)_{x \in Q_0}, (V(a))_{a \in Q_1})$  is defined as the function  $\underline{d} : Q_0 \longrightarrow \mathbb{Z}$  given by  $\underline{d}(x) = \dim$  $(V_x)$ .

A representation V is thus determined by the dimension vector  $\underline{d}$  and the maps  $V_{ta} \rightarrow V_{ha}$ . Given representations  $V = ((V_i)_{i \in Q_0}, (V(a))_{a \in Q_1})$  and  $W = ((W_i)_{i \in Q_0}, (W(a))_{a \in Q_1})$  of Q, a morphism  $\Phi : V \rightarrow W$  is a collection of K-linear maps  $\phi_i : V_i \rightarrow W_i$  such that for every  $a \in Q_1$ , the square

$$V_{ta} \xrightarrow{V(a)} V_{ha}$$

$$\downarrow \phi_{ta} \qquad \qquad \downarrow \phi_{ha}$$

$$W_{ta} \xrightarrow{W(a)} W_{ha}$$

commutes.

With this definition of morphisms, the collection of all representations of a quiver Q(over K) forms an abelian category which we denote by  $\operatorname{Rep}_K(Q)$  [ASS06, Chapter 3]. The full subcategory of  $\operatorname{Rep}_K(Q)$  consisting of finite-dimensional representations will be denoted by  $\operatorname{rep}_K(Q)$ .

The **representation space**  $\operatorname{Rep}(Q, \underline{d})$  of a quiver Q is the collection of all representations of Q of fixed dimension vector  $\underline{d}$ . Note that we can think of  $\operatorname{Rep}(Q, \underline{d})$  as the set  $\prod_{a \in Q_1} \operatorname{Hom}(K^{d_{ta}}, K^{d_{ha}})$ . Thus,  $\operatorname{Rep}(Q, \underline{d})$  is a finite dimensional K-vector space with an affine structure.

We can introduce a non-symmetric bilinear form, called the **Euler form**, on the space of dimension vectors of representations of a quiver Q as follows. For  $\alpha, \beta \in \mathbb{N}^{|Q_0|}$  define

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha)$$
(1.1.1)

In particular, this gives

$$\langle \alpha, \alpha \rangle = \sum_{x \in Q_0} \alpha(x)^2 - \sum_{a \in Q_1} \alpha(ta) \alpha(ha)$$
 (1.1.2)

Let  $Gl(\alpha) = \prod_{i \in Q_0} Gl(\alpha_i, K)$ . Then  $\sum_{x \in Q_0} \alpha(x)^2 = \dim Gl(\alpha)$  and

$$\sum_{a \in Q_1} \alpha(ta) \alpha(ha) = \dim \operatorname{Rep}_K(Q, \alpha).$$
 Thus we have

$$\langle \alpha, \alpha \rangle = \dim Gl(\alpha) - \dim \operatorname{Rep}_K(Q, \alpha)$$
 (1.1.3)

### **1.2** Auslander-Reiten quiver

Gabriel [Gab72] proved that the set of isomorphism classes of indecomposable representations of Q is in bijective correspondence with the set of positive roots  $R^+$  of the corresponding root systems. Under this correspondence, simple roots correspond to simple objects. Every representation of Q can be written uniquely (upto permutation of factors) as a direct sum of indecomposable representations

$$V = \bigoplus_{\alpha \in R^+} m_\alpha X_\alpha$$

(where  $m_{\alpha}$  = multiplicity of  $X_{\alpha}$  in V). The indecomposable representations can be obtained as the vertices of the Auslander-Reiten quiver of Q.

**Definition 1.2.1.** Let A be a basic and connected finite dimensional algebra. The Auslander-Reiten quiver  $\Gamma(\text{mod } A)$  of mod A is defined as:

- (1) The vertices are the isomorphism classes [X] of indecomposables modules X in mod A.
- (2) For vertices [M], [N] the arrows  $[M] \to [N]$  are in bijective correspondence with the vectors of a basis of the K-vector space of the irreducible morphisms  $M \to N$ .

If Q is a finite acyclic quiver then the path algebra KQ is a basic, connected, finitedimensional K-algebra. The category mod KQ is representation-finite, which means there are finitely many isomorphism classes of indecomposable KQ-modules. As a result, the quiver  $\Gamma(\text{mod } KQ)$  is a finite quiver. Also, every irreducible morphism  $f: M \to N$  is either a monomorphism or epimorphism and if M = N, then f must be an isomorphism since M is finite-dimensional K-vector space. Thus  $\Gamma(\text{mod } KQ)$  has no loops. For a finite acyclic quiver Q (without relations) the simple, indecomposable projective and indecomposable injective modules over KQ are easy to describe.

(1) The simple modules correspond to the representations  $((S_i)_{i \in \mathbb{Q}_0}, (S_a)_{a \in Q_1})$  as *i* varies over  $Q_0$  such that

$$(S_i)_j = \begin{cases} K & i = j \\ 0 & i \neq j \end{cases}$$
$$S_a = 0 \quad \forall a \in Q_1.$$

- (2) The indecomposable projectives correspond to representations ((P<sub>i</sub>)<sub>i∈Q<sub>0</sub></sub>, (P<sub>a</sub>)<sub>a∈Q<sub>1</sub></sub>) such that (P<sub>i</sub>)<sub>j</sub> = K<sup>n</sup> where n is the dimension of the vector space having as basis the set of all paths from i to j. If a : i → j then P<sub>a</sub> : P<sub>i</sub> → P<sub>j</sub> is given by right multiplication by a.
- (3) The indecomposable injectives correspond to representations ((I<sub>i</sub>)<sub>i∈Q0</sub>, (I<sub>a</sub>)<sub>a∈Q1</sub>) such that (I<sub>i</sub>)<sub>j</sub> = K<sup>m</sup> where m is the dimension of the vector space with basis the set of all paths from j to i. For an arrow a : i → j then I<sub>a</sub> : I<sub>i</sub> → I<sub>j</sub> is given by the dual of left multiplication by a.

Thus the number of simples equals the number of projectives equals the number of injectives equals the number of vertices in Q.

Notation 1.2.2. An element of mod KQ corresponds to a representation of Q. We use this correspondence to write the modules in mod KQ. We retain the shape of Q and at vertex i we write  $K^n$  or n where n is the dimension of the vector space at i. For instance, if Q is  $1 \to 2$  we will use either 10 or K0 to denote  $S_1$ , 01 or 0K to denote  $S_2$  and 11 or KK to denote  $P_1 = I_2$ .

**Example 1.2.3.** Let Q be the quiver  $1 \stackrel{a}{\leftarrow} 2 \stackrel{b}{\rightarrow} 3 \stackrel{c}{\leftarrow} 4$ . The underlying graph of Q is the Dynkin diagram  $A_4$ . The number of positive roots in the corresponding root system is 10, so by Gabriel's theorem there are 10 isomorphism classes of indecomposable modules in mod KQ. We list all the simple, indecomposable projective and indecomposable injective modules in mod KQ:

Simple	Projective	Injective
$S_1 = K000$	$P_1 = K000$	$I_1 = KK00$
$S_2 = 0K00$	$P_2 = KKK0$	$I_2 = 0K00$
$S_3 = 00K0$	$P_3 = 00K0$	$I_3 = 0KKK$
$S_4 = 000K$	$P_4 = 00KK$	$I_4 = 000K$

 $\Gamma(\mod KQ)$  is



Figure 1.1: AR quiver of source-sink  $A_4$ 

To construct an Auslander-Reiten quiver it is useful to know how to construct almost split sequences. A good account of the theory of almost split sequences can be found in [ASS06].

## Chapter 2

### **Orbit Closures**

In Section 1.2 we noted that for a quiver Q the space of quiver representations  $\operatorname{Rep}(Q, \underline{d})$  can be viewed as an affine space. The algebraic group  $Gl(\underline{d})$  acts on  $\operatorname{Rep}(Q, \underline{d})$  by simultaneous change of basis giving rise to orbits. The Zariski closures of these orbits are the objects of our study.

In this chapter we review some of the known results about orbit closures. For most part we will review only those concepts which we use later. For a fuller description of the study of orbit closures we refer to the excellent survey article of Zwara [Zwa11]. Section 2.1 contains the basic definitions and examples of orbit closures. These varieties are singular in general. In Section 2.2 we discuss the problem of studying the singularities in orbit closures. In our later work we make use of a particular desingularization of the orbit closures. We explain this desingularization in section 2.2.1.

#### 2.1 Varieties of representations

We fix a finite quiver  $Q = (Q_0, Q_1)$  and let  $\underline{d} \in \mathbb{N}^{|Q_0|}$ . Define  $Gl(\underline{d})$  to be the product  $\prod_{x \in Q_0} \operatorname{GL}(d(x), K)$ . We want to consider the following action of  $Gl(\underline{d})$  on  $\operatorname{Rep}(Q, \underline{d})$ 

$$((g_x)_{x \in Q_0}, (V(a))_{a \in Q_1}) \longmapsto (g_{ha}V(a)g_{ta}^{-1})_{a \in Q_1}$$

The orbits of this action are the isomorphism classes of representations of Q.

**Example 2.1.1.** Let Q be the quiver  $1 \xrightarrow{a} 2$  and let  $\underline{d} = (d_1, d_2)$ . Then  $\operatorname{Rep}(Q, (d_1, d_2))$  is the space of all representations of Q of the form

$$V_1 \xrightarrow{V(a)} V_2$$

where dim  $V_1 = d_1$ , dim  $V_2 = d_2$  and V(a) is a  $d_2 \times d_1$  matrix over K. The entries of the matrix V(a) determine the representation, hence  $\operatorname{Rep}(Q, (d_1, d_2))$  is isomorphic to the affine space  $\mathbb{A}^{d_1 d_2}$ . In this case  $Gl(\underline{d}) = Gl(d_1, K) \times Gl(d_2, K)$ . The action of  $Gl(d_1, d_2)$ on  $\operatorname{Rep}(Q, (d_1, d_2))$  is given by

$$(g_1, g_2) \circ V = g_2(V(a))g_1^{-1}.$$

The orbits of this action are determined by the rank of the matrix V(a). Let  $0 \le r \le \min\{d_1, d_2\}$ . The orbits are described by

$$O_r = \{ V \in \operatorname{Rep}(Q, (d_1, d_2)) \mid \operatorname{rank} V(a) = r \}.$$

Thus  $O_r$  consists of  $d_2 \times d_1$  matrices of rank r. The closure of  $O_r$  consists of all  $d_2 \times d_1$ matrices of rank at most r

$$\overline{O}_r = \bigcup_{i \le r} O_i = \{ V \in \operatorname{Rep}(Q, (d_1, d_2)) \mid \operatorname{rank} V(a) \le r \}.$$

These are the well-known **determinantal varieties**. The orbit closure  $\overline{O}_r$  is generated by  $(r+1) \times (r+1)$  minors of matrices  $V(a) \in \overline{O}_r$ .

For  $V \in \operatorname{Rep}(Q,\underline{d})$  we will denote the orbit of V by  $O_V$ . The orbits are irreducible smooth varieties which are open in their closure in  $\operatorname{Rep}(Q,\underline{d})$ . The orbit closure  $\overline{O}_V$  is not smooth in general.

**Lemma 2.1.2.** Let  $V \in \operatorname{Rep}(Q, \alpha)$ . Then

$$\operatorname{codim}\overline{O}_V = \operatorname{dim} \operatorname{Ext}^1(V, V)$$

Proof. By the Orbit-Stabilizer theorem we have-

$$\dim \overline{O}_V = \dim O_V$$
  
= dim  $Gl_K(\alpha) - \dim \operatorname{Stab}(V)$   
= dim  $Gl_K(\alpha) - \dim \operatorname{Aut}_A(V, V)$   
= dim  $Gl_K(\alpha) - \dim \operatorname{Hom}_A(V, V)$ 

since  $\operatorname{Aut}_A(V)$  is open in  $\operatorname{Hom}_A(V, V)$ . So

$$\operatorname{codim} \overline{O}_{V} = \dim \operatorname{Rep}(Q, \alpha) - \dim Gl(\underline{d}) + \dim \operatorname{Hom}_{A}(V, V)$$

$$= \sum_{a \in Q_{1}} \alpha(ta)\alpha(ha) - \sum_{x \in Q_{0}} \alpha(x)^{2} + \dim \operatorname{Hom}_{A}(V, V)$$

$$= -\langle \alpha, \alpha \rangle + \dim \operatorname{Hom}_{A}(V, V)$$

$$= \dim \operatorname{Ext}^{1}(V, V) - \dim \operatorname{Hom}_{A}(V, V) + \dim \operatorname{Hom}_{A}(V, V)$$

$$= \dim \operatorname{Ext}^{1}(V, V).$$

For a more useful description of  $O_V$  and  $\overline{O}_V$  we describe a partial order on the orbits. Let  $V, W \in \operatorname{Rep}(Q, \underline{d})$ . We say that  $V \leq_{deg} W$  (i.e. V is a degeneration of W) if the orbit of W is contained in the closure of the orbit of V (i.e.  $O_W \subset \overline{O}_V$ ). This introduces a partial order on the orbits. Riedtmann [Rie98] introduced another partial order given by  $V \leq_{Hom} W$  if dim  $\operatorname{Hom}_Q(X, V) \leq \dim \operatorname{Hom}_Q(X, W)$  for all indecomposables X in  $\operatorname{Rep}(Q, \underline{d})$ . The connection between these two partial orders is given by

**Theorem 2.1.3.** (Bongartz [Bon96]) If A is a representation-directed, finite dimensional, associative K-algebra then the partial orders  $\leq_{deg}$  and  $\leq_{Hom}$  coincide.

Since  $\operatorname{Rep}(Q, \underline{d})$  satisfies the hypotheses of this theorem, the orbit of  $V \in \operatorname{Rep}(Q, \underline{d})$  is given by

$$O_V = \{ W \in \operatorname{Rep}(Q, \underline{d}) \mid \dim \operatorname{Hom}_Q(X, V) = \dim \operatorname{Hom}_Q(X, W) \}$$
(2.1.1)

and the corresponding orbit closure is

$$\overline{O}_V = \{ W \in \operatorname{Rep}(Q, \underline{d}) \mid \dim \operatorname{Hom}_Q(X, V) \le \dim \operatorname{Hom}_Q(X, W) \}$$
(2.1.2)

where X varies over all indecomposables in  $\operatorname{Rep}(Q, \underline{d})$ .

If Q is a Dynkin quiver, it is known that  $\operatorname{Rep}(Q, \underline{d})$  consists of finitely many isomorphism classes of indecomposables hence the action of  $Gl(\underline{d})$  admits finitely many orbits.  $\operatorname{Rep}(Q, \underline{d})$  is irreducible implies there exists a dense  $Gl(\underline{d})$ -orbit. The orbit closures of Dynkin quivers contain many interesting varieties. For example the Schubert varieties in partial flag varieties are among them.

**Example 2.1.4.** Let us consider the partial flag variety  $\operatorname{Flag}(r_1, r_2, \ldots, r_s; K^n)$ . The Schubert varieties are the orbits of the group B of upper-triangular matrices in  $\operatorname{Gl}(n, \mathbb{C})$  acting on  $\operatorname{Flag}(r_1, r_2, \ldots, r_s; K^n)$  (we identify  $\operatorname{Flag}(r_1, r_2, \ldots, r_s; K^n)$  with

 $Gl(n,\mathbb{C})/P(r_1,\ldots,r_s)$ ). Let Q be the quiver

$$x_1 \to x_2 \to \dots x_{n-1} \to u \leftarrow y_s \leftarrow \dots y_2 \leftarrow y_1$$

and the dimension vectors  $\beta(x_i) = i, \beta(u) = n, \beta(y_j) = r_j$ . Then the intersection of the orbit closure with the open set of representations with all linear maps being injective gives the fibered product  $Y \times_B Gl(n, \mathbb{C})$  for some Schubert variety Y. All Schubert varieties can be obtained in this way. Thus the study of the singularities of Schubert varieties of type  $A_n$  is part of the study of the singularities of orbit closures for quivers of type  $A_n$ .

### 2.2 Singularities in orbit closures

The algebraic group  $GL(\underline{d})$  acts on  $Rep(Q,\underline{d})$ ; for  $V \in Rep(Q,\underline{d})$ , let  $\overline{O}_V$  denote the closure of an orbit  $O_V$ . Then  $\overline{O}_V$  is a subvariety of  $Rep(Q,\underline{d})$ . It is an interesting problem to study the type of singularities that occur in these orbit closures. The geometry of such orbit closures was first studied by Abeasis, Del Fra and Kraft in [ADFK81]. They proved for the case of equioriented  $A_n$  (over fields of characteristic zero) that the orbit closures are normal, Cohen-Macaulay and have rational singularities. This result was generalized to fields of arbitrary characteristic by Lakshmibai and Magyar in [LM98]. They show using standard monomial theory that the defining ideals of orbit closures in case of equioriented  $A_n$  are reduced, so the singularities of  $\overline{O}_V$  are identical to those of Schubert varieties. This implies that the orbit closures are normal, Cohen-Macaulay etc. This result was generalized to orbit closures for arbitrary quivers of type  $A_n$  and  $D_n$  by Bobinski and Zwara in [BZ01] and [BZ02]. They make use of certain hom-controlled functors to reduce the general case to a special one and draw their conclusions by comparing the special case to Schubert varieties.

Our approach to studying orbit closures is calculating resolutions of their defining ideals. We use these resolutions to draw conclusions about the geometric properties of orbit closures. To calculate resolutions we will employ Weyman's geometric technique. A prerequisite for this technique is the existence of a desingularization which satisfies some more properties as described in Chapter 3. The next section describes the desingularization we will use.

#### 2.2.1 Desingularization

In [Rei03], Reineke describes an explicit method of constructing desingularizations of orbit closures of representations of Q. The desingularizations depend on certain directed partitions of the isomorphism classes of indecomposable objects (or equivalently of the set of positive roots  $R^+$ )-

**Definition 2.2.1.** A partition  $\mathcal{I}_* = (\mathcal{I}_1, \cdots, \mathcal{I}_s)$ , where  $R^+ = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_s$ , is called directed if:

- 1.  $Ext_O^1(X_\alpha, X_\beta) = 0$  for all  $\alpha, \beta \in \mathfrak{I}_t$  for  $t = 1, \cdots, s$ .
- 2.  $Hom_Q(X_\beta, X_\alpha) = 0 = Ext_Q^1(X_\alpha, X_\beta)$  for all  $\alpha \in \mathfrak{I}_t, \beta \in \mathfrak{I}_u, t < u$

These conditions can be expressed in terms of the Euler form as-

- 1.  $\langle \alpha, \beta \rangle = 0$  for  $\alpha, \beta \in \mathfrak{I}_t$  for  $t = 1, \cdots, s$
- 2.  $\langle \alpha, \beta \rangle \geq 0 \geq \langle \beta, \alpha \rangle$  for  $\alpha \in \mathfrak{I}_t, \beta \in \mathfrak{I}_u, t < u$

Let Q be a Dynkin quiver and consider its Auslander-Reiten quiver  $\Gamma(\mod KQ)$ . A partition of indecomposables exists because the category of finite-dimensional representations is directed; in particular, we can choose a sectional tilting module and let  $\mathcal{I}_t$  be its Coxeter translates. We fix a partition  $\mathcal{I}_*$  of  $\Gamma(\mod KQ)$ . Then the indecomposable representations  $X_{\alpha}$  are the vertices of  $\Gamma(\mod KQ)$ . For a representation  $V = \bigoplus_{\alpha \in R^+} m_{\alpha} X_{\alpha}$ , we define representations

$$V_{(t)} := \bigoplus_{\alpha \in \mathfrak{I}_t} m_\alpha X_\alpha, \quad t = 1, \cdots, s$$

Then  $V = V_{(1)} \oplus \cdots \oplus V_{(s)}$ . Let  $\underline{d}_t = \dim V_{(t)}$ . We consider the incidence variety

$$Z_{\mathfrak{I}_{*},V} \subset \prod_{x \in Q_{0}} Flag(d_{s}(x), d_{s-1}(x) + d_{s}(x), \cdots, d_{2}(x) + \cdots + d_{s}(x), V_{x}) \times Rep_{K}(Q, \underline{d})$$

defined as

$$Z_{\mathcal{I}_{*},V} = \{ ((R_{s}(x) \subset R_{s-1}(x) \subset \cdots \subset R_{2}(x) \subset V_{x}), V) \mid \forall a \in Q_{1}, \forall t, \ V(a)(R_{t}(ta)) \subset R_{t}(ha) \}$$

$$(2.2.1)$$

In this case we say that Z is a (s-1)-step desingularization.

**Theorem 2.2.2** (Reineke [Rei03]). Let Q be a Dynkin quiver,  $\mathfrak{I}_*$  a directed partition of  $R^+$ . Then the second projection

$$q: Z_{\mathfrak{I}_*,V} \longrightarrow Rep_K(Q,\underline{d})$$

makes  $Z_{\mathfrak{I}_{*},V}$  a desingularization of the orbit closure  $\overline{O}_{V}$ . More precisely,  $q(Z_{\mathfrak{I}_{*},V}) = \overline{O}_{V}$ and q is a proper birational isomorphism of  $Z_{\mathfrak{I}_{*},V}$  and  $\overline{O}_{V}$ .

In the next section, we will realize  $Z_{\mathcal{I}_{*},V}$  as the total space of a vector bundle over  $\prod_{x \in Q_0} Flag(d_s(x), d_{s-1}(x) + d_s(x), \cdots, d_2(x) + \cdots + d_s(x), V_x).$ 

### Chapter 3

### The geometric technique

We refer to the method used for constructing the resolution as the geometric technique (also referred to as the Kempf-Lascoux-Weyman geometric technique in recent literature). The general idea is to construct a desingularization Z of  $\overline{O}_V$  such that Z is the total space of a suitable vector bundle. Using the results of Kempf [Kem75] on collapsing of vector bundles, Lascoux [Las78] gave the construction of a minimal resolution of determinantal ideals for generic matrices. He made effective use of the combinatorics of representations of the general linear group and Bott's vanishing theorem for the cohomology of homogeneous vector bundles. The geometric technique provides a generalisation of Lascoux's construction.

The content of this chapter is based on the book 'Cohomology of vector bundles and syzygies' by Jerzy Weyman [Wey03].

Let E be an *n*-dimensional vector space over an algebraically closed fieldu K. The **Grassmannian**  $\operatorname{Gr}(r, E)$  is the set of all *r*-dimensional subspaces of E. It can be embedded in the projective space  $\mathbb{P}(\wedge^r E)$  using the Plücker embedding thus making  $\operatorname{Gr}(r, E)$ 

a projective variety. The general linear group  $\operatorname{Gl}(n, E)$  acts on  $\operatorname{Gr}(r, E)$  via its natural action on E.  $\operatorname{Gr}(r, E)$  is the unique orbit of this action.  $\operatorname{Gl}(n, E)$  also acts on  $\mathbb{P}(\wedge^r E)$  via its linear action on  $\wedge^r E$  and the Plucker embedding is equivariant with respect to this action. Thus  $\operatorname{Gr}(r, E)$  is a homogeneous  $\operatorname{Gl}_n$ -space and as a consequence it is non-singular of dimension d(n-d).

Let  $E \times \operatorname{Gr}(r, E) \xrightarrow{p} \operatorname{Gr}(r, E)$  be the projection onto the second coordinate. This construction defines a trivial vector bundle over  $\operatorname{Gr}(r, E)$  of dimension n. An important role is played by the tautological subbundles and factorbundles of the trivial vector bundle. The **tautological subbundle**  $\mathcal{R}$  is defined to be the variety

$$\mathcal{R} = \{ (x, R) \in E \times \operatorname{Gr}(r, E) \mid x \in R \}.$$

The **tautological factor bundle**  $\Omega$  is defined to be the quotient  $E \times \operatorname{Gr}(r, E)/\Re$ . Thus we have an exact sequence of vector bundles over  $\operatorname{Gr}(r, E)$ 

$$0 \to \mathcal{R} \to E \times \operatorname{Gr}(r, E) \to \mathcal{Q} \to 0$$

As a result we have for every  $R \in Gr(r, E)$  an exact sequence of fibres

$$0 \to R \to E \to E/R \to 0$$

Thus the dimensions of  $\mathcal{R}$  and  $\mathcal{Q}$  are r and n-r respectively.

Now we describe the geometric technique which we will use to construct the resolutions of orbit closures. This technique is applicable to subvarieties Y of an affine space X which admit a desingularization Z such that Z is the total space of a subbundle of the trivial vector bundle  $X \times \mathcal{V}$  over some projective variety  $\mathcal{V}$ . In this situation the structure sheaf  $\mathcal{O}_Z$  can be resolved by  $\mathcal{O}_{X \times \mathcal{V}}$ -modules using a Koszul complex. The pushforward of this Koszul complex by the desingularization map gives a complex  $\mathbf{F}_{\bullet}$  which is supported in Y. This is the complex we wish to calculate.

To be precise, let  $\mathcal{V}$  be a projective variety of dimension m and  $X = \mathbb{A}_K^N$ . Let  $\mathcal{E}$  denote the trivial vector bundle  $X \times \mathcal{V} \xrightarrow{p} \mathcal{V}$  of dimension N. Z is a subset of  $X \times \mathcal{V}$  such that the vector bundle  $\mathcal{S}$  given by  $Z \to \mathcal{V}$  is a subbundle of  $\mathcal{E}$ . Let  $q : X \times \mathcal{V} \to X$  be the projection and suppose Y = q(Z). At this point we do not assume that Z is a desingularization of Y.

We have an exact sequence of vector bundles

$$0 \to \mathbb{S} \to \mathbb{E} \to \mathbb{T} \to 0$$

over  $\mathcal{V}$  where  $\mathcal{T}$  is the quotient bundle  $\mathcal{E}/S$ . Let the dimensions of S and  $\mathcal{T}$  be s and t respectively. The following is Proposition 5.1.1 in [Wey03].

**Theorem 3.0.3.** Let  $\xi$  denote the dual vector bundle  $\mathfrak{T}^*$ . The Koszul complex

$$\mathbf{K}(\xi)_{\bullet}: 0 \to \bigwedge^{t}(p^{*}\xi) \to \dots \to \bigwedge^{2}(p^{*}\xi) \to p^{*}\xi \to \mathcal{O}_{X \times \mathcal{V}} \to \mathcal{O}_{Z} \to 0$$

is a locally free resolution of  $\mathcal{O}_Z$  as an  $\mathcal{O}_{X \times \mathcal{V}}$ -module. The differentials of this complex are homogeneous of degree 1 in the coordinate functions on X.

Let A = K[X]. The main theorem of the geometric technique asserts that we can

use  $\mathbf{K}(\xi)_{\bullet}$  to construct the free complex  $\mathbf{F}_{\bullet}$  of A-modules with homology supported in Y. In ideal cases,  $\mathbf{F}_{\bullet}$  gives a free resolution of the defining ideal of Y. Let A denote the coordinate ring of X. The following theorem gives the existence of  $\mathbf{F}_{\bullet}$ .

**Theorem 3.0.4.** The terms of the complex  $\mathbf{F}_{\bullet}$  are free graded A-modules given by

$$\mathbf{F}_i = \bigoplus_{j \ge 0} H^j(\mathcal{V}, \bigwedge^{i+j} \xi) \otimes_K A(-i-j)$$

**Theorem 3.0.5.** Let q' = q|Z. Suppose q' is a birational isomorphism. Then the following properties hold:

- (a) The module  $q'(\mathcal{O}_Z)$  is the normalization of K[Y].
- (b) If R<sup>i</sup>q'<sub>\*</sub>O<sub>Z</sub> = 0 for i > 0 then F<sub>•</sub> is a finite free resolution of the normalization of K[Y].
- (c) If  $R^i q'_* \mathfrak{O}_Z = 0$  for i > 0 and  $\mathbf{F}_0 = H^0(\mathfrak{V}, \bigwedge^0 \xi) \otimes A = A$ , then Y is normal and has rational singularities.

In order to calculate the cohomology terms in  $\mathbf{F}_i$  we apply Bott's algorithm (Theorem 3.0.6). The flag variety  $\mathcal{V} = \prod_{x \in Q_0} \operatorname{Gr}(d_2(x), V_x)$  is a homogeneous space for Gl(n, K) (which we will denote henceforth by  $Gl_n$ ). This makes it possible to describe vector bundles on  $\mathcal{V}$  in terms of weights of  $Gl_n$ -representations [Wey03, Proposition 4.1.3]. We denote by  $L(\alpha)$  the vector bundle corresponding to weight  $\alpha$  and by  $S_\beta$  the Schur module corresponding to the weight  $\beta$ . The Bott's theorem for cohomology of vector bundles yields the following algorithm in case of  $\mathcal{V}$ .

**Theorem 3.0.6** (Bott's algorithm [Wey03]). Let  $\alpha = (\alpha_1, ..., \alpha_n)$ . The permutation  $\sigma_i = (i, i + 1)$  acts on the set of weights in the following way:

$$\sigma_i \cdot \alpha = (\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1} - 1, \alpha_i + 1, \alpha_{i+2}, ..., \alpha_n).$$
(3.0.1)

If  $\alpha$  is a nonincreasing, then  $R^0h_*L(\alpha) = S_{(\alpha)}\xi$  and  $R^ih_*L(\alpha) = 0$  for i > 0. If  $\alpha$  is not a partition, then we start to apply the exchanges of type (3.0.1), trying to move bigger number to the right past the smaller number. Two possibilities can occur:

- 1.  $\alpha_{i+1} = \alpha_i + 1$  when the exchange of type (3.0.1) leads to the same sequence. In this case  $R^i h_* L(\alpha) = 0$  for all  $i \ge 0$ .
- 2. After applying say j exchanges, we transform  $\alpha$  into a nonincreasing sequence  $\beta$ . Then  $R^i h_* L(\alpha) = 0$  for  $i \neq j$  and  $R^j h_* L(\alpha) = S_{(\beta)} \xi$

The process of applying Bott's algorithm to weights of the form  $(0^k, \lambda)$  plays an important role in all our calculations and proofs, so it is useful to introduce some notation. Notation 3.0.7. Whenever we apply Bott's algorithm for the exchanges, we will refer to it as 'Bott exchanges'.

- A partition λ = (λ<sub>1</sub>, λ<sub>2</sub>, ..., λ<sub>n</sub>) is a non-decreasing sequence of non-negative integers. The Young diagram corresponding to partition λ consists of λ<sub>i</sub> boxes in the *i*th row. The conjugate partition λ' is the partition (λ'<sub>1</sub>, λ'<sub>2</sub>, ..., λ'<sub>m</sub>) where λ'<sub>j</sub> is the number of boxes in the *j*th column. We will denote the last row of a Young tableau λ by λ<sub>last</sub>.
- 2. We denote by  $[0^k, \lambda]$  the end result after applying Bott exchanges to a weight  $(0^k, \lambda)$ .

3.  $N_{\lambda}$  will denote the number of Bott exchanges required to go from  $(0^k, \lambda)$  to  $[0^k, \lambda]$ .

With this notation, applying Bott's algorithm for the weight  $(0^k, \lambda)$  gives us exactly one of the following results -

- a. During any of the successive Bott exchanges, we arrive at sequence of the form  $(\cdots, m, m + 1, \cdots)$ ; in this case, the next exchange will leave the sequence unchanged, so this is the first case of Bott's algorithm. Then we say  $[0^k, \lambda]$  is zero.
- b. If the above case does not occur and we reach a non-increasing sequence after  $N_{\lambda}$ Bott exchanges, then we say  $[0^k, \lambda]$  is the resulting sequence  $(\lambda_1 - k, \lambda_2 - k, \cdots, \lambda_p - k, p^k, \lambda_{p+1}, \cdots, \lambda_r)$ . Then  $N_{\lambda} = pk$ .

**Example 3.0.8.** Suppose  $\lambda = (4, 4, 3, 2)$  and k = 2 so that we want to apply Bott's algorithm to the sequence (0, 0, 4, 4, 3, 2). Exchanging  $\lambda_1$  with the 2 zeroes gives (2, 1, 1, 4, 3, 2) which is not a non-increasing sequence. The first increase occurs at  $\lambda_2$ , so we exchange  $\lambda_2$  with the 2 1's to get (2, 2, 2, 2, 3, 2). Repeating the same exercise for  $\lambda_3$  we see that an exchange between the 4th and 5th terms leads to no change in the sequence. Thus [0, 0, 4, 4, 3, 2] = 0 which is indicative of the fact that the contribution of the vector bundle corresponding to weight  $(0^k, \lambda)$  is zero.

**Example 3.0.9.** As another example consider  $\mu = (7, 5, 1, 1)$  and k = 3. Consider the sequence (0, 0, 0, 7, 5, 1, 1). Applying Bott exchanges to  $\mu_1$  and the zeroes gives the sequence (4, 1, 1, 1, 5, 1, 1). This is not a non-increasing sequence, so we apply Bott exchanges to 5 and the three 1's to get (4, 2, 2, 2, 2, 1, 1) which is non-increasing. Thus the contribution of the vector bundle  $\mathcal{E}$  corresponding to the weight (0, 0, 0, 7, 5, 1, 1) is  $S_{4,2,2,2,2,1,1}\mathcal{E}^*$  with  $N_{\mu} = 6$ .
Now we show how to associate a vector bundle  $\xi$  to the desingularization described in Section 2.2.1. Let  $\mathcal{R}_t(x)$  and  $\mathcal{Q}_t(x)$  denote the tautological subbundle and factorbundle respectively on  $Flag(d_t(x), d_{t-1}(x) + d_t(x), \dots, d_2(x) + \dots + d_t(x), V_x)$ . Note that

$$Rep_K(Q,\underline{d}) = \bigoplus_{a \in Q_1} \operatorname{Hom}(V_{ta}, V_{ha}) = \bigoplus_{a \in Q_1} V_{ta}^* \otimes V_{ha}.$$

Then the desingularization given by

$$Z_{\mathfrak{I}_{*},V} = \{ (V, R_{s}(x) \subset R_{s-1}(x) \subset \cdots \subset R_{2}(x) \subset V_{x}) \mid \forall a \in Q_{1}, \forall t, \ V(a)(R_{t}(ta)) \subset R_{t}(ha) \}$$

is the total space of a vector bundle  $\eta$  over  $\mathcal{V}$ , where  $\mathcal{V}$  is the product of flag varieties  $\prod_{x \in Q_0} Flag(d_s(x), d_{s-1}(x) + d_s(x), \cdots, d_2(x) + \cdots + d_s(x), V_x). \eta \text{ is a subbundle of the triv-}$ ial vector bundle  $\mathcal{E}$  given by

$$\bigoplus_{a \in Q_1} V_{ta}^* \otimes V_{ha} \times \mathcal{V} \to \mathcal{V}$$

Define  $\xi$  to be the dual of  $\mathcal{E}/\eta$ . Then

$$\xi = \bigoplus_{a \in Q_1} \left( \sum_{t=1}^s \mathcal{R}_t(ta) \otimes \mathcal{Q}_t(ha)^* \right) \subset \bigoplus_{a \in Q_1} V_{ta} \otimes V_{ha}^*$$

To calculate the terms of  $\mathbf{F}_{\bullet}$  as in Theorem 3.0.4, we need to calculate the exterior powers of  $\xi$ . This is a difficult problem in general since  $\xi$  is not semisimple. However if we restrict to the case s = 1, then  $\xi = \bigoplus_{a \in Q_1} \mathcal{R}_1(ta) \otimes \mathcal{Q}_1(ha)^*$  so that  $\xi$  is semisimple. This allows us to apply Cauchy's formula to calculate exterior powers of  $\xi$ . For this, let  $\underline{\lambda}$  be a tuple of partitions  $\lambda(a)$  associated to each arrow  $a \in Q_1$  and let  $|\underline{\lambda}| = \sum_{a \in Q_1} |\lambda(a)|$ . Then

$$\bigwedge^t \xi = \bigoplus_{|\underline{\lambda}|=t} \bigwedge^t \xi(\underline{\lambda})$$

where

$$\bigwedge^{t} \xi(\underline{\lambda}) = S_{\lambda(a)} \mathcal{R}_1(ta) \otimes S_{\lambda(a)'} \mathcal{Q}_1(ha)^*$$

For this reason we restrict to orbit closures that admit a 1-step desingularization. With this restriction the projective variety  $Flag(d_s(x), d_{s-1}(x)+d_s(x), \dots, d_2(x)+\dots+d_t(x), V_x)$ is the Grassmannian  $Gr(d_2(x), V_x)$  for every  $x \in Q_0$ . Thus all the orbit closures we will consider henceforth are those that admit a 1-step desingularization.

## Chapter 4

# Non-equioriented quiver of type $A_3$

The first case we study is that of a non-equioriented quiver of type  $A_3$ . This case is nice in the sense that every orbit admits a 1-step desingularization. This case also yields well to an exploration of the Gorenstein property. The results of this chapter have been announced in the paper [Sut11b].

In Section 4.1 we demonstrate the calculation of the complex  $\mathbf{F}_{\bullet}$  and using it we derive some geometric properties of orbit closures. In Section 4.2 we use the calculations of the previous section to give a closed form for the minimal generators of the generating ideal of an orbit closure. Section 4.3 deals with the last term of  $\mathbf{F}_{\bullet}$ . We find a necessary and sufficient condition for the orbit closure to be Gorenstein.

#### 4.1 Calculation of $F_{\bullet}$

We will work with non-equioriented quiver  $Q = A_3$  in the form  $1 \xrightarrow{a} 3 \xleftarrow{b} 2$ . We can assume this orientation without loss of generality because the other orientation is covered by the equivalence of categories  $\operatorname{Rep}_K(Q)$  and  $\operatorname{Rep}_K(Q^{op})$ .

Recall that any representation of Q can be expressed uniquely as a direct sum of indecomposable representations of Q. The Auslander-Reiten quiver of Q lists all the indecomposables along with the irreducible maps between them. We will denote the indecomposable representations by their dimension vectors, for example, 110 will stand for the representation  $K \to K \leftarrow 0$ . With this notation we have We can construct a



Figure 4.1: AR quiver of  $1 \xrightarrow{a} 3 \xleftarrow{b} 2$ 

partition  $(\mathcal{J}_1, \mathcal{J}_2)$  of this quiver as described in Section 2.2.1 which has the form shown in Figure 4.2. The part on the left is  $\mathcal{J}_1$  and that on the right is  $\mathcal{J}_2$ .



Figure 4.2: Partition

The fact that we have partitioned the AR quiver into two parts means that every orbit will admit a 1-step desingularization. This important point distinguishes the case of non-equioriented  $A_3$  quiver. Note that this is the only 2-part partition possible for the AR quiver of  $1 \xrightarrow{a} 3 \xleftarrow{b} 2$ . These facts also hold true for the quiver with the opposite orientation. Now let  $V = V_1 \xrightarrow{V_a} V_3 \xleftarrow{V_b} V_2$  be a representation of Q. By the unique decomposition theorem,  $V = a(010) \oplus b(011) \oplus c(110) \oplus d(111) \oplus e(100) \oplus f(001)$  where the non-negative integers a, b, c, d, e, f denote the multiplicities with which the corresponding indecomposable representations appear as a summand of V. Then the dimension vector of V is  $\alpha = (b + d + f, a + b + c + d, c + d + e)$ . Reineke's construction of the desingularization Z dictates that  $\beta = (d + f, d, d + e)$ . (In the notation of Section 2.2,  $\beta = d_2$  and  $\alpha = d_1 + d_2$ ). Using the above partition we get the desingularization  $Z \subset \operatorname{Rep}(Q, \alpha) \times \operatorname{Gr}(d + f, V_1) \times \operatorname{Gr}(d, V_3) \times \operatorname{Gr}(d + e, V_2)$  of  $\overline{O}_V$  given by

$$Z = \{ (R_1, R_2, R_3) \in \operatorname{Gr}(d + f, V_1) \times \operatorname{Gr}(d, V_3) \times \operatorname{Gr}(d + e, V_2) \mid ((R_x)_{x \in Q_0}, V(a), V(b)) \in \operatorname{Rep}(Q, \beta) \}$$
(4.1.1)

or equivalently by

$$Z = \{ (V_a, V_b) \in \operatorname{Hom}(V_1, V_3) \times \operatorname{Hom}(V_2, V_3) \mid V(a)(R_1) \subset R_3 \text{ and } V(b)(R_2) \subset R_3 \}$$
(4.1.2)

We may visualize Z as being of the form

$$\begin{array}{ccc} V_1 \longrightarrow V_3 \longleftarrow V_2 \\ \cup & \cup & \cup \\ R_1 \longrightarrow R_3 \longleftarrow R_2 \end{array}$$

with dimension vectors of the rows being  $\alpha = (b + d + f, a + b + c + d, c + d + e)$  and  $\beta = (d + f, d, d + e)$ . Let  $Q_x := V_x/R_x$  and  $\gamma_x = \alpha_x - \beta_x$  so that dim  $Q_x = \gamma_x$ .

Let  $\mathcal{R}_x$  and  $\mathcal{Q}_x$  denote respectively the tautological subbundle and factor bundle of the trivial vector bundle  $V_x \times \operatorname{Gr}(\beta_x, V_x) \xrightarrow{p} \operatorname{Gr}(\beta_x, V_x)$  for  $1 \le x \le 3$ . By definition the fibers

of a point  $R_x \in \operatorname{Gr}(\beta_x, V_x)$  with respect to vector bundles  $\mathcal{R}_x$  and  $\mathcal{Q}_x$  are  $R_x$  and  $Q_x$ respectively. Identify the vector space  $\operatorname{Hom}(V, W)$  with  $V^* \otimes W$ . Under this identification the desingularization Z can be viewed as being the total space of a vector bundle  $\eta$  which is a subbundle of the trivial vector bundle

$$\mathcal{E} = (V_1^* \otimes V_3 \oplus V_2^* \otimes V_3) \times \prod_{x \in Q_0} \operatorname{Gr}(\beta_x, V_x) \to \prod_{x \in Q_0} \operatorname{Gr}(\beta_x, V_x).$$

For calculating the complex  $\mathbf{F}_{\bullet}$  we consider the vector bundle which is dual to the factorbundle  $\mathcal{E}/\eta$  given by

$$\xi = \mathcal{R}_1 \otimes \mathcal{Q}_3^* \oplus \mathcal{R}_2 \otimes \mathcal{Q}_3^* \tag{4.1.3}$$

Let us denote  $\prod_{x \in Q_0} Gr(\beta_x, V_x)$  by  $\mathcal{V}$ . By Theorem 3.0.6 the terms of the free resolution  $\mathbf{F}_{\bullet}$  resolving the structure sheaf of Z are

$$F_i = \bigoplus_{j \ge 0} H^j(\mathcal{V}, \bigwedge^{i+j} \xi) \otimes A[-i-j]$$
(4.1.4)

Note that by Cauchy's formula we have

$$\bigwedge^{t} \xi = \bigoplus_{|\lambda|+|\mu|=t} S_{\lambda} \mathcal{R}_{1} \otimes S_{\mu} \mathcal{R}_{2} \otimes S_{\lambda'} \mathcal{Q}_{3}^{*} \otimes S_{\mu'} \mathcal{Q}_{3}^{*}$$
(4.1.5)

To calculate  $H^{j}(\mathcal{V}, \bigwedge^{i+j} \xi)$  we apply Bott's algorithm to the weights

$$(0^{\gamma_1},\lambda), (0^{\gamma_2},\mu), (-\nu, 0^{\beta_3})$$

for all  $S_{\nu}$  occuring in  $S_{\lambda'} \otimes S_{\mu'}$ . Suppose  $N_{\lambda} = u\gamma_1$ ,  $N_{\mu} = v\gamma_2$  and  $N_{\nu} = w\beta_3$ . Explicitly -

$$\begin{array}{l} (0^{\gamma_1},\lambda) = (\underbrace{0,\cdots,0}_{\gamma_1},\lambda_1,\lambda_2,\cdots) & \xrightarrow{u\gamma_1 \text{ Bott}}_{\text{exchanges}} [0^{\gamma_1},\lambda] \\ &= (\lambda_1 - \gamma_1,\lambda_2 - \gamma_1,\cdots,\lambda_u - \gamma_1, \underbrace{u,\cdots,u}_{\gamma_1},\lambda_{u+1},\cdots) \\ (0^{\gamma_2},\mu) = (\underbrace{0,\cdots,0}_{\gamma_2},\mu_1,\mu_2,\cdots) & \xrightarrow{v\gamma_2 \text{ Bott}}_{\text{exchanges}} [0^{\gamma_2},\mu] \\ &= (\mu_1 - \gamma_2,\mu_2 - \gamma_2,\cdots,\mu_v - \gamma_2,\underbrace{v,\cdots,v}_{\gamma_2},\mu_{v+1},\cdots) \end{array}$$

We write the third weight in its dual form -

$$(-\nu, 0^{\beta_3}) = (\cdots, -\nu_2, -\nu_1, \underbrace{0, \cdots, 0}_{\beta_3}) \xrightarrow{w\beta_3 \text{ Bott}} [-\nu, 0^{\beta_3}]$$
$$= (\cdots, -\nu_{w+1}, \underbrace{w, \cdots, w}_{\beta_3}, -\nu_w - \beta_3, \cdots, \nu_1 - \beta_3)$$

Then the total number of exchanges N equals  $u\gamma_1 + v\gamma_2 + w\beta_3$ . We summarize this in-

**Proposition 4.1.1.** The terms of the complex  $F_{\bullet}$  are given by -

$$\mathbf{F}_{i} = \bigoplus_{t=1}^{\dim \xi} \bigoplus_{|\lambda|+|\mu|=t} c_{\lambda',\mu'}^{\nu} (S_{[0^{\gamma_{1}},\lambda]}V_{1} \otimes S_{[0^{\gamma_{2}},\mu]}V_{2} \otimes S_{[-\nu,0^{\beta_{3}}]}V_{3}^{*})$$

where  $S_{\nu} \subset S_{\lambda'} \otimes S_{\mu'}$  and  $|\lambda| + |\mu| - N = i$ .

Since the term  $|\lambda| + |\mu| - N$  occurs often, we give it a name -

**Definition 4.1.2.** Let  $\lambda(a)$  be partition associated to arrow  $a \in Q_1$  and let

 $\underline{\lambda} = (\lambda(a))_{a \in Q_1}$ . Define

$$D(\underline{\lambda}) = \sum_{a \in Q_1} |\lambda(a)| - N \tag{4.1.6}$$

In this chapter the tuple  $\underline{\lambda}$  will be  $(\lambda, \mu)$ , that is we associate parition  $\lambda$  to arrow a and  $\mu$  to arrow b of  $Q : 1 \xrightarrow{a} 3 \xleftarrow{b} 2$ . We denote by  $\nu$  a partition occuring in the Littlewood-Richardson product of  $\lambda$  and  $\mu$ . From the earlier discussion it is clear that the triple (u, w, v) depends on the partitions  $(\lambda, \mu)$ . We denote the triple (u, w, v) by  $\underline{u}(\underline{\lambda})$ .

From Proposition 4.1.1, it is clear that in order to calculate the terms  $F_i$  of the resolution, we need to calculate  $D(\underline{\lambda})$ . Due to the number of variables involved and the peculiar form of exchanges required, the calculation of a closed formula for  $D(\underline{\lambda})$  is not easy in general. Our key result is Proposition 4.1.4 which gives us a lower bound for  $D(\underline{\lambda})$  in terms of the Euler form of quiver Q. First we prove a lemma which is an easy exercise in counting boxes-

**Lemma 4.1.3.** Let  $\lambda$  be a Young tableau. Then for all a and b,

$$\lambda_1 + \lambda_2 + \dots + \lambda_a \le ab + (\lambda'_{b+1} + \dots + \lambda'_{last}).$$

*Proof.* We consider three cases:

Case (1)  $\lambda'_{b+1} = a$ . Then

$$\lambda_1 + \lambda_2 + \dots + \lambda_a = ab + \lambda'_{b+1} + \dots + \lambda'_{last}$$

Case (2)  $\lambda'_{b+1} > a$ . In this case  $\lambda'_{b+1}, \lambda'_{b+2}, \cdots \lambda'_{last}$  contribute more boxes so that

$$\lambda_1 + \lambda_2 + \dots + \lambda'_a \le ab + \lambda'_{b+1} + \dots + \lambda'_{last}$$

Case (3)  $\lambda'_{b+1} < a$ . Here the rectangle *ab* contributes more boxes, so that

$$\lambda_1 + \lambda_2 + \dots + \lambda_a \le ab + \lambda'_{b+1} + \dots + \lambda'_{last}$$

By symmetry we also have for all a and b:

$$\lambda_1' + \lambda_2' + \dots + \lambda_a' \le ab + (\lambda_{b+1} + \dots + \lambda_{last})$$

$$(4.1.7)$$

**Proposition 4.1.4.** Let Q be the non-equivalent of quiver  $A_3$ . Let  $\underline{\lambda}$  be a tuple of partitions associated to arrows of Q and let  $\underline{u}(\lambda) \in \mathbb{N}^{|Q_0|}$  be a vector that depends on  $\underline{\lambda}$ . If  $\langle, \rangle$  denotes the Euler form on Q then

$$D(\underline{\lambda}) \ge \langle \underline{u}(\lambda), \underline{u}(\lambda) \rangle$$

Proof. Since  $[0_1^{\gamma}, \lambda] = (\lambda_1 - \gamma_1, \lambda_2 - \gamma_1, \dots, \lambda_u - \gamma_1, u^{\gamma_1}, \lambda_{u+1}, \dots)$  is a non-increasing sequence, we have that each of  $\lambda_1 - \gamma_1, \dots, \lambda_u - \gamma_1$  is greater than (or equal to) u, which means each of  $\lambda_1, \dots, \lambda_u$  is greater than (or equal to)  $u + \gamma_1$ . Thus  $\lambda_1 + \dots + \lambda_u \ge u^2 + u\gamma_1$ . Similarly  $\mu_1 + \dots + \mu_v \ge v^2 + v\gamma_2$  and  $\nu_1 + \dots + \nu_w \ge w^2 + w\beta_3$ . By Lemma 4.1.3 we get-

$$w.u \ge (\lambda'_1 + \dots + \lambda'_w) - (\lambda_{u+1} + \dots + \lambda_{last})$$

$$w.v \ge (\mu'_1 + \dots + \mu'_w) - (\mu_{v+1} + \dots + \mu_{last})$$
So,  $w(u+v) \ge (\lambda'_1 + \dots + \lambda'_w + \mu'_1 + \dots + \mu'_w) - (\lambda_{u+1} + \dots + \lambda_{last} + \mu_{v+1} + \dots + \mu_{last})$ 

$$\ge \nu_1 + \dots + \nu_w - (\lambda_{u+1} + \dots + \lambda_{last} + \mu_{v+1} + \dots + \mu_{last})$$
thus  $\nu_1 + \dots + \nu_w \le w(u+v) + (\lambda_{u+1} + \dots + \lambda_{last} + \mu_{v+1} + \dots + \mu_{last})$ 

Therefore

$$(u^{2} + u\gamma_{1}) + (v^{2} + v\gamma_{2}) + (w^{2} + w\beta_{3}) \leq \lambda_{1} + \dots + \lambda_{u} + \mu_{1} + \dots + \mu_{v}$$
$$+ \nu_{1} + \dots + \nu_{w}$$
$$\leq \lambda_{1} + \dots + \lambda_{u} + \mu_{1} + \dots + \mu_{v}$$
$$+ w(u + v) + \lambda_{u+1} + \dots + \lambda_{last}$$
$$+ \mu_{v+1} + \dots + \mu_{last}$$
$$= w(u + v) + |\lambda| + |\mu|$$

So we have 
$$|\lambda| + |\mu| \ge (u^2 + u\gamma_1) + (v^2 + v\gamma_2) + w(w + \beta_3 - u - v)$$
  
=  $u\gamma_1 + v\gamma_2 + w\beta_3 + (u^2 + v^2 + w^2 - uw - vw)$   
=  $u\gamma_1 + \gamma_2 + w\beta_3 + \langle (u, w, v), (u, w, v) \rangle$ 

In their paper [BZ01], Bobinski and Zwara proved the normality of orbit closures for Dynkin quivers of type  $A_n$  with arbitrary orientation. Using the above proposition we can derive the normality of orbit closures in our case -

**Corollary 4.1.5.** In the case of quiver  $Q: 1 \rightarrow 2 \leftarrow 3$  the orbit closures are normal, Cohen-Macaulay with rational singularities.

*Proof.* We have that  $\langle (u, w, v), (u, w, v) \rangle \ge 0$  since it is the Euler form of a Dynkin quiver Q. Then from Proposition 4.1.1 and Proposition 4.1.4,  $F_i = 0$  for i < 0.

Also,  $\langle (u, w, v), (u, w, v) \rangle = 0$  if and only if u = v = w = 0 in which case  $\lambda = \mu = \nu = 0$ . Thus  $F_0 = 0$ . By Theorem 3.0.5, this implies that the orbit closure is normal with rational singularities.

Remark 4.1.6. For purposes of calculation, it is useful to record some simple observations regarding the sizes of partitions  $\lambda$ ,  $\mu$  and  $\nu$ . From Equation (4.1.5) it is clear that when calculating  $\bigwedge^t \xi$ , we only need to consider those partitions  $\lambda$ ,  $\mu$ ,  $\nu$  such that  $\lambda$  is contained in a dim  $Q_3 \times \dim R_1$  rectangle,  $\mu$  is contained in a dim  $Q_3 \times \dim R_2$  rectangle and  $\nu$  is contained in a dim  $(R_1 + R_2) \times \dim Q_3$  rectangle. Thus the largest possible contributing triples are  $(\lambda, \mu, \nu) = (\gamma_3^{\beta_1}, \gamma_3^{\beta_2}, (\beta_1 + \beta_2)^{\gamma_3})$  (the notation  $\alpha^{\beta}$  stands for the rectangular partition  $(\alpha, \alpha, \dots, \alpha)$  of length  $\beta$ ).

**Example 4.1.7.** Let  $V = 010 \oplus 011 \oplus 110 \oplus 111 \oplus 100 \oplus 001$  and I be the defining ideal of  $\overline{O}_V$ . Then  $\alpha = (3,4,3)$  and  $\beta = (2,1,2)$ .  $A = Sym(V_1 \otimes V_3^*) \oplus Sym(V_2 \otimes V_3^*)$  and dim  $\xi = \dim(R_1 \otimes Q_3^* \oplus R_2 \otimes Q_3^*) = 12$ . Hence we need to calculate  $\wedge^0 \xi, \wedge^1 \xi, \ldots, \wedge^{12} \xi$ . Let  $\xi_1 = R_1 \otimes Q_3^*$  and  $\xi_2 = R_2 \otimes Q_3^*$ 

$$\wedge^{1}\xi = (\wedge^{1}\xi_{1} \otimes \wedge^{0}\xi_{2}) \oplus \wedge^{0}\xi_{1} \otimes \wedge^{1}\xi_{2})$$
$$= [(S_{1}R_{1} \otimes S_{1}Q_{3}^{*}) \otimes (S_{0}R_{2} \otimes S_{0}Q_{3}^{*})] \oplus [(S_{0}R_{1} \otimes S_{0}Q_{3}^{*}) \otimes (S_{1}R_{2} \otimes S_{1}Q_{3}^{*})]$$
$$= [S_{1}R_{1} \otimes S_{0}R_{2} \otimes S_{1}Q_{3}^{*}] \oplus [S_{0}R_{1} \otimes S_{1}R_{2} \otimes S_{1}Q_{3}^{*}]$$

The weight associated to the first summand is (0, 1, 0; 0, 0, 0; 0, 0, -1, 0) and weight associated to the second summand is (0, 0, 0; 0, 1, 0; 0, 0, -1, 0). Applying Bott's algorithm we see that none of these terms contribute to any of the  $F_i$ . For an example of a contributing weight we calculate  $\wedge^3 \xi$ . From Remark 4.1.6, we know that  $\lambda$  is contained in the rectangle  $(3^2)$ ,  $\mu$  is contained in  $(3^2)$  and  $\nu$  is contained in  $(4^3)$ .

$$\begin{split} \wedge^{3}\xi &= (\wedge^{3}\xi_{1} \otimes \wedge^{0}\xi_{2}) \oplus (\wedge^{2}\xi_{1} \otimes \wedge^{1}\xi_{2}) \oplus (\wedge^{1}\xi_{1} \otimes \wedge^{2}\xi_{2}) \oplus (\wedge^{0}\xi_{1} \otimes \wedge^{3}\xi_{2}) \\ &= [(S_{(2,1)}R_{1} \otimes S_{(0)}R_{2} \otimes S_{(2,1)}Q_{3}^{*})] \oplus [(S_{(3)}R_{1} \otimes S_{(0)}R_{2} \otimes S_{(1,1,1)}Q_{3}^{*})] \\ &\oplus [S_{(2)}R_{1} \otimes S_{(1)}R_{2} \otimes S_{(2,1)}Q_{3}^{*})] \oplus [S_{(2)}R_{1} \otimes S_{(1)}R_{2} \otimes S_{(1,1,1)}Q_{3}^{*})] \\ &\oplus [S_{(1,1)}R_{1} \otimes S_{(1)}R_{2} \otimes S_{(2,1)}Q_{3}^{*})] \oplus [S_{(1,1)}R_{1} \otimes S_{(1)}R_{2} \otimes S_{(3)}Q_{3}^{*})] \\ &\oplus [S_{(1)}R_{1} \otimes S_{(2)}R_{2} \otimes S_{(1,1,1)}Q_{3}^{*})] \oplus [S_{(1)}R_{1} \otimes S_{(2)}R_{2} \otimes S_{(2,1)}Q_{3}^{*})] \\ &\oplus [S_{(1)}R_{1} \otimes S_{(1,1)}R_{2} \otimes S_{(2,1)}Q_{3}^{*})] \oplus [S_{(1)}R_{1} \otimes S_{(1,1)}R_{2} \otimes S_{(3)}Q_{3}^{*})] \\ &\oplus [(S_{(0)}R_{1} \otimes S_{(3)}R_{2} \otimes S_{(1,1,1)}Q_{3}^{*})] \oplus [(S_{(0)}R_{1} \otimes S_{(2,1)}R_{2} \otimes S_{(2,1)}Q_{3}^{*})] \\ &\oplus [(S_{(0)}R_{1} \otimes S_{(3)}R_{2} \otimes S_{(1,1,1)}Q_{3}^{*})] \oplus [(S_{(0)}R_{1} \otimes S_{(2,1)}R_{2} \otimes S_{(2,1)}Q_{3}^{*})] \end{split}$$

The weights associated to the summands in that order are:

$(0\ 2\ 1; 0\ 0\ 0; 0 - 1 - 2\ 0),$	$(0 \ 3 \ 0; 0 \ 0 \ 0; -1 - 1 - 1 \ 0),$	$(0\ 2\ 0; 0\ 1\ 0; 0-1-2\ 0)$
$(0 \ 2 \ 0; 0 \ 1 \ 0; -1 - 1 \ -1 \ 0),$	$(0\ 1\ 1; 0\ 1\ 0; 0 - 1 - 2\ 0),$	$(0\ 1\ 1; 0\ 1\ 0; 0\ 0-3\ 0)$
$(0\ 1\ 0; 0\ 2\ 0; -1 - 1\ -1\ 0),$	$(0\ 1\ 0; 0\ 2\ 0; 0 - 1 - 2\ 0),$	$(0\ 1\ 0; 0\ 1\ 1; 0 - 1 - 2\ 0)$
$(0\ 1\ 0; 0\ 1\ 1; 0\ 0 - 3\ 0),$	$(0 \ 0 \ 0; 0 \ 3 \ 0; -1 - 1 - 1 \ 0)$ ,	$(0 \ 0 \ 0; 0 \ 2 \ 1; 0 - 1 - 2 \ 0)$

Applying Bott exchanges to each weight we see that only the first and last summands contribute the non-zero terms  $(\wedge^3 V_1 \otimes \wedge^3 V_3^* \otimes A(-3))$  and  $(\wedge^3 V_2 \otimes \wedge^3 V_3^* \otimes A(-3))$  to  $\mathbf{F}_1$ . Continuing in this manner we get the resolution:

A

 $\uparrow$ 

 $(\wedge^{3}V_{1} \otimes \wedge^{3}V_{3}^{*} \otimes A(-3)) \oplus (\wedge^{3}V_{2} \otimes \wedge^{3}V_{3}^{*} \otimes A(-3)) \oplus (\wedge^{2}V_{1} \otimes \wedge^{2}V_{2} \otimes \wedge^{4}V_{3}^{*} \otimes A(-4))$ 

 $\uparrow$ 

 $(S_{211}V_1 \otimes \wedge^4 V_3^* \otimes A(-4)) \oplus (S_{211}V_2 \otimes \wedge^4 V_3^* \otimes A(-4)) \oplus$  $(\wedge^3 V_1 \otimes \wedge^2 V_2^* \otimes S_{2111}V_3^* \otimes A(-5)) \oplus (\wedge^2 V_1 \otimes \wedge^3 V_2 \otimes S_{2111}V_3^* \otimes A(-5)) \oplus$  $\wedge^3 V_1 \otimes \wedge^3 V_2 \otimes S_{222}V_3^* \otimes A(-6)$ 

 $\uparrow$ 

 $(S_{211}V_1 \otimes \wedge^3 V_2 \otimes S_{2221}V_3^* \otimes A(-7)) \oplus (\wedge^3 V_1 \otimes S_{211}V_2 \otimes S_{2221}V_3^* \otimes A(-7)) \oplus (\wedge^2 V_1 \otimes S_{222}V_2^* \otimes S_{2222}V_3^* \otimes A(-8)) \oplus (S_{222}V_1 \otimes \wedge^2 V_2 \otimes S_{2222}V_3^* \otimes A(-8)) \oplus (\wedge^3 V_1 \otimes \wedge^3 V_2 \otimes S_{3111}V_3^* \otimes A(-6))$ 

 $\uparrow$ 

 $(S_{211}V_1 \otimes S_{211}V_2 \otimes S_{2222}V_3^* \otimes A(-8)) \oplus (S_{222}V_1 \otimes \wedge^3 V_2 \otimes S_{3222}V_3^* \otimes A(-9)) \oplus (\wedge^3 V_1 \otimes S_{222}V_2 \otimes S_{3222}V_3^* \otimes A(-9))$ 

 $\uparrow$ 

 $(S_{222}V_1 \otimes S_{222}V_2 \otimes S_{3333}V_3^* \otimes A(-12))$ 

#### 4.2 Minimal generators of the defining ideal

Let  $V \in Rep(Q, \underline{d}), V = a(010) \oplus b(011) \oplus c(110) \oplus d(111) \oplus e(100) \oplus f(001)$ . Then

rank 
$$\phi = b + d$$
, rank  $\psi = c + d$ , rank  $(\phi | \psi) = b + c + d$ 

We will denote these ranks by p, q, r respectively. Hence N = ub + vc + wd.

We consider orbits admitting a Reineke desingularization given by the partition in Figure 4.2. The following result is the main theorem of this section. It describes the first term  $\mathbf{F}_1$  of the resolution  $\mathbf{F}_{\bullet}$ . In particular, it says that the summands of  $\mathbf{F}_1$  are obtained by contributions from  $\bigwedge^{rank(\phi)+1} \xi$ ,  $\bigwedge^{rank(\psi)+1} \xi$  and  $\bigwedge^{rank(\phi)\psi)+1} \xi$ . As a result we will have that the generators of the defining ideal are determinantal, in the sense that they are maximal minors of  $\phi$ ,  $\psi$  and  $\phi | \psi$ .

**Theorem 4.2.1.**  $\mathbf{F}_1 = H^p(\mathcal{V}, \bigwedge^{p+1} \xi) \oplus H^q(\mathcal{V}, \bigwedge^{q+1} \xi) \oplus H^r(\mathcal{V}, \bigwedge^{r+1} \xi).$ 

*Proof.* From Proposition 4.1.1, we have that

$$\mathbf{F}_1 = \bigoplus_{t=1}^{\dim \xi} \bigoplus_{|\lambda|+|\mu|=t} c_{\lambda',\mu'}^{\nu} (S_{[0^b,\lambda]} V_1 \otimes S_{[0^c,\mu]} V_2 \otimes S_{[-\nu,0^d]} V_3^*)$$

where  $S_{\nu} \subset S_{\lambda'} \otimes S_{\mu'}$  and  $D(\underline{\lambda}) = 1$ . Also by Proposition 4.1.4,

$$\begin{split} D(\underline{\lambda}) \geq \langle (u,w,v), (u,w,v) \rangle \\ i.e. & 1 \geq \langle (u,w,v), (u,w,v) \rangle \end{split}$$

But Q is Dynkin, so the Euler form  $E_Q(u, w, v) = \langle (u, w, v), (u, w, v) \rangle > 0$ , so

$$\langle (u, w, v), (u, w, v) \rangle = 1$$

By a theorem of Gabriel [Gab75], there is a one-to-one correspondence between the roots of the quadratic  $E_Q = 1$  and dimension vectors of indecomposables in mod KQ when KQis representation-finite. Thus, (u, w, v) is one of (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (0, 1, 1)and (1, 1, 1). We analyze these triples to prove our proposition. Recall that the weights of  $\bigwedge^i \xi$  are of the form

$$(0^b, \lambda), (0^c, \mu), (-\nu, 0^d)$$

where  $|\lambda| + |\mu| = i$ .

(1) (u, w, v) = (1, 0, 0): in this case N = b, so  $|\lambda| + |\mu| = b + 1$ . u = 1 implies that  $\lambda = (b + 1, 0 \cdots, 0)$ , so  $\mu = 0$ . This implies  $\nu = \lambda'$ , but w = 0, so we will get a contributing triple only when d = 0. In that case  $p = \gamma_1$  and

$$H^p(\mathcal{V},\bigwedge^{p+1}\xi) = \wedge^{p+1}V_1 \otimes \wedge^{p+1}V_3^*$$

is the only contribution to  $\mathbf{F}_1$ .

(2) (u, w, v) = (0, 1, 0): here N = d. So |λ| + |μ| = |ν| = d + 1. Also w = 1 implies ν must be (d + 1, 0, ..., 0). So a contributing triple occurs only when b = c = 0. Then r = d and we get contributing triples (1<sup>k</sup>; 1<sup>l</sup>; d + 1) where k + l = d + 1. The contribution to F<sub>1</sub> is

$$H^{r}(\mathcal{V},\bigwedge^{r+1}\xi) = \wedge^{k}V_{1} \otimes \wedge^{l}V_{2} \otimes \wedge^{r+1}V_{3}^{*}$$

(3) (u, w, v) = (0, 0, 1): this case is analogous to the first one. A contributing triple occurs only when d = 0 in which case the contribution to  $\mathbf{F}_1$  is

$$H^{q}(\mathcal{V},\bigwedge^{q+1}\xi) = \wedge^{q+1}V_{2} \otimes \wedge^{q+1}V_{3}^{*}.$$

(4) (u, w, v) = (1, 1, 0): this implies N = b + d = p.  $D(\underline{\lambda}) = 1$  implies  $|\lambda| + |\mu| - N = 1$ , so  $|\lambda| + |\mu| = |\nu| = b + d + 1$ . u = 1 implies  $\lambda$  is of the form  $(b + 1, 1^k, 0, ...)$ , similarly w = 1 implies  $\nu$  is of the form  $(d + 1, 1^l, 0, ...)$  (thus both  $\lambda$  and  $\nu$  are hooks). Then  $|\nu| = b + d + 1$  implies l = b.

Since v = 0 we know that there are zero exchanges for the weight  $(0^c, \mu)$ . This can happen if either  $\mu = 0$  or c = 0. If  $\mu = 0$ , then  $\nu = \lambda'$  and

$$H^{p}(\mathcal{V},\bigwedge^{p+1}\xi) = S_{[0^{b},\lambda]}V_{1} \otimes S_{[-\nu,0^{d}]}V_{3}^{*}$$
$$= \wedge^{p+1}V_{1} \otimes \wedge^{p+1}V_{3}^{*}$$

If c = 0, then  $\mu = \nu \setminus \lambda = (1^{d-k})$ . In this case

$$H^{p}(\mathcal{V},\bigwedge^{p+1}\xi) = S_{[0^{b},\lambda]}V_{1} \otimes S_{\mu}V_{2} \otimes S_{[-\nu,0^{d}]}V_{3}^{*}$$
$$= \wedge^{b+k+1}V_{1} \otimes \wedge^{d-k}V_{2} \otimes \wedge^{p+1}V_{3}^{*}$$

(5) (u, w, v) = (0, 1, 1): this case is analogous to the previous one. u = 0 implies either

 $\lambda = 0$  or b = 0. If  $\lambda = 0$ , then  $\nu = \mu'$  and

$$H^{q}(\mathcal{V}, \bigwedge^{q+1} \xi) = S_{[0^{c},\mu]} V_{2} \otimes S_{[-\nu,0^{d}]} V_{3}^{*}$$
$$= \wedge^{q+1} V_{2} \otimes \wedge^{q+1} V_{3}^{*}$$

If b = 0, then  $\lambda = \nu \setminus \mu = (1^{d-k})$ . In this case

$$H^{q}(\mathcal{V},\bigwedge^{q+1}\xi) = S_{\lambda}V_{1} \otimes S_{[0^{c},\mu]}V_{2} \otimes S_{[-\nu,0^{d}]}V_{3}^{*}$$
$$= \wedge^{d-k}V_{1} \otimes \wedge^{c+k+1}V_{2} \otimes \wedge^{q+1}V_{3}^{*}$$

(6) (u, w, v) = (1, 1, 1) in this case N = b + c + d = r.  $\lambda$  and  $\mu$  are hooks of the form:

$$\lambda = (b+1, 1^k, 0, \dots), \ \mu = (c+1, 1^l, 0, \dots).$$

Since  $\nu$  is such that  $S_{\nu} \subset S_{\lambda'} \otimes S_{\mu'}$ ,  $\nu$  is also a hook of the form  $(d+1, 1^m, 0, ...)$ . Since  $|\lambda| + |\mu| = |\nu| = b + c + d + 1$ , we must have k + l = d - 1 and m = b + c. Thus

$$H^{r}(\mathcal{V},\bigwedge^{r+1}\xi) = S_{[0^{b},\lambda]}V_{1} \otimes S_{[0^{c},\mu]}V_{2} \otimes S_{[-\nu,0^{d}]}V_{3}^{*}$$
$$= \bigoplus_{k+l=d-1} \wedge^{b+k+1}V_{1} \otimes \wedge^{c+l+1}V_{2} \otimes \wedge^{b+c+d+1}V_{3}^{*}$$

By Cauchy's formula, this term is a direct summand of  $\bigwedge^{r+1}([V_1 \oplus V_2] \otimes V_3^*)$ .  $\Box$ 

**Corollary 4.2.2.** Let rank  $(\phi) = p$ , rank  $(\psi) = q$ , rank  $(\phi + \psi) = r$ . The minimal generators of the defining ideal are determinantal:  $(p + 1) \times (p + 1)$  minors of  $\phi$ , the  $(q + 1) \times (q + 1)$  minors of  $\psi$  and the  $(r + 1) \times (r + 1)$  minors of  $\phi | \psi$ , taken by choosing

b+k+1 columns of  $\phi$  and c+l+1 columns of  $\psi$ , where k+l=d-1.

Proof. The defining ideal of the orbit closure  $\overline{O}_V$  is generated by the image of the map  $F_1 \xrightarrow{\delta} A$ . By Theorem 4.2.1, the image of the differential map  $\delta$  is generated by  $(p+1) \times (p+1)$ -minors of the matrix corresponding to the linear map  $\phi$ ,  $(q+1) \times (q+1)$ -minors of the matrix corresponding to the linear map  $\psi$  and  $(r+1) \times (r+1)$ -minors of the matrix corresponding to the linear map  $\psi$  and  $(r+1) \times (r+1)$ -minors of the matrix corresponding to the linear map  $\psi$ .

In Example 4.1.7, we found

$$\mathbf{F}_1 = (\wedge^3 V_1 \otimes \wedge^3 V_3^* \otimes A(-3)) \oplus (\wedge^3 V_2 \otimes \wedge^3 V_3^* \otimes A(-3)) \oplus (\wedge^2 V_1 \otimes \wedge^2 V_2 \otimes \wedge^4 V_3^* \otimes A(-7))$$

Fixing a basis for vector spaces  $V_1$ ,  $V_2$  and  $V_3$ , the minimal generators of the defining ideal are  $3 \times 3$  minors of the  $4 \times 3$  matrices  $\phi$  and  $\psi$  and  $4 \times 4$  minors of the map  $\phi | \psi : V_1 \oplus V_2 \to V_3$ , obtained by choosing 2 columns of  $\phi$  and 2 columns of  $\psi$ .

### 4.3 $F_{top}$ and classification of Gorenstein orbits

Let Q be a Dynkin quiver. We denote the last term of the resolution  $\mathbf{F}_{\bullet}$  by  $\mathbf{F}_{top}$ . Let  $t = \dim \xi$ , where  $\xi$  is the vector bundle defined in Equation (4.1.3). The top exterior power of  $\xi(a)$  contributes the term

$$S_{[0^{d_1(ta)}, d_1(ha)^{d_2(ta)}, \cdots, (d_1(ha) + \dots + d_{s-1}(ha))^{d_s(ta)}]}(ta)$$

$$\otimes S_{[(-d_2(ta) - \dots - d_s(ta))^{d_1(ha)}, \cdots, -d_s(ta)^{d_{s-1}(ha)}, 0^{d_s(ha)}]}(ha)^*$$
(4.3.1)

Thus the contribution of the top exterior power of  $\xi$  is given by

$$\bigotimes_{x \in Q_0} S_{(k_1(x)^{d_1(x)}, \dots, k_s(x)^{d_s(x)})}(x)$$
(4.3.2)

where

$$k_p(x) = \sum_{a \in Q_1; ta=x} \sum_{u < p} d_u(ha) - \sum_{a \in Q_1; ha=x} \sum_{u > p} d_u(ta)$$
(4.3.3)

First, we give a sufficient condition for the orbit closure  $\overline{O}_V$  to be Gorenstein in case of any Dynkin quiver Q. The condition that for every  $x \in Q_0$ , the number

$$k_p(x) - \sum_{u < p} d_u(x) + \sum_{u > p} d_u(x)$$
(4.3.4)

is independent of p  $(p = 1, 2, \dots, s)$ , is equivalent to the the condition that  $\bigwedge^t \xi$ , the top exterior power of  $\xi$ , contributes a trivial representation to  $\mathbf{F}_{top}$ . We show that the latter condition, together with normality, implies that the corresponding orbit closure is Gorenstein. First we show that the condition (4.3.4) is equivalent to the property that the  $\tau$ -orbits in the Auslander-Reiten quiver are constant-

**Lemma 4.3.1.** Let  $\tau$  denote the Auslander-Reiten translate and suppose  $\underline{d}(x) = (d_u(x))$ (for  $u = 1, 2, \dots, s$ ) are dimensions of the flag at vertex x in the desingularization Z. Then

$$\langle \underline{e}_x, \underline{d}_p(x) \rangle = -\langle \underline{d}_{p+1}(x), \underline{e}_x \rangle$$

for all  $x \in Q_0$  and  $p = 1, 2, \dots, s - 1$ , where  $\underline{e}_x$  is the dimension vector of the simple representation supported at x.

*Proof.* Condition (4.3.4) translates to the equations-

$$k_{p+1}(x) - k_p(x) = d_p(x) + d_{p+1}(x)$$
(4.3.5)

for  $x \in Q_0$  and  $p = 1, 2, \dots, s - 1$ . This is equivalent to-

$$\sum_{a \in Q_1; ta=x} d_t(ha) + \sum_{a \in Q_1; ha=x} d_{p+1}(ta) = d_{p+1}(x) + d_p(x)$$
(4.3.6)

for all  $x \in Q_0$  and  $p = 1, 2, \dots, s - 1$ . These conditions can be expressed in terms of Euler form as follows-

$$\begin{array}{rcl} \langle \underline{e}_x, \underline{d}_p \rangle & = & d_p(x) - \sum_{\substack{a \in Q_1 \\ ta = x}} d_p(ha) \\ \\ & = & \sum_{\substack{a \in Q_1 \\ ha = x}} d_{p+1}(ta) - d_{p+1}(x) \\ \\ & = & -\langle \underline{d}_{p+1}, \underline{e}_x \rangle \end{array}$$

Thus,

$$\langle \underline{e}_x, \underline{d}_p \rangle = - \langle \underline{d}_{p+1}, \underline{e}_x \rangle$$

where  $\underline{e}_x$  is the dimension vector of the simple representation supported at x.

**Lemma 4.3.2.** Let  $m = \dim \mathcal{V}$  and  $t = \dim \xi$ . Then

Proof.

$$\operatorname{codim} \overline{O}_V = \dim X - \dim \overline{O}_V$$
$$= \dim X - \dim Z$$
$$= \dim X - (\dim X + m - t)$$
$$= t - m$$

**Lemma 4.3.3.** Suppose  $\bigwedge^t \xi$  contributes a trivial representation to  $\mathbf{F}_{t-m}$ . Then the resolution  $\mathbf{F}_{\bullet}$  is self-dual. In particular,  $\mathbf{F}_{t-m} \cong \mathbf{F}_0^*$ .

*Proof.* If  $H^m(\mathcal{V}, \bigwedge^t \xi)$  is a trivial representation then  $\bigwedge^t \xi \cong \omega_{\mathcal{V}}$ , where  $\omega_{\mathcal{V}}$  denotes the canonical sheaf on  $\mathcal{V}$ . This implies that  $\omega_{\mathcal{V}} \otimes \bigwedge^t \xi^* \cong \bigwedge^0 \xi \cong K$ . Then for  $0 \le i \le m$ ,

$$\mathbf{F}_{t-m-i} = \bigoplus_{j\geq 0} H^{m-j}(\mathcal{V}, \bigwedge^{t-i-j} \xi)$$

$$\cong \bigoplus_{j\geq 0} H^{j}(\mathcal{V}, \omega_{\mathcal{V}} \otimes \bigwedge^{t-i-j} \xi^{*})^{*} \quad \text{(by Serre duality)}$$

$$\cong \bigoplus_{j\geq 0} H^{j}(\mathcal{V}, \omega_{\mathcal{V}} \otimes \bigwedge^{t} \xi^{*} \otimes \bigwedge^{i+j} \xi)^{*}$$

$$\cong \bigoplus_{j\geq 0} H^{j}(\mathcal{V}, \bigwedge^{i+j} \xi)^{*}$$

$$= \mathbf{F}_{i}^{*} \qquad \Box$$

**Theorem 4.3.4.** Assume that for each  $p = 1, 2, \dots, s - 1$  we have  $\underline{d}_{p+1} = \tau^{+} \underline{d}_{p}$ . Then the complex  $\mathbf{F}_{\bullet}$  is self-dual. If the incidence variety comes from Reineke desingularization and the corresponding orbit closure is normal with rational singularities, then it is also Gorenstein.

*Proof.* If the  $\tau$ -orbits of an AR quiver are constant then by Lemma 4.3.1,  $\bigwedge^t \xi$  contributes a trivial representation to  $\mathbf{F}_{t-m}$ . Then applying Lemma 4.3.3 we get that  $\mathbf{F}_{t-m} \cong \mathbf{F}_0^* \cong A^*$ , therefore dim  $\mathbf{F}_{t-m} = 1$ .

In particular, for our case of non-equioriented  $A_3$  this says that the orbits with multiplicities satisfying a = d, b = e and c = f are Gorenstein.

Next, we investigate necessary conditions for for the orbit closure  $\overline{O}_V$  to be Gorenstein in case of non-equioriented  $A_3$ . Recall that for our case of non-equioriented  $A_3$ , we have desingularization-

$$\begin{array}{ccc} V_1 \longrightarrow V_3 \longleftarrow V_2 \\ \cup & \cup & \cup \\ R_1 \longrightarrow R_3 \longleftarrow R_2 \end{array}$$

As before, let  $V = a(010) \oplus b(011) \oplus c(110) \oplus d(111) \oplus e(100) \oplus f(001)$  be a representation of  $A_3$ . Then

$$d_1 = (b, a + b + c, c); \quad d_2 = (d + f, d, d + e)$$

From (4.3.2) the weights for  $\bigwedge^t \xi$  are:

$$(0^{b}, (a+b+c)^{d+f}), (0^{c}, (a+b+c)^{d+e}), ((-2d-e-f)^{a+b+c}, 0^{d}).$$

For the case of non-equivriented  $A_3$ , we investigate the following question: in what cases does  $\bigwedge^t \xi$  contribute a non-zero representation? To which term  $F_i$  does  $\bigwedge^t \xi$  contribute? First we show that a contribution from  $\bigwedge^t \xi$  always goes to  $\mathbf{F}_{t-m}$ .

**Lemma 4.3.5.** If the weight of the  $\bigwedge^t \xi$  gives a non-zero partition after Bott exchanges, then the corresponding representation is a summand of  $\mathbf{F}_{t-m}$ .

*Proof.* It is enough to show that  $D(\underline{\lambda}) = \operatorname{codim} \overline{O}_V$  for  $\lambda = ((a + b + c)^{d+f})$  and  $\mu = ((a + b + c)^{d+e})$ . We apply Bott's algorithm to each weight to get:

$$[0^b, (a+b+c)^{d+f}] = ((a+c)^{d+f}, (d+f)^b)$$
 after  $b(d+f)$  Bott exchanges,

$$[0^c, (a+b+c)^{d+e}] = ((a+b)^{d+e}, (d+e)^c) \text{ after } c(d+e) \text{ Bott exchanges},$$

 $[(-2d-e-f)^{a+b+c}, 0^d] = ((-a-b-c)^d, (-d-e-f)^{a+b+c}) \text{ after } d(a+b+c) \text{ Bott exchanges.}$ 

$$D(\underline{\lambda}) = [(d+f)(a+b+c)] + [(d+e)(a+b+c)]$$
  
-  $[b(d+f) + c(d+e) + d(a+b+c)]$   
=  $ad + ae + af + be + cf$   
=  $codim \overline{O}_V$   
=  $t - m$ 

Next we list the cases in which  $\bigwedge^t \xi$  contributes a non-zero term. Observe that a contribution will occur whenever the Bott exchanges give a non-increasing sequence for every term of

$$(0^b, (a+b+c)^{d+f}), (0^c, (a+b+c)^{d+e}), ((-2d-e-f)^{a+b+c}, 0^d)$$

Also, note that if any of b, c or d are zero, then there are no exchanges for the corresponding term in the weight. We base our cases on this observation.

**Proposition 4.3.6.**  $\bigwedge^t \xi$  contributes to  $\mathbf{F}_{t-m}$  in the following cases when the correspond-

Cases	Conditions		
$b = 0, \ c = 0, \ d = 0$	no condition		
$b \neq 0, \ c = 0, \ d = 0$	$a+c \ge d+f$		
$b = 0, \ c \neq 0, \ d = 0$		$a+b \ge d+e$	
$b = 0, \ c = 0, \ d \neq 0$			$d+e+f \ge a+b+c$
$b = 0, \ c \neq 0, \ d \neq 0$		$a+b \ge d+e \ ,$	$d + e + f \ge a + b + c$
$b \neq 0, \ c = 0, \ d \neq 0$	$a+c \ge d+f \;,$		$d+e+f \ge a+b+c$
$b \neq 0, \ c \neq 0, \ d = 0$	$a+c \ge d+f,$	$a+b \ge d+e$	
$b \neq 0, \ c \neq 0, \ d \neq 0$	$a+c \ge d+f,$	$a+b \ge d+e,$	$d+e+f \ge a+b+c$

ing conditions are satisfied:

Table 4.1: Cases when  $\bigwedge^t \xi$  contributes to  $\mathbf{F}_{t-m}$ 

For the cases listed above, we calculate the representation that  $\bigwedge^t \xi$  contributes to  $\mathbf{F}_{t-m}$ :

Case	Weight of $\bigwedge^t \xi$	Corresponding term in $\mathbf{F}_{t-m}$
$b = 0, \ c = 0, \ d = 0$	$(a^{f}; a^{e}; (-e-f)^{a})$	$S_{(a^f)}V_1\otimes S_{(a^e)}V_2$
		$\otimes S_{((-e-f)^a)}V_3^*$
$b \neq 0, \ c = 0, \ d = 0$	$(0^b, (a+b)^f; (a+b)^e;$	$S_{(a^f,f^b)}V_1 \otimes S_{((a+b)^e)}V_2$
	$(-e-f)^{a+b})$	$\otimes S_{((-e-f)^{a+b})}V_3^*$
$b=0, \ c\neq 0, \ d=0$	$((a+c)^f; 0^c, (a+c)^e;$	$S_{((a+c)^f)}V_1 \otimes S_{(a^e,e^c)}V_2$
	$(-e-f)^{a+c})$	$\otimes S_{((-e-f)^{a+c})}V_3^*$
$b = 0, \ c = 0, \ d \neq 0$	$(a^{d+f}; a^{d+e};$	$S_{(a^{d+f})}V_1 \otimes S_{(a^{d+e})}V_2$
	$(-2d-e-f)^a, 0^d)$	$\otimes S_{(-a^d,(-d-e-f)^a)}V_3^*$
Continued on next page		

Case	Weight of $\bigwedge^t \xi$	Corresponding term in $\mathbf{F}_{t-m}$
$b = 0, \ c \neq 0, \ d \neq 0$	$((a+c)^{d+f}; 0^c, (a+c)^{d+e};$	$S_{((a+c)^{d+f})}V_1 \otimes S_{(a^{d+e},(d+e)^c)}V_2$
	$(-2d - e - f)^{a+c}, 0^d)$	$\otimes S_{((-a-c)^d,(-d-e-f)^{a+c})}V_3^*$
$b \neq 0, \ c = 0, \ d \neq 0$	$(0^b, (a+b)^{d+f}; (a+b)^{d+e};)$	$S_{(a^{d+f},(d+f)^b)}V_1 \otimes S_{((a+b)^{d+e})}V_2$
	$(-2d - e - f)^{a+b}, 0^d)$	$\otimes S_{((-a-b)^d,(-d-e-f)^{a+b})}V_3^*$
$b \neq 0, \ c \neq 0, \ d = 0$	$(0^b, (a+b+c)^f;$	$S_{((a+c)^f, f^b)}V_1 \otimes S_{((a+b)^e, e^c)}V_2$
	$0^c, (a+b+c)^e;$	$\otimes S_{((-e-f)^{a+b+c})}V_3^*$
	$(-e-f)^{a+b+c})$	
$b \neq 0, \ c \neq 0, \ d \neq 0$	$(0^b, (a+b+c)^{d+f};$	$S_{((a+c)^{d+f},(d+f)^b)}V_1$
	$0^{c}, (a+b+c)^{d+e};$	$\otimes S_{((a+b)^{d+e},(d+e)^c)}V_2$
	$(-2d - e - f)^{a+b+c}, 0^d)$	$\otimes S_{((-a-b-c)^d,(-d-e-f)^{a+b+c})}V_3^*$

Table 4.2: Term contributed by  $\bigwedge^t \xi$ 

Since  $\overline{O}_V$  is Cohen-Macaulay by Corollary 4.1.5, it is Gorenstein if and only if  $\mathbf{F}_{t-m}$  is 1-dimensional. It is known that determinantal varieties are Gorenstein if and only if the top term in their Lascoux resolution is 1-dimensional. So we investigate only those orbit closures which are not determinantal varieties arising from 1 map. We list such cases after Theorem 4.3.7.

**Theorem 4.3.7.** A non-determinantal orbit closure  $\overline{O}_V$  is Gorenstein if and only if V is in an orbit with multiplicities satisfying one of the following conditions:

- (1) a = d, b = e, c = f
- (2) a = d + e, b = 0, c = f

(3) a = d + e, b = f = 0

(4) 
$$a = d + f, c = 0, b = e$$

(5) a = d + f, c = e = 0

Proof. Part (1) follows from Theorem 4.3.4 and Table 4.2. For instance, in the case  $b \neq 0$ ,  $c \neq 0$ ,  $d \neq 0$  the term  $H^m(\mathcal{V}, \bigwedge^t \xi)$  is 1-dimensional if and only if a + c = d + f, a + b = d + e and a + b + c = d + e + f that is if and only if a = d, b = e and c = f. For the remaining parts, note that (2) is symmetric to (4) and (3) is symmetric to (5), so it suffices to prove (2) and (3).

For part (2), note that the weight of  $\bigwedge^t \xi$  is

$$((d+e+c)^{d+c}; 0^c, (d+e+c)^{d+e}; (-2d-e-c)^{d+e+c}, 0^d)$$

Calculating  $D(\underline{\lambda})$  shows that  $H^m(\mathcal{V}, \bigwedge^t \xi)$  is non-zero and dim  $H^m(\mathcal{V}, \bigwedge^t \xi) = 1$ . So by Lemma 4.3.3, the complex  $\mathbf{F}_{\bullet}$  is self-dual in this case.  $\mathbf{F}_0 = A$  implies  $\mathbf{F}_{t-m}$  is 1dimensional, hence Gorenstein.

Finally, to prove part (3) we show combinatorially that there exists a unique triple  $\underline{\lambda} = (\lambda, \mu, \nu)$  for which  $D(\underline{\lambda}) = t - m$ . Notice that for this case we have

$$t - m = (d + e + c)(2d + e) - d(d + e + c) - c(d + e) = (d + e)^{2}.$$

Claim 1:  $D((d+e)^d; (d+e+c)^{d+e}; (2d+e)^{d+e}, (d+e)^c) = t - m.$ 

$$D((d+e)^d; (d+e+c)^{d+e}; (2d+e)^{d+e}, (d+e)^c) = (d+e)(2d+e+c)$$
$$-c(d+e) - d(d+e)$$
$$= (d+e)^2$$

Also note that  $((2d+e)^{d+e}, (d+e)^c)$  is the unique term in the Littlewood-Richardson product of  $((d+e)^d)$  and  $((d+e+c)^{d+e})$  which satisfies conditions ...

Claim 2: If  $\underline{\hat{\lambda}} = (\hat{\lambda}, \hat{\mu}, \hat{\nu})$  is any other contributing triple, then  $D(\underline{\hat{\lambda}}) < t - m$ .

Observe that  $\nu$  has 2 corner boxes either of which can be removed to obtain a smaller  $\hat{\nu}$ . Suppose we remove the first corner box. This corresponds to removing one corner box from  $\mu$ . The next triple contributing a 1-dimensional representation is  $(\hat{\lambda}, \hat{\mu}, \hat{\nu}) = ((d+e-1)^d; (d+e+c)^{d+e-1}, d+e-1; (2d+e-1)^{d+e-1}, (d+e-1)^{c+1})$  with number of exchanges decreased by c + d. Then

$$D(\hat{\underline{\lambda}}) = (d+e-1)(2d+e+c-1) - c(d+e-1) + d(d+e-1)$$
$$= (d+e-1)^2 < t-m$$

On the other hand if we remove the second corner box, this corresponds to removing a box from  $\mu$  and the next contributing triple is again  $((d+e-1)^d; (d+e+c)^{d+e-1}, d+e-1; (2d+e)^{d+e-1}, (d+e-1)^{c+1})$ . Thus, removing boxes from either corner results in a triple with  $D(\hat{\lambda}) < t - m$ .

Thus, the  $((d+e)^d; (d+e+c)^{d+e}; (2d+e)^{d+e}, (d+e)^c)$  is the unique triple that contributes to

 $F_{t-m}$ ; applying Bott exchanges to the corresponding weight we get that the contribution is a trivial representation. By Lemma 4.3.3 and the fact that  $\overline{O}_V$  is Cohen-Macaulay, we are done.

Finally, we give a list of orbits that can occur if the orbit closure is Gorenstein. These are the determinantal orbits mentioned earlier. Since it is enough to specify the multiplicities a, b, c, d, e, f to specify an orbit, we present the orbits in the shape of the AR quiver (Figure 4.1) with multiplicities in place of indecomposables.

b $e$	b $e$	b $e$
a = 0	0 0	a 0
0   f = a	$0 \qquad f$	0 0
$0 \qquad e = a$	0 0	() e
a 0	a = 0	0 0
c $f$	c $f$	c $f$
0 e	0 e	$b \qquad e = b$
a = d + e + f $d$	0 $d$	0 $d$
$0 \qquad f$	c $f = c$	$0 \qquad f$
$b \qquad e = b$	$b \qquad e = a + b$	<i>b</i> 0
a 0	a 0	a 0
$c \qquad f = c$	c ()	$c \qquad f = a + c$
$b \qquad e=b$	<i>b</i> 0	b 0
0 $d$	$a \qquad d = a$	0 $d$
$c \qquad f = c$	c 0	$c \qquad f = c$
	$b \qquad e = b$	
	0 $d$	
	c 0	

We present the analysis of a few cases here and the rest are similar. The orbit



Figure 4.3: Example of determinantal orbit closure

corresponds to the representation  $V = a(010) \oplus b(011) \oplus e(100) \oplus a(001)$ . The dimension vector of V is  $\underline{d} = (e, a + b, a + b)$  so that V is a representation of the form

 $K^e \xrightarrow{\phi} K^{a+b} \xleftarrow{\psi}_{\operatorname{rank}=0} K^{a+b}$ . Thus  $\overline{O}_V$  is the determinantal variety generated by  $(b+1) \times (b+1)$  minors of  $\phi$ .

For another example, consider the orbit in Figure 4.4. A representation in this orbit

$$\begin{array}{ccc} 0 & e \\ a = d + e + f & d \\ 0 & f \end{array}$$

Figure 4.4: Example of determinantal orbit closure

is given by  $W = (d + e + f)(010) \oplus d(111) \oplus e(100) \oplus f(001)$  and has the form

 $\begin{array}{c} K^{d+f} \xrightarrow{\phi} K^{2d+e+f} \xleftarrow{\psi}_{\operatorname{rank}=d} K^{d+e}. \ \overline{O}_W \text{ is the determinantal variety generated by } (d+1) \times (d+1) \text{ minors of the } (2d+e+f) \times (2d+e+f) \text{ minors of the matrix } \phi | \psi : W_1 \oplus W_2 \to W_3. \end{array}$ 

As a final example, consider the following orbit. A representation in the orbit of Figure 4.5 has the form  $K^{a+b+c} \xrightarrow{\phi} K^{a+b+c} \xleftarrow{\psi}_{\operatorname{rank}=c} K^c$ . The corresponding orbit closure is a determinantal variety generated by  $(b+1) \times (b+1)$  minors of  $\phi$ .



Figure 4.5: Example of determinantal orbit closure

## Chapter 5

# Source-sink quivers

In this chapter we calculate resolutions for another class of Dynkin quivers which we term as 'source-sink quivers'. These are quivers with every vertex being either a source or a sink. For such Dynkin quivers we can generalize the lower bound for  $D(\underline{\lambda})$  obtained in Proposition 4.1.4. This enables us to use the geometric technique to calculate resolutions for orbit closures which admit a 1-step desingularization. We use this resolution to conclude the normality of such orbit closures.

The first section contains the main results of the chapter. We provide some examples of the calculation in the second section.

The contents of this chapter have appeared in [Sut11a].

### 5.1 Main results

To prove the main theorem (Theorem 5.1.3) we need a couple of lemmas about Young tableaux. This first of this is proved in Chapter 4 and we only recall the statement here:

**Lemma 5.1.1.** Let  $\lambda$  be a Young tableau. Then for all a and b,

$$\lambda_1 + \lambda_2 + \dots + \lambda_a \le ab + (\lambda'_{b+1} + \dots + \lambda'_{last}).$$

The next lemma is one of the well known Horn-type inequalities for triples of partitions [Ful98].

**Lemma 5.1.2.** Suppose  $\nu$  is one of the partitions occuring in Littlewood-Richardson product of  $\lambda$  and  $\mu$ . Then

$$\nu_1 + \nu_2 + \dots + \nu_k \le (\lambda_1 + \lambda_2 + \dots + \lambda_k) + (\mu_1 + \mu_2 + \dots + \mu_k).$$

Let  $Q = (Q_0, Q_1)$  be an source-sink Dynkin quiver. Fix a representation V of Q. Let  $(\lambda(a))_{a \in Q_1}$  be a  $|Q_1|$ -tuple of partitions. Consider the variety Z obtained as a 1-step desingularization of  $\overline{O}_V$ . We have as before the vector bundle  $\xi$  defined by

$$\xi = \bigoplus_{a \in Q_1} \mathcal{R}_{ta} \otimes \mathcal{Q}_{ha}^* \tag{5.1.1}$$

so that

$$\bigwedge^{t} \xi = \bigoplus_{\sum_{a \in Q_{1}} k_{a}=t} \bigwedge^{k_{a}} (\mathcal{R}_{ta} \otimes \mathcal{Q}_{ha}^{*})$$

$$= \bigoplus_{\sum_{a \in Q_{1}} |\lambda(a)|=t} \left[ \bigotimes_{a \in Q_{1}} S_{\lambda(a)} \mathcal{R}_{ta} \otimes S_{\lambda(a)'} \mathcal{Q}_{ha}^{*} \right] \qquad \text{(by Cauchy's formula)}$$

$$= \bigoplus_{\sum_{a \in Q_{1}} |\lambda(a)|=t} \bigotimes_{x \in Q_{0}} \left[ \bigotimes_{a \in Q_{1} \mid ta=x} S_{\lambda(a)} \mathcal{R}_{x} \otimes \bigotimes_{a \in Q_{1} \mid ha=x} S_{\lambda(a)'} \mathcal{Q}_{x}^{*} \right] \qquad (5.1.2)$$

When calculating the resolution  $\mathbf{F}_{\bullet}$  of  $q_*(\mathcal{O}_Z)$  we are concerned with the difference

$$D(\underline{\lambda}) := \sum_{a \in Q_1} |\lambda(a)| - N$$

where N is the total number of Bott exchanges required in the process of obtaining a partition from the weights described below. Our main theorem is an inequality involving the above difference and the Euler form of Q. It is a generalization of the inequality obtained for  $D(\underline{\lambda})$  in Proposition 4.1.4.

Let  $Q' \subset Q_0$  be the set of all source vertices and  $Q'' \subset Q_0$  be the set of all sink vertices. Let  $\lambda(a)$  be a non-increasing sequence associated to every arrow  $a \in Q_1$ . With this notation, the exterior power  $\bigwedge^t \xi$  in Equation (5.1.2) can be viewed as

$$\bigwedge^{t} \xi = \bigoplus_{\sum_{a \in Q_{1}} |\lambda(a)| = t} \left[ \left( \bigotimes_{x \in Q'} \bigotimes_{a \in Q_{1} | ta = x} S_{\lambda(a)} \mathcal{R}_{x} \right) \otimes \left( \bigotimes_{x \in Q''} \bigotimes_{a \in Q_{1} | ha = x} S_{\lambda(a)'} \mathcal{Q}_{x}^{*} \right) \right]$$
(5.1.3)

Thus we have one summand for every  $|Q_1|$ -tuple of non-increasing sequences  $(\lambda(a))_{a \in Q_1}$ . It will be useful to let this tuple of partitions also stand for the summand it corresponds to. So we write

$$\bigwedge^t \xi = \bigoplus_{|\underline{\lambda}|=t} \bigwedge^t \xi(\underline{\lambda})$$

where

$$\bigwedge^{t} \xi(\underline{\lambda}) = (\bigotimes_{x \in Q'} \bigotimes_{a \in Q_1 \mid ta = x} S_{\lambda(a)} \mathcal{R}_x) \otimes (\bigotimes_{x \in Q''} \bigotimes_{a \in Q_1 \mid ha = x} S_{\lambda(a)'} \mathcal{Q}_x^*)$$
(5.1.4)

If x is a vertex with more than one incoming or outgoing vertices then the corresponding term in the right hand side of Equation (5.1.4) is calculated using the Littlewood-Richardson rule for tensor products. Recall that we use the notation  $\lambda(a_1a_2)$  to denote a Young tableau occuring in the Littlewood-Richardson product of Young tableaux  $\lambda(a_1)$ and  $\lambda(a_2)$ .

To calculate the resolution  $\mathbf{F}_{\bullet}$  we associate a weight to each summand of  $\bigwedge^{t} \xi(\underline{\lambda})$ . Each summand consists of tensor products of terms of the form  $S_{\lambda(-)}R_x$  (for  $x \in Q'$ ) and  $S_{\lambda(-)}Q_x^*$  (for  $x \in Q''$ ). If  $x \in Q'$  the associated sequence is  $(0^{\gamma_x}, \lambda(a_1a_2...a_k))$  where  $a_1, a_2, \cdots a_k$  are all the outgoing arrows at x; if  $x \in Q''$ , the sequence is

 $(-\lambda(b_1b_2\dots b_l)', 0^{\beta_x})$  where  $b_1, b_2, \dots b_l$  are all the incoming arrows at x. We can now state the main theorem.

**Theorem 5.1.3.** Let Q be a Dynkin quiver with source-sink orientation and let  $\langle, \rangle$  be the Euler form on Q. Let  $\underline{\lambda}$  be a tuple of partitions associated to the arrows of Q and  $\underline{u}(\underline{\lambda}) \in \mathbb{N}^{|Q_0|}$  be a vector associated to  $\underline{\lambda}$ . Then

$$D(\underline{\lambda}) \ge \langle \underline{u}(\underline{\lambda}), \underline{u}(\underline{\lambda}) \rangle.$$

*Proof.* To calculate  $D(\underline{\lambda})$  we apply Bott's algorithm to the weights described above and count the total number of exchanges N. There is one weight associated to every vertex; let  $N_x$  denote the number of Bott exchanges at vertex x.

If x is a source, the weight at x is of the form  $(0^{\gamma_x}, \lambda(I_x))$  where  $I_x = a_{i_1}a_{i_2} \dots a_{i_k}$ such that  $a_{i_1}, a_{i_2}, \dots a_{i_k}$  are all the arrows incident at x. Then  $N_x = \gamma_x u_x$  where  $u_x$  is the largest number such that  $\lambda(I)_{u_x} - \gamma_x \ge u_x$ .

Similarly, if y is a sink, then weight at y is of the form  $(-\lambda(J_y)', 0^{\beta_y})$ , where  $J_y = b_{j_1}b_{j_2}\dots b_{j_l}$  such that  $b_{j_1}, b_{j_2},\dots b_{j_l}$  are all the arrows incident at y. In this case  $N_y = \beta_y u_y$ 

where  $u_y$  is the largest number such that  $-\lambda(J)_{u_y} + \beta_y \leq u_y$ . Thus

$$N = \sum_{x \in Q'} N_x + \sum_{y \in Q''} N_y = \sum_{x \in Q'} \gamma_x u_x + \sum_{y \in Q''} \beta_y u_y$$
(5.1.5)

Note that if  $u_x$  is the largest number such that  $\lambda(I)_{u_x} - \gamma_x \ge u_x$  then

$$\lambda(I_x)_1 \ge \lambda(I_x)_2 \ge \cdots \lambda(I_x)_{u_x} \ge \gamma_x + u_x$$

implies

$$\lambda(I_x)_1 + \lambda(I_x)_2 + \dots + \lambda(I_x)_{u_x} \ge u_x(\gamma_x + u_x) = u_x^2 + \gamma_x u_x$$
(5.1.6)

For similar reasons we have

$$\lambda(J_y)'_1 + \lambda(J_y)'_2 + \dots + \lambda(J_y)'_{u_y} \ge u_y(\beta_y + u_y) = u_y^2 + \beta_y u_y$$
(5.1.7)

On the other hand we have by Lemma 5.1.2 that

$$\lambda(I_x)_1 + \dots + \lambda(I_x)_{u_x} \le \sum_{\substack{a_{i_k} \\ x \longrightarrow}} (\lambda(a_{i_k})_1 + \dots + \lambda(a_{i_k})_{u_x})$$

Combining this with Inequality (5.1.6) gives

$$\sum_{\substack{a_{i_k}\\x \longrightarrow}} (\lambda(a_{i_k})_1 + \dots + \lambda(a_{i_k})_{u_x}) \ge u_x^2 + \gamma_x u_x$$
(5.1.8)

for every pair  $(x, I_x)$  with  $x \in Q'$ .

Similarly

$$\lambda(J_y)'_1 + \dots + \lambda(J_y)'_{u_y} \leq \sum_{\substack{b_{j_k} \\ \longrightarrow y}} (\lambda(b_{j_k})_1 + \dots + \lambda(b_{j_k})_{u_y})$$

together with Inequality (5.1.7) implies

$$\sum_{\substack{b_{j_k}\\ \to y}} (\lambda(b_{j_k})_1 + \dots + \lambda(b_{j_k})_{u_y}) \ge u_y^2 + \beta_y u_y$$
(5.1.9)

for every pair  $(y, J_y)$  with  $y \in Q''$ .

Using Lemma 5.1.1 we get a further upper bound on the right hand side terms of Inequality (5.1.9): if  $b_{j_k}$  is an arrow from  $x_k$  to y then

$$u_{x_{k}}u_{y} + \lambda(b_{j_{k}})_{u_{x_{k}}+1} + \lambda(b_{j_{k}})_{u_{x_{k}}+2} + \dots + \lambda(b_{j_{k}})_{last} \ge \lambda(b_{j_{k}})'_{1} + \dots + \lambda(b_{j_{k}})'_{u_{y}}$$

for every k = 1, 2, ..., l. So for every pair  $(y, J_y)$  we get inequalities

$$\sum_{\substack{b_{j_k}\\x_k \to y}} (u_{x_k} u_y + \lambda(b_{j_k})_{u_{x_k}+1} + \lambda(b_{j_k})_{u_{x_k}+2} + \dots + \lambda(b_{j_k})_{last}) \ge u_y^2 + \beta_y u_y$$
(5.1.10)

Adding the inequalities in (5.1.8) and (5.1.10) for all pairs  $(x, I_x)_{x \in Q'}$  and  $(y, I_y)_{y \in Q''}$ , we get

$$\sum_{a \in Q_1} |\lambda(a)| + \sum_{x \stackrel{a}{\to} y} u_x u_y \ge \sum_{x \in Q'} (u_x^2 + \gamma_x u_x) + \sum_{y \in Q''} (u_y^2 + \beta_y u_y) = \sum_{x \in Q_0} u_x^2 - N \quad (5.1.11)$$
which means

$$\sum_{a \in Q_1} |\lambda(a)| - N \ge \sum_{x \in Q_0} u_x^2 - \sum_{\substack{x \stackrel{a}{\to} y}} u_x u_y = \langle \underline{u}(\underline{\lambda}), \underline{u}(\underline{\lambda}) \rangle$$
(5.1.12)

As a result, we can prove the normality of a class of orbit closures arising from sourcesink Dynkin quivers.

**Corollary 5.1.4.** Let Q be a Dynkin quiver with source-sink orientation, V be a representation of Q such that the orbit closure  $\overline{O}_V$  admits a 1-step desingularization Z. Then  $\overline{O}_V$  is normal and has rational singularities.

*Proof.* Q is Dynkin implies  $\langle \underline{u}(\underline{\lambda}), \underline{u}(\underline{\lambda}) \rangle > 0$  for all  $\underline{\lambda}$ . Theorem 5.1.3 implies that the terms  $\mathbf{F}_i$  of the resolution  $\mathbf{F}_{\bullet}$  are zero for i < 0 and  $F_0 = A$ . By Theorem 3.0.5 it follows that the orbit closure is normal and has rational singularities.

The following corollary concerns orbit closures arising from extended Dynkin quivers.

**Corollary 5.1.5.** Let Q be an extended Dynkin quiver with source-sink orientation. If V is a representation of Q such that the orbit closure  $\overline{O}_V$  admits a 1-step desingularization Z then  $\mathbf{F}_{\bullet}$  is a minimal free resolution of the normalization of  $\overline{O}_V$ .

*Proof.* If Q is extended Dynkin, then  $\langle \underline{u}(\underline{\lambda}), \underline{u}(\underline{\lambda}) \rangle \geq 0$  for all  $\underline{\lambda}$ . This implies  $\mathbf{F}_i = 0$  for i < 0. The result then follows from Theorem 3.0.5.

#### 5.2 Examples

**Example 5.2.1.** Consider  $Q = A_4$  with orientation as in the figure below. Figure 1.1 shows the Auslander-Reiten quiver of Q. Let V be the direct sum of indecomposables with dimension vectors (1, 0, 0, 0), (1, 1, 1, 0), (0, 0, 1, 0), (0, 0, 1, 1), (0, 1, 1, 0), (1, 1, 1, 1)

and (1, 1, 0, 0). V admits a 1-step desingularization with dimension vectors  $\alpha = (4, 4, 5, 2)$ and  $\beta = (2, 3, 2, 1)$ . The coordinate ring of the affine space Rep(Q, (4, 4, 5, 2)) is

$$A = Sym(V_2 \otimes V_1^*) \oplus Sym(V_2 \otimes V_3^*) \oplus Sym(V_4 \otimes V_3^*)$$

Let  $R_i$  denote the subspace of  $V_i$  of dimension  $\beta_i$  and let  $Q_i := V_i/R_i$ . Then

$$\xi = R_2 \otimes Q_1^* \oplus R_2 \otimes Q_3^* \oplus R_4 \otimes Q_3^*$$

$$\bigwedge^{t} \xi = \bigoplus_{\sum_{i=1}^{3} |\lambda(i)|=t} S_{\lambda(1)'}Q_{1}^{*} \otimes S_{\lambda(12)}R_{2} \otimes S_{\lambda(23)'}Q_{3}^{*} \otimes S_{\lambda(3)}R_{4}$$

The resolution of  $\overline{O}_V$  is-

A  $\uparrow$   $(\wedge^4 V_2 \otimes \wedge^4 V_3^* \otimes A(-4)) \oplus (\wedge^4 V_1^* \otimes \wedge^4 V_2 \otimes A(-4))$   $(\wedge^3 V_2 \otimes \wedge^5 V_3^* \otimes \wedge^2 V_4 \otimes A(-5) \oplus (\wedge^4 V_1^* \otimes S_{2221} V_2 \otimes \wedge^5 V_3^* \otimes \wedge^2 V_4 \otimes A(-5))$   $\uparrow$   $(S_{2111} V_2 \otimes \wedge^5 V_3^* \otimes A(-5)) \oplus (\wedge^4 V_1^* \otimes S_{2222} V_2 \otimes \wedge^4 V_3^* \otimes A(-8))$   $(\wedge^4 V_1^* \otimes S_{3222} V_2 \otimes \wedge^5 V_3^* \otimes A(-9))$ 

 $\uparrow$ 

$$(\wedge^4 V_1^* \otimes S_{3222} V_2 \otimes \wedge^5 V_3^* \otimes A(-9))$$
$$\oplus (\wedge^4 V_1^* \otimes S_{2222} V_2 \otimes S_{21111} V_3^* \otimes \wedge^2 V_4 \otimes A(-10))$$
$$\oplus (S_{2222} V_2 \otimes S_{22222} V_3^* \otimes \wedge^2 V_4 \otimes A(-10))$$

 $\uparrow$ 

 $\wedge^4 V_1^* \otimes S_{3333} V_2 \otimes S_{22222} V_3^* \otimes \wedge^2 V_4 \otimes A(-14)$ 

**Example 5.2.2.** Let  $Q = D_5$  with the orientation in Figure 5.2. Figure 5.3 is the



Figure 5.2:  $D_5$ 

Auslander-Reiten quiver of Q. The marked indecomposables are summands of V, the circled ones are in  $\mathcal{J}_1$  and the boxed ones are in  $\mathcal{J}_2$ . Let V be the direct sum of the



Figure 5.3: AR quiver of  $D_5$ 

chosen indecomposables having dimension vectors (1, 0, 0, 0, 0), (1, 1, 1, 0, 0), (0, 0, 1, 0, 1), (0, 0, 1, 1, 1) and (1, 2, 2, 1, 1). V admits a 1-step desingularization with dimension vectors  $\alpha = (3, 3, 5, 2, 3)$  and  $\beta = (1, 2, 3, 2, 2)$ . Then  $A = Sym(V_2 \otimes V_1^*) \oplus Sym(V_2 \otimes V_3^*) \oplus$  $Sym(V_4 \otimes V_3^*) \oplus Sym(V_5 \otimes V_3^*)$ .

Let  $R_i$  denote the subspace of  $V_i$  of dimension  $\beta_i$  and let  $Q_i := V_i/R_i$ . Then

$$\xi = R_2 \otimes Q_1^* \oplus R_2 \otimes Q_3^* \oplus R_4 \otimes Q_3^* \oplus R_5 \otimes Q_3^*$$
 and

$$\bigwedge^{\iota} \xi = \bigoplus_{\sum_{i=1}^{3} |\lambda(i)|=t} S_{\lambda(1)'} Q_1^* \otimes S_{\lambda(12)} R_2 \otimes S_{\lambda(234)'} Q_3^* \otimes S_{\lambda(3)} R_4 \otimes S_{\lambda(4)} R_5$$

The resolution of  $\overline{O}_V$  is-

A

 $\uparrow$ 

$$(\wedge^{3}V_{1}^{*} \otimes \wedge^{3}V_{2} \otimes A(-3)) \oplus (\wedge^{3}V_{2} \otimes \wedge^{5}V_{3}^{*} \otimes \wedge 2V_{4} \otimes A(-5))$$
$$\oplus (\wedge^{5}V_{3}^{*} \otimes \wedge^{2}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-5))$$
$$\oplus (\wedge^{2}V_{1}^{*} \otimes \wedge^{3}V_{2} \otimes \wedge^{5}V_{3}^{*} \otimes \wedge^{1}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-7))$$
$$\oplus (\wedge^{2}V_{1}^{*} \otimes \wedge^{3}V_{2} \otimes \wedge^{5}V_{3}^{*} \otimes \wedge^{2}V_{4} \otimes \wedge^{2}V_{5} \otimes A(-7))$$

 $\uparrow$ 

$$\begin{array}{c} \oplus (\wedge^{3}V_{1}^{*} \otimes \wedge^{3}V_{2} \otimes \wedge^{5}V_{3}^{*} \otimes \wedge^{2}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-8)) \\ \oplus (\wedge^{3}V_{1}^{*} \otimes S_{211}V_{2} \otimes \wedge^{5}V_{3}^{*} \otimes \wedge^{1}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-8)) \\ \oplus (\wedge^{3}V_{1}^{*} \otimes S_{211}V_{2} \otimes \wedge^{5}V_{3}^{*} \otimes \wedge^{2}V_{4} \otimes \wedge^{2}V_{5} \otimes A(-8)) \\ \oplus (\wedge^{3}V_{1}^{*} \otimes S_{222}V_{2} \otimes \wedge^{5}V_{3}^{*} \otimes \wedge^{2}V_{4} \otimes A(-8)) \\ \oplus (\wedge^{2}V_{1}^{*} \otimes \wedge^{3}V_{2} \otimes S_{21111}V_{3}^{*} \otimes \wedge^{2}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-8)) \\ \oplus (\wedge^{3}V_{2} \otimes S_{2222}V_{3}^{*} \otimes S_{22}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-10)) \\ \oplus (\wedge^{2}V_{1}^{*} \otimes S_{222}V_{2} \otimes S_{22222}V_{3}^{*} \otimes S_{21}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-12)) \\ \oplus (\wedge^{2}V_{1}^{*} \otimes S_{222}V_{2} \otimes S_{22222}V_{3}^{*} \otimes S_{22}V_{4} \otimes \wedge^{2}V_{5} \otimes A(-12)) \end{array}$$

 $\uparrow$ 

$$\begin{array}{l} \oplus (\wedge^{3}V_{1}^{*} \otimes S_{211}V_{2} \otimes S_{21111}V_{3}^{*} \otimes \wedge^{2}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-9) \\ \oplus (\wedge^{2}V_{1}^{*} \otimes S_{222}V_{2} \otimes S_{32222}V_{3}^{*} \otimes S_{22}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-13)) \\ \oplus (\wedge^{3}V_{1}^{*} \otimes S_{322}V_{2} \otimes S_{22222}V_{3}^{*} \otimes S_{22}V_{4} \otimes \wedge^{2}V_{5} \otimes A(-13)) \\ \oplus (\wedge^{3}V_{1}^{*} \otimes S_{322}V_{2} \otimes S_{22222}V_{3}^{*} \otimes S_{21}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-13)) \\ \oplus (\wedge^{3}V_{1}^{*} \otimes S_{222}V_{2} \otimes S_{22222}V_{3}^{*} \otimes S_{22}V_{4} \otimes \wedge^{3}V_{5} \otimes A(-13)) \\ \end{array}$$

 $\uparrow$ 

 $(\wedge^3 V_1^* \otimes S_{322} V_2 \otimes S_{32222} V_3^* \otimes S_{22} V_4 \otimes \wedge^2 V_5 \otimes A(-14))$ 



Figure 5.4:  $E_6$ 

**Example 5.2.3.**  $Q = E_6$  with the orientation as in Figure 5.4. Figure 5.5 is the Auslander-Reiten quiver of Q. The marked indecomposables are summands of V, the one in the circle being  $I_1 \in \mathcal{J}_1$  and the one in the square being  $I_2 \in \mathcal{J}_2$ . Let  $V = I_1 \oplus I_2$ . Then V admits a 1-step desingularization with dimension vectors  $\alpha = \underline{\dim} V = (1, 3, 4, 3, 1, 2)$  and  $\beta = (1, 2, 3, 2, 1, 1)$ . The coordinate ring  $A = Sym(V_1 \otimes V_2^*) \oplus Sym(V_3 \otimes V_2^*) \oplus Sym(V_3 \otimes V_4^*) \oplus Sym(V_3 \otimes V_6^*) \oplus Sym(V_5 \otimes V_4^*)$  and

$$\xi = R_1 \otimes Q_2^* \oplus R_3 \otimes Q_2^* \oplus R_3 \otimes Q_4^* \oplus R_3 \otimes Q_6^* \oplus R_5 \otimes Q_4^*.$$

$$\bigwedge^{\iota} \xi = \bigoplus_{\sum_{i=1}^{5} |\lambda(i)|=t} S_{\lambda(1)} R_1 \otimes S_{\lambda(12)'} Q_2^* \otimes S_{\lambda(235)} R_3 \otimes S_{\lambda(34)'} Q_4^* \otimes S_{\lambda(4)} R_5 \otimes S_{\lambda(5)'} Q_6^*$$

The resolution of  $\overline{O}_V$  is-

A

 $\uparrow$ 

 $(V_1 \otimes \wedge^3 V_2^* \otimes \wedge^4 V_3 \otimes \wedge^2 V_6^* \otimes A(-5)) \oplus (\wedge^4 V_3 \otimes \wedge^3 V_4^* \otimes V_5 \otimes \wedge^2 V_6^* \otimes A(-5))$  $\oplus (V_1 \otimes \wedge^3 V_2^* \otimes \wedge^4 V_3 \otimes \wedge^3 V_4^* \otimes V_5 \otimes A(-6))$  $\oplus (V_1 \otimes \wedge^3 V_2^* \otimes S_{2221} V_3 \otimes \wedge^3 V_4^* \otimes \wedge^2 V_6^* \otimes A(-8))$  $\oplus (\wedge^3 V_2^* \otimes S_{2222} V_3 \otimes \wedge^3 V_4^* \otimes V_5 \wedge^2 V_6^* \otimes A(-8))$  $\oplus (\wedge^3 V_2^* \otimes S_{2221} V_3 \otimes \wedge^3 V_4^* \otimes V_5 \wedge^2 V_6^* \otimes A(-8))$  $\oplus (\wedge^3 V_2^* \otimes S_{2221} V_3 \otimes \wedge^3 V_4^* \otimes V_5 \wedge^2 V_6^* \otimes A(-8))$ 

#### $\uparrow$

 $(V_1 \otimes \wedge^3 V_2^* \otimes S_{2222} V_3 \otimes \wedge^3 V_4^* \otimes S_{21} V_6^* \otimes A(-9))$  $\oplus (V_1 \otimes \wedge^3 V_2^* \otimes S_{2221} V_3 \otimes \wedge^3 V_4^* \otimes V_5 \otimes S_{21} V_6^* \otimes A(-9))$  $\oplus (V_1 \otimes \wedge^3 V_2^* \otimes S_{2221} V_3 \otimes S_{211} V_4^* \otimes V_5 \otimes \wedge^2 V_6^* \otimes A(-9))$  $\oplus (V_1 \otimes S_{211} V_2^* \otimes S_{2222} V_3 \otimes \wedge^3 V_4^* \otimes V_5 \otimes \wedge^2 V_6^* \otimes A(-9))$  $\oplus (V_1 \otimes S_{211} V_2^* \otimes S_{2221} V_3 \otimes \wedge^3 V_4^* \otimes V_5 \otimes \wedge^2 V_6^* \otimes A(-9))$  $\oplus (\wedge^3 V_2^* \otimes S_{2222} V_3 \otimes \wedge^3 V_4^* \otimes V_5 \otimes S_{21} V_6^* \otimes A(-9))$  $\oplus (\wedge^3 V_2^* \otimes S_{2222} V_3 \otimes S_{211} V_4^* \otimes V_5 \otimes \wedge^2 V_6^* \otimes A(-9))$  $\oplus (\wedge^3 V_2^* \otimes S_{2222} V_3 \otimes S_{211} V_4^* \otimes V_5 \otimes \wedge^2 V_6^* \otimes A(-9))$ 

 $\uparrow$ 

 $(V_{1} \otimes \wedge^{3}V_{2}^{*} \otimes S_{2222}V_{3} \otimes S_{211}V_{4}^{*} \otimes V_{5} \otimes S_{21}V_{6}^{*} \otimes A(-10))$   $\oplus (V_{1} \otimes S_{211}V_{2}^{*} \otimes S_{2222}V_{3} \otimes \wedge^{3}V_{4}^{*} \otimes V_{5} \otimes S_{21}V_{6}^{*} \otimes A(-10))$  $\oplus (V_{1} \otimes S_{211}V_{2}^{*} \otimes S_{2222}V_{3} \otimes S_{211}V_{4}^{*} \otimes V_{5} \otimes \wedge^{2}V_{6}^{*} \otimes A(-10))$ 



Figure 5.5: AR quiver of  $E_6$ 

## Chapter 6

# Equioriented $A_n$

In this chapter we describe briefly the incidence varieties introduced in [Sch92, DSW07]. We work in the case of equioriented  $A_n$ . These varieties determine orbit closures under some conditions (Proposition 6.0.11). In this case we can use the geometric technique to calculate resolutions and prove normality of orbit closures in a larger class: orbit closures admitting a 1-step desingularization are contained in the class of orbit closures arising from incidence varieties. This provides a different proof of the result in [ADFK81] about the geometry orbit closures of type  $A_n^{eq}$ .

Let  $x_1, x_2, \dots, x_n$  denote vertices and  $a_1, a_2, \dots, a_{n-1}$  denote arrows of the quiver  $Q = A_n^{eq}$  with  $ta_i = x_i$  and  $ha_i = x_{i+1}$ . Fix two dimension vectors  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and  $\beta = (\beta_1, \beta_2, \dots, \beta_n)$  and let  $\gamma = (\alpha_i - \beta_i)_{i \in [1,n]}$ . Let  $V_i = K^{\alpha_i}$  and identify  $\operatorname{Rep}_K(A_n, \alpha)$ with  $\bigoplus_{i=1}^{n-1} Hom_K(V_i, V_{i+1})$ . Let  $\operatorname{Gr}(\beta, \gamma)$  denote  $\prod_{i=1}^n \operatorname{Gr}(\beta_i, V_i)$ . Consider the sequence of tautological vector bundles

$$0 \to \mathcal{R}_i \to V_i \times \operatorname{Gr}(\beta_i, V_i) \to \mathcal{Q}_i \to 0$$

on  $Gr(\beta_i, V_i)$ . The incidence variety

$$Z(\beta,\gamma) = \{ (V,(R_x)) \in \operatorname{Rep}(A_n,\alpha) \times Gr(\beta,\gamma) \mid V(a)(R_{ta}) \subset R_{ha}, \forall a \in Q_1 \}$$

was defined by Schofield in [Sch92]. Let p: Rep $(A_n, \alpha) \times \operatorname{Gr}(\beta, \gamma) \to \operatorname{Gr}(\beta, \gamma)$  be the projection map.  $Z(\beta, \gamma)$  is a zero set of a cosection of the vector bundle  $p^*\xi$  on Rep $(A_n, \alpha) \times \operatorname{Gr}(\beta, \gamma)$ , where  $\xi$  is a vector bundle on  $\operatorname{Gr}(\beta, \gamma)$  defined as:

$$\xi = \bigoplus_{a \in Q_1} \mathcal{R}_{ta} \otimes \mathcal{Q}_{ha}^*.$$

The following result is proved in [Sch92].

**Theorem 6.0.4.** 1. The first projection  $q: Z(\beta, \gamma) \to Rep(A_n, \alpha)$  is a proper map.

- 2.  $p: Z(\beta, \gamma) \to Gr(\beta, \gamma)$  be the second projection.  $(Z(\beta, \gamma), Gr(\beta, \gamma), p)$  is a vector bundle.
- 3. dim  $Z(\beta, \gamma)$  dim  $Rep(Q, \alpha) = \langle \beta, \alpha \rangle$

Let  $Y(\beta, \gamma) := q(Z(\beta, \gamma))$ . The incidence variety  $Z(\beta, \gamma)$  is irreducible and q is proper so that  $Y(\beta, \gamma)$  is irreducible. This implies that  $Y(\beta, \gamma) = \overline{O}_V$  for some  $V \in \operatorname{Rep}(Q, \alpha)$ . If the generic fibre of q is a point then  $Z(\beta, \gamma)$  is a desingularization of  $Y(\beta, \gamma)$ . In such cases we can use the geometric technique to calculate the resolution  $\mathbf{F}(\beta, \gamma)_{\bullet}$  of the normalization of Y.

To calculate a resolution using the geometric technique we need a lower bound on  $D(\underline{\lambda})$ as in the earlier cases of Chapter 4 and Chapter 5. This is the content of Proposition 6.0.7. We now describe the combinatorics required to obtain the bound. Consider the vector bundle  $\xi = \bigoplus_{a \in Q_1} \mathcal{R}_{ta} \otimes \mathcal{Q}_{ha}^*$  defined earlier. To calculate  $\mathbf{F}(\beta, \gamma)_{\bullet}$ we need to calculate  $\bigwedge^t \xi$ . The exterior powers of  $\xi$  decompose according to Cauchy formula. To describe this decomposition we need some notation. For an (n-1)-tuple of partitions  $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n-1)})$  we denote by  $|\underline{\lambda}|$  the sum  $\sum_{i=1}^{n-1} |\lambda^{(i)}|$ . With this notation we have

$$\bigwedge^{s} \xi = \bigoplus_{|\underline{\lambda}|=s} \xi(\underline{\lambda})$$

where

$$\xi(\underline{\lambda}) = \bigotimes_{i=1}^{n-1} (S_{\lambda^{(i)}} \mathcal{R}_i \otimes S_{\lambda^{(i)}} \mathcal{Q}_{i+1}^*)$$

We rearrange the terms to get

$$\xi(\underline{\lambda}) = \bigotimes_{i=1}^{n} (S_{\lambda^{(i)}} \mathcal{R}_i \otimes S_{\lambda^{(i-1)}} \mathcal{Q}_i^*)$$

with the convention  $\lambda^{(0)} = (0), \lambda^{(n)} = (0)$ . Note that each  $\lambda^{(i)}$  is contained in a  $\beta_i \times \gamma_{i+1}$ rectangle (which is same as saying  $\lambda^{(i)} \leq (\beta_i)^{\gamma_i}$ , that is,  $\lambda_j^{(i)} \leq \beta_j$  for all j and  $\lambda_k^{(i)'} \leq \gamma_{k+1}$ for all k). The contribution of each summand  $\xi(\underline{\lambda})$  to cohomology of  $\bigwedge^s \xi$  can be calculated by applying Bott's theorem to the weights

$$\nu^{(i)} = (-\lambda_{\gamma_i}^{(i-1)\prime}, \dots, -\lambda_1^{(i-1)\prime}, \lambda_1^{(i)}, \dots, \lambda_{\beta_i}^{(i)})$$

for i = 1, ..., n. If the resulting contribution is  $S_{\mu^{(i)}}V_i$  for each i = 1, ..., n then the summand  $\xi(\underline{\lambda})$  contributes the term  $\bigotimes_{i=1}^n S_{\mu^{(i)}}V_i$  to the cohomology of  $\bigwedge^s \xi$ . In order to calculate the terms of the complex  $F(\beta; \gamma)_{\bullet}$  explicitly let us look more closely at the contribution of the term  $\xi(\underline{\lambda})$ . Assume this contribution is nonzero. Applying the Bott

theorem to the weight  $\nu^{(i)}$  means we have to find permutations  $w(i) \in \sum_{\alpha_i}$  and dominant weights  $\mu^{(i)} = w(i)(\nu^{(i)} + \rho^{(i)}) - \rho^{(i)}$  where  $\rho^{(i)} = (\alpha_i - 1, \alpha_i - 2, \dots, 1, 0)$ . This action can be interpreted as the Bott exchanges described in Chapter 3.

We define operations which associate to terms in  $F(\beta; \gamma)_i$  certain terms in  $F(\beta; \gamma)_{i-1}$ or in  $F(\beta; \tilde{\gamma})_i$ , where  $\tilde{\gamma}$  differs from  $\gamma$  at the *i*th coordinate. Let us denote by  $\mathcal{P}(\beta, \gamma)(u)$ the set of functions  $\underline{\lambda}$  such that the term  $\xi(\underline{\lambda})$  gives a nonzero contribution to  $F(\beta; \gamma)_u$ . From the description of the term  $F(\beta; \gamma)_u$ , it is clear that  $\underline{\lambda} \in \mathcal{P}(\beta, \gamma)(u)$  if and only if

$$D(\underline{\lambda}) := \sum_{i=1}^{n-1} |\lambda^{(i)}| - \sum_{i=1}^{n} N_{\nu^{(i)}} = u.$$

Here  $N_{\nu^{(i)}}$  is the number of Bott exchanges applied to  $\nu^{(i)}$ .

Let  $\xi(\underline{\lambda})$  be a term giving a nonzero contribution to  $F(\beta;\gamma)_{\bullet}$ . Consider the weight

$$\nu^{(i)} = (-\lambda^{(i-1)\prime}_{\gamma_i}, \dots, -\lambda^{(i-1)\prime}_1, \lambda^{(i)}_1, \dots, \lambda^{(i)}_{\beta_i}).$$

We say that a corner box in the *t*-th row of  $\lambda^{(i)}$  is **linked** to a corner box in *s*-th column  $\lambda^{(i-1)}$  if  $\lambda_s^{(i-1)'} + \lambda_t^{(i)} = s + t$ . It is clear that the corner box in  $\lambda^{(i)}$  can be linked to at most one corner box of  $\lambda^{(i-1)}$  and it can be linked to at most one corner box of  $\lambda^{(i+1)}$ . This definition includes the partitions  $\lambda^{(0)} = \lambda^{(n+1)} = (0)$  which we treat as having one corner box each.

Remark 6.0.5. If  $\lambda_t^{(i)}$  is linked to  $\lambda_s^{(i-1)'}$  then it means that after (s+t-1) Bott exchanges  $\lambda_t^{(i)}$  is exchanged with  $\lambda_s^{(i-1)'}$  and we get equality  $\lambda_t^{(i)} - (s+t-1) = \lambda_s^{(i-1)'} + 1$ . This means that the corner boxes of  $\lambda_t^{(i)}$  and  $\lambda_s^{(i-1)'}$  are essential for counting the number of exchanges in the sense that deleting *either one* of these boxes leads to a weight that contributes

zero. On the other hand, if we delete *both* these corner boxes then we get a weight that contributes a non-zero term with one less exchange, that is after  $N_{\nu^{(i)}} - 1$  exchanges.

For any fixed corner box x, either x is linked to some corner box or it is not linked to any. In the latter case, we associate to  $\underline{\lambda}$  a term  $\underline{\hat{\lambda}}$  be deleting x. More precisely, if  $\underline{\lambda}$ is an element of  $\mathcal{P}(\beta, \gamma)(u)$  such that for certain i  $(1 \leq i \leq n-1)$ , x is a corner box in  $\lambda^{(i)}$  which is not linked to any of the corner boxes in  $\lambda^{(i-1)}$  or  $\lambda^{(i+1)}$ . Then we define  $\underline{\hat{\lambda}}$  by setting  $\hat{\lambda}^{(j)} = \lambda^{(j)}$  for  $j \neq i$  and  $\hat{\lambda}^{(i)}$  is obtained from  $\lambda^{(i)}$  by removing x. Then  $|\underline{\hat{\lambda}}| = |\underline{\lambda}| - 1$ while  $\sum_{i=1}^{n} N_{\hat{\nu}^{(i)}} = \sum_{i=1}^{n} N_{\nu^{(i)}}$  so that  $\underline{\hat{\lambda}} \in \mathcal{P}(\beta, \gamma)(u-1)$ . This determines for each u functions  $d(\gamma^{(i)}) : \mathcal{P}(\beta, \gamma)(u) \to \mathcal{P}(\beta, \gamma)(u-1)$  defined on certain subset of  $\mathcal{P}(\beta, \gamma)(u)$ .

If x is linked to some corner box, we consider the chain of linked boxes containing x. For this, let  $\underline{\lambda}$  be a term in  $\mathcal{P}(\beta, \gamma)(u)$  such that for certain  $0 \leq i \leq j \leq n$  there are corner boxes  $x_k$  of  $\lambda^{(k)}$  for  $i \leq k \leq j$  such that  $x_i$  is not linked to a corner box in  $\lambda^{(i-1)}$ ,  $x_j$  is not linked to a corner box in  $\lambda^{(j+1)}$  and  $x_k$  is linked to  $x_{k+1}$  for  $i \leq k \leq j-1$ . This describes a chain of linked corner boxes starting at  $x_i$  in  $\lambda^{(i)}$  and ending at  $x_j$  in  $\lambda^{(j)}$ . A chain can be one of the following three types:

- (I) the chain is linked to either  $\lambda^{(0)}$  or  $\lambda^{(n)}$
- (II) the chain contains a corner box linked to zero
- (III) the chain is neither linked to  $\lambda^{(0)}$  or  $\lambda^{(n)}$  nor contains a corner box linked to zero.

Our strategy is to preserve the chains of type (I) and remove all unlinked boxes as well as chains of type (II) and (III) to get an extremal term (a simpler 'representative' of  $\underline{\lambda}$ ), which we will denote by  $\hat{\lambda}$  and call the extremal term. If we have a chain of type (III), we define the term  $\underline{\hat{\lambda}}$  by setting

$$\hat{\lambda}^{(k)} = \begin{cases} \lambda^{(k)}, & \text{if } k < i \text{ or } k > j \\ \lambda^{(k)} \text{ minus the box } x_k, & \text{if } i \le k \le j \end{cases}$$

(i.e. we remove the entire chain of linked corner boxes). Then it is clear by definition that the term  $\underline{\hat{\lambda}} \in \mathcal{P}(\beta, \gamma)(u-1)$ . This determines for each u functions  $d(i, j) = d(\gamma^{(j)}) \circ$  $\dots \circ d(\gamma^{(i)}) : \mathcal{P}(\beta, \gamma)(u) \to \mathcal{P}(\beta, \gamma)(u-1)$  defined on certain subset of  $\mathcal{P}(\beta, \gamma)(u)$ .

Suppose now that  $\lambda \in \mathcal{P}(\beta, \gamma)(u)$  contains a chain of type (II) and the corner box  $x_r$  of  $\lambda^{(r)}$  is linked to zero in  $\lambda^{(r-1)'}$ , for some  $i \leq r \leq j$ . Then

$$\nu^{(r)} = (0, \dots, -\lambda_2^{(r-1)'}, -\lambda_1^{(r-1)'}, \lambda_1^{(r)}, \lambda_2^{(r)}, \dots, \lambda_{\beta_r}^{(r)})$$

(here  $\lambda^{(r-1)'}$  has  $\gamma_r$  parts with  $\lambda_{\gamma_r}^{(r-1)'} = 0$ ). To this  $\lambda^{(r-1)'}$  we associate the term with  $\gamma_r - 1$  parts by deleting the last zero:

$$\hat{\lambda}^{(r-1)'} = (\lambda_1^{(r-1)'}, \lambda_2^{(r-1)'}, \dots, \lambda_{\gamma_r-1}^{(r-1)'})$$

Now define the term  $\underline{\hat{\lambda}}$  as follows

$$\hat{\lambda}^{(k)} = \begin{cases} \lambda^{(k)}, & \text{if } k < i \text{ or } k > j \\\\ \lambda^{(k)} \text{ minus the box } x_k, & \text{if } i \le k \le j, \ k \ne r-1 \\\\ \hat{\lambda}^{(r-1)} & \text{if } k = r-1 \end{cases}$$

Since the number of boxes removed equals the number of exchanges reduced,  $D(\hat{\underline{\lambda}}) = u$ 

and  $\underline{\hat{\lambda}} \in \mathcal{P}(\beta, \hat{\gamma})(u)$  where  $\hat{\gamma} < \gamma$ . Let  $\hat{d}(\gamma^{(r)}) : \mathcal{P}(\beta, \gamma)(u) \to \mathcal{P}(\beta, \hat{\gamma})(u)$  denote the function that maps  $\lambda^{(r)}$  to  $\hat{\lambda}^{(r)}$ . Then for each u we have functions  $\hat{d}_{\gamma}(i, j) = \hat{d}(\gamma^{(j)}) \circ \ldots \circ \hat{d}(\gamma^{(i)}) :$  $\mathcal{P}(\beta, \gamma)(u) \to \mathcal{P}(\beta, \hat{\gamma})(u)$  defined on certain subset of  $\mathcal{P}(\beta, \gamma)(u)$ . Analogously, we can define maps  $\hat{d}_{\beta}(i, j) : \mathcal{P}(\beta, \gamma)(u) \to \mathcal{P}(\hat{\beta}, \gamma)(u)$  where  $\hat{\beta} < \beta$ .

We call the term  $\underline{\lambda} \in \mathcal{P}(\beta, \gamma)(u)$  **extremal** if it is not in the domain of any of the functions d(i, j),  $\hat{d}_{\gamma}(i, j)$  or  $\hat{d}_{\beta}(i, j)$  for  $1 \leq i \leq j \leq n-1$ .

Let  $\underline{\lambda}$  be an element of  $\mathcal{P}(\beta, \gamma)(u)$ . If  $\lambda^{(i)} = (0)$  for some *i* then the calculation of such term can be obtained by calculating corresponding terms in complexes  $F(\beta; \gamma)_{\bullet}$  for quivers  $A_m$  for m < n. Therefore we call the term  $\underline{\lambda} \in \mathcal{P}(\beta, \gamma)(u)$  **proper** if  $\lambda^{(i)} \neq (0)$  for  $1 \leq i \leq n-1$ .

**Lemma 6.0.6.** If  $\underline{\lambda} \in \mathcal{P}(\beta, \gamma)(u)$  is a proper extremal term, then  $\underline{\lambda}$  is of the form  $((t_i + \gamma_1)^{t_i}, t_i^{\beta_n})$  for  $1 \leq t_i \leq n-1$ .

Proof. Since  $\underline{\lambda}$  contains no unlinked boxes or chains of linked corners of type (II) or (III), for every  $1 \leq i \leq n-1$  there are only two corner boxes in  $\lambda^{(i)}$ : one contained in the chain linked to  $\lambda^{(0)} = 0^{\gamma_1}$  and the other contained in the chain linked to  $\lambda^{(n)} = 0^{\beta_n}$ .

First consider the chain linked to  $\lambda^{(0)} = 0^{\gamma_1}$ . We denote the corner box in the *s*-th row and *t*-th column of  $\lambda^{(i)}$  by  $\lambda_{st}^{(i)}$ . Now suppose  $\lambda_{jk}^{(1)}$  is linked to  $\lambda^{(0)}$ . Then by the definition of linked boxes  $\lambda_j^{(1)} = \gamma_1 + j$  (so that  $k = j + \gamma_1$ ). We let  $t_1 = j$ . If  $\lambda_{jk}^{(1)}$  is linked to  $\lambda_{pq}^{(2)}$ , then  $\lambda_k^{(1)'} + \lambda_p^{(2)} = k + p$  i.e.  $j + \lambda_p^{(2)} = \gamma_1 + j + p$ . Let  $t_2 = p$ . Then  $\lambda_p^{(2)} = t_2 + \gamma_1$ . Continuing in this manner, for every  $1 \le i \le n - 1$  we get  $t_i$  such that  $\lambda^{(i)} = t_i + \gamma_1$ .

Similarly, by considering the chain linked to  $\lambda^{(n)} = 0^{\beta_n}$ , we get that there exist  $s_i$  for  $1 \le i \le n-1$  such that  $\lambda^{(i)'} = s_i + \beta_n$ . Since each  $\lambda^{(i)}$  contains at most 2 corner boxes we get  $t_i = s_i$ .

**Proposition 6.0.7.** Let  $\underline{\lambda} \in \mathcal{P}(\beta, \gamma)(u)$ . Then there exists a proper extremal term  $\underline{\hat{\lambda}} \in \mathcal{P}(\beta, \gamma')(u')$  for some  $\gamma' \leq \gamma$  and  $u' \leq u$  such that:

$$D(\underline{\lambda}) \ge D(\underline{\hat{\lambda}})$$

(here  $\gamma' \leq \gamma$  iff  $\gamma'_i \leq \gamma_i, \ 1 \leq i \leq n.$ )

Proof. Starting with  $\underline{\lambda} \in \mathcal{P}(\beta, \gamma)(u)$ , by applying the maps  $d(\gamma^{(i)})$  and d(i, j) for different values of i and j, we arrive at a term  $\underline{\hat{\lambda}}$  some  $\mathcal{P}(\beta, \gamma)(u')$  where  $u' \leq u$ . If  $\underline{\hat{\lambda}}$  contains no chain of type (II) then it is the required extremal term. If  $\hat{\lambda}^{(i)}$  has a corner box linked to zero, we apply  $\hat{d}(\gamma^{(i)})$  (or  $\hat{d}(\beta^{(i)})$ ) to get a term in  $\mathcal{P}(\beta, \hat{\gamma})(u')$  (or  $\mathcal{P}(\hat{\beta}, \gamma)(u')$  resp.)which we also denote by  $\underline{\hat{\lambda}}$ . Continuing in this manner we obtain an extremal  $\underline{\hat{\lambda}}$  satisfying the required properties.

**Proposition 6.0.8.** Let  $\underline{\lambda} \in \mathcal{P}(\beta, \gamma)(u)$ . Then

$$D(\underline{\lambda}) \ge \langle (t_1, t_2, \dots, t_{n-1}), (t_1, t_2, \dots, t_{n-1}) \rangle$$

where  $\langle \cdot, \cdot \rangle$  is the Euler form of equioriented quiver  $A_{n-1}$ .

*Proof.* We show that  $D(\hat{\underline{\lambda}}) = \langle (t_1, t_2, \dots, t_{n-1}), (t_1, t_2, \dots, t_{n-1}) \rangle$ , then the result follows from the previous lemma.

We have  $\hat{\lambda}^{(i)} = ((t_i + \gamma_1)^{t_i}, t_i^{\beta_n})$  implies  $|\hat{\lambda}^{(i)}| = t_i^2 + t_i(\gamma_1 + \beta_n)$ . Also

$$\hat{\nu}^{(i)} = (-(t_{(i-1)})^{\gamma_1}, \ -(t_{i-1}+\beta_n)^{t_{i-1}}, \ (t_i+\gamma_1)^{t_i}, \ (t_i)^{\beta_n})$$

So  $N_{\hat{\nu}^{(i)}} = t_i(t_{i-1} + \gamma_1) + \beta_n(t_{i-1}).$ 

$$D(\hat{\lambda}) = \sum_{1}^{n-1} |\hat{\lambda}^{(i)}| - \sum_{1}^{n} N_{\nu^{(i)}}$$
  
=  $\sum_{1}^{n-1} [t_i^2 + t_i(\gamma_1 + \beta_n)] - \sum_{2}^{n-1} [t_i(t_{i-1} + \gamma_1) + \beta_n(t_{i-1})] - \gamma_1 t_1 - \beta_n t_{n-1}$   
=  $\sum_{1}^{n-1} t_i^2 - \sum_{2}^{n-1} t_i t_{i-1}$   
=  $\langle (t_1, t_2, \dots, t_{n-1}), (t_1, t_2, \dots, t_{n-1}) \rangle$ 

**Theorem 6.0.9.** Let Q be an equivarianted quiver of type  $A_n$  and  $\beta, \gamma$  be two dimension vectors. Let  $\alpha = \beta + \gamma$  and  $V \in \text{Rep}(Q, \alpha)$ . Then

- (1)  $F(\beta, \gamma)_i = 0$  for i < 0.
- (2)  $F(\beta, \gamma)_0 = A.$
- (3) The summands of  $F(\beta, \gamma)_1$  are of the form  $\bigwedge^{\gamma_i + \beta_j + 1} V_i \otimes \bigwedge^{\gamma_i + \beta_j + 1} V_j^*$  where  $1 \le i < j \le n$ .

Proof. Since  $Q = A_n^{eq}$ , the Euler form is positive definite. From Lemma 6.0.8 it follows that any term  $\underline{\lambda}$  contributes to non-negative degrees in  $\mathbf{F}_{\bullet}$ . This implies  $F(\beta, \gamma)_i = 0$  for i < 0. Also the proper extremal terms occur in positive degrees so the only term in degree 0 is the trivial term  $\lambda^{(i)} = 0$  for  $1 \le i \le n - 1$ . So  $F(\beta, \gamma)_0 = A$ .

If a term in  $\mathcal{P}(\beta, \gamma)$  is not extremal, then it becomes a trivial term after applying one of the maps d(i, j) or  $\hat{d}(i, j)$ . This implies the term itself is trivial. If a term  $\underline{\lambda} \in \mathcal{P}(\beta, \gamma)$  is extremal then  $D(\underline{\lambda}) = 1$  that is

$$\langle (t_1, t_2, \dots, t_{n-1}), (t_1, t_2, \dots, t_{n-1}) \rangle = 1$$

By a theorem of Gabriel [Gab75], the roots of this quadratic form are in one-one correspondence with the dimension vectors of indecomposable representations of  $A_n$ . So the roots are  $t_{i,j} = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{i \text{ to } j}, 0, \ldots, 0)$  for  $1 \leq i \leq j \leq (n-1)$ . The extremal term  $\underline{\lambda}(i, j)$  corresponding to the root  $t_{i,j}$  is given by

$$\lambda^{(k)}(i,j) = \begin{cases} 0 & \text{if } k < i \text{ or } k > j-1 \\ (1+\gamma_i, 1^{\beta^j}) & \text{if } i \le k \le j-1 \end{cases}$$

As a result,

$$\nu^{(k)} = \begin{cases} (0^{\gamma_k - \gamma_i - 1}, -1^{\gamma_i}, -1 - \beta_j, 1 + \gamma_1, 1^{\beta_j}, 0^{\beta_k - \beta_j - 1}) & \text{if } i + 1 \le k \le j - 1 \\ (0^{\gamma_i}, 1 + \gamma_i, 1^{\beta_j}, 0^{\beta_i - \beta_j - 1}) & \text{if } k = i \\ (0^{\gamma_j - \gamma_i - 1}, -1^{\gamma_i}, -1 - \beta_j, 0^{\beta_j}) & \text{if } k = j \\ (0^{\gamma_k + \beta_k}) & \text{otherwise} \end{cases}$$

On applying Bott exchanges to these, we see that the terms corresponding to k = i and k = j give the partitions  $(1^{\gamma_i + \beta_j + 1})$  and  $(-1^{\gamma_i + \beta_j + 1})$  respectively (after  $\gamma_i + \beta_j$  exchanges) while the remaining terms give the trivial partition. So the contribution from  $\underline{\lambda}(i, j)$  is  $\bigwedge^{\gamma_i + \beta_j + 1} V_i \otimes \bigwedge^{\gamma_i + \beta_j + 1} V_j^*$  where  $1 \le i < j \le n$ .

Thus the terms contributing to  $F(\beta, \gamma)_1$  correspond to vanishing of  $\gamma_i + \beta_j$  minors of

the composition  $V(a_{j-1}) \circ \cdots \circ V(a_i)$ .

We determine the pairs  $\alpha, \beta$  for which  $Y(\beta, \gamma) = q(Z(\beta, \gamma))$  determines an orbit in  $Rep(A_n, \alpha)$ . For these orbits the collapsing technique gives a resolution of the defining ideal of the orbit closures  $\overline{Y(\beta, \gamma)}$ .

Let  $V = ((V_i = K^{\alpha_i})_{i \in Q_0}, (V_a)_{a \in Q_1})$  be a representation of  $A_n$ . Let

$$S_{i,j} = 0 \to \cdots K \to K \to \cdots K \to 0 \to \cdots 0$$

denote the indecomposable representation of  $A_n$  with K in positions i to j and 0 elsewhere. Then  $V = \bigoplus_{1 \le i \le j \le n} m_{i,j} S_{i,j}$ , where  $m_{i,j}$  denotes the multiplicity of  $S_{i,j}$  in V. Let  $r_{i,i} = \dim V_i$  and  $r_{i,j} = \operatorname{rank}(V_i \to V_j)$  for i < j. Then V is determined by the multiplicities  $m_{i,j}$  or by the ranks  $r_{i,j}$ . Given any one set of conditions, we can obtain the other set by elementary algebraic operations. The following result states the formula for this translation.

Lemma 6.0.10. Rank-multiplicity relations:

$$r_{i,j} = \sum_{\substack{k \le i \le j \le l \\ m_{i,j} = r_{i,j} - r_{i,j+1} - r_{i-1,j} + r_{i-1,j+1}}} m_{k,l}$$

Now  $q(Z(\beta, \gamma))$  determines the orbit of V whenever the multiplicities  $m_{i,j} \ge 0$ . This condition translates into conditions on  $\beta$  and  $\gamma$ . As a result we have the following

**Proposition 6.0.11.** The image  $q(Z(\beta, \gamma))$  determines an orbit in  $Rep(A_n, \alpha)$  whenever

$$\beta_1 \ge \beta_2 \ge \cdots \beta_n \quad and \quad \gamma_1 \le \gamma_2 \le \cdots \gamma_n$$

*Proof.* From Theorem 6.0.9 we know that  $r_{i,j} = \gamma_i + \beta_j$ . If i > 1 and j < n,

$$m_{i,j} = r_{i,j} - r_{i,j+1} - r_{i-1,j} + r_{i-1,j+1}$$
  
=  $\gamma_i + \beta_j - \gamma_i - \beta_{j+1} - \gamma_{i-1} - \beta_j + \gamma_{i-1} + \beta_{j+1}$   
= 0

If i = 1,  $m_{1,j} = r_{1,j} - r_{1,j+1} = \beta_j - \beta_{j+1}$  so that  $m_{1,j} \ge 0$  implies  $\beta_j \ge \beta_{j+1}$  for all  $1 \le j \le n$ .

If j = n,  $m_{i,n} = r_{i,n} - r_{i-1,n} = \gamma_i - \gamma_{i-1}$  so that  $m_{i,n} \ge 0$  implies  $\gamma_i \ge \gamma_{i-1}$  for all  $1 \le i \le n$ .

**Corollary 6.0.12.** Let  $\beta, \gamma$  be dimension vectors satisfying conditions in Proposition 6.0.11. Suppose the generic fiber of q is a point. Then the complex  $\mathbf{F}(\beta, \gamma)_{\bullet}$  is a minimal free resolution of  $Y(\beta, \gamma)$ . Moreover,  $Y(\beta, \gamma)$  is normal, Cohen-Macaulay and has rational singularities.

*Proof.* If the generic fibre of q is a point then q is a birational isomorphism and  $Z(\beta, \gamma)$  is a desingularization of  $Y(\beta, \gamma)$ . The result then follows from Theorem 6.0.9 and Theorem 3.0.5.

## Summary and future work

#### 6.1 Summary

This thesis work demonstrates the use of Weyman's geometric technique to studying orbit closures of representations of Dynkin quivers. To make our calculations explicit and algorithmic, we have restricted to orbit closures admitting a 1-step desingularization (Section 2.2.1).

Let Q be a Dynkin quiver, V be a representation of Q. The set of representations of Qof a fixed dimension vector  $\underline{d}$  are denoted by  $\operatorname{Rep}(Q, \underline{d})$ . The orbits of  $Gl(\underline{d})$  in  $\operatorname{Rep}(Q, \underline{d})$ determine isomorphism classes of representations of Q. Let  $\overline{O}_V$  be the closure of the orbit of  $V \in \operatorname{Rep}(Q, \underline{d})$ . We wish to study the geometry of orbit closures by calculating the resolutions of their defining ideals. Our work relies on Theorem 3.0.4 which is the main theorem of the geometric technique. It turns out that showing a minimal free resolution exists for  $\overline{O}_V$  amounts to showing that a certain difference estimate  $D(\underline{\lambda})$  is non-negative (here  $\underline{\lambda}$  is a tuple of partitions associated to arrow set of Q). We achieve this by showing something stronger, namely that  $D(\underline{\lambda})$  is bounded below by Euler form of Q evaluated at a vector  $\underline{u}(\underline{\lambda})$ . This is our key result.

In Chapter 4 we use this technique to carry out explicit calculations for orbit closures

arising from representations of non-equioriented  $A_3$ . The resolutions enable us to read off certain geometric properties of orbit closures. This case is special because we can find a partition of the corresponding Auslander-Reiten quiver into 2 parts, thus enabling every orbit closure to admit a 1-step desingularization. In short, the results of Chapter 4 are applicable to *all orbit closures* arising in the representation space of non-equioriented  $A_3$ .

In Chapter 5 we explore representations of a source-sink quiver Q. We prove our key result in this context (Theorem 5.1.3) which ascertains the existence of minimal free resolutions and normality for orbit closures corresponding to Q. Since Q can be of any Dynkin type, this proves in particular the normality of orbit closures (admitting 1-step desingularization) for source-sink Dynkin quivers of type  $E_6$ ,  $E_7$  and  $E_8$ .

In Chapter 6 we prove the key result for equioriented quiver  $A_n$ . This gives an alternate proof of normality for orbit closures (admitting 1-step desingularization) of type  $A_n$ . We also consider the more general case of Schofield's incidence varieties  $Z(\beta, \gamma)$ .

To summarize, the lower bound on  $D(\underline{\lambda})$  by the Euler form is the main combinatorial argument that yields all the interesting results. As of now we have this estimate for quivers with source-sink orientation and for equioriented  $A_n$ .

#### 6.2 Future work

Our hope is that the bound on  $D(\underline{\lambda})$  can be generalized to Dynkin quivers with arbitrary orientation. Having done so, we will be able to prove the existence of minimal free resolutions and normality for orbit closures (admitting 1-step desingularization) of *all* Dynkin quivers.

The next step towards achieving a complete picture for all orbit closures arising from

Dynkin quivers would be to get rid of the condition of 1-step desingularization. For this we need to know how to calculate exterior powers of non-semisimple vector bundles. This is work in progress.

As mentioned earlier, Schofield's incidence varieties give rise to certain varieties in  $\operatorname{Rep}(Q,\underline{d})$  which are orbit closures under some additional conditions. Proposition 6.0.11 lists these conditions in the case of equioriented  $A_n$ . It is an interesting problem to find these conditions for other Dynkin quivers, namely, we would like to know: when are the varieties determined by  $Z(\beta, \gamma)$  orbit closures?

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