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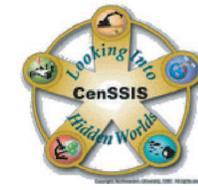
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# STABILIZATION OF CONSTRAINED OPTIMIZATION PROBLEMS FOR ELASTIC MODULUS RECONSTRUCTION

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## Abstract

Inverse problems are often formulated as constrained optimization problems. For example, one may wish to find the parameter distribution that provides the best match to the measured data. The Lagrange multiplier method is usually used in the formulation of such problems, resulting in a two-field variational formulation that is often called a mixed formulation. Typical straightforward discretizations of this problem, for example classical Galerkin finite element method, are frequently non-convergent and can cause severe problems. These include oscillations, locking, singular matrices, and spurious nonphysical solutions. This is the case even when precisely the same method for the forward problem gives optimally accurate discrete solutions and shows optimal convergence.

We show in the context of inverse elasticity how to address such difficulties by residual based stabilization. We formulated the inverse problem to reconstruct the elastic modulus distribution of soft tissue as a constrained minimization problem for which we seek the stationary point of a Lagrangian. We discretized the resulting equations using the Galerkin approximation of the finite element method. Stabilization was considered here by adding to the weak form a perturbation term based on the residual form of the Euler-Lagrange equation. In this way stability was enhanced without upsetting consistency. Preliminary results obtained when solving the inverse problem for various cases show that the stabilization terms employed in this formulation enable us to deal with the spurious solutions that appear in the absence of stabilization. The purpose of this study is to find the best stabilization terms in order to solve accurately the inverse problem.

## Introduction

Inverse problems are often formulated as optimization problems [3]: One seeks the parameter distribution which, when used in a forward model of the imaging problem, gives the best match possible to the measured data. That is, the “true” parameter distribution is thought to be that which optimally predicts the data. Different inverse problem solution strategies differ by how “best match” is defined mathematically, or by how the optimal solution is reached, but nearly all share the goal of optimizing the fit to the data.

In practice, the inverse problem is solved as follows:

1. One guesses a parameter distribution.
2. One solves the “forward problem” for this guessed parameter distribution. This forward solve requires a convergent, stable numerical method.
3. One compares the predicted field to the measured field. If they agree satisfactorily, the guessed parameter distribution is taken to be correct. If the agreement is unsatisfactory, the parameter distribution is updated and the cycle is repeated.

Step 2 requires the solution of the forward problem on a discrete mesh. One expects that as that mesh is refined, the solution approaches that of the continuous equations. This is true of a convergent numerical method. We’ve found that even if a stable, optimally convergent numerical method is used in step 2, *the discrete parameter distribution often does not converge to the exact solution with mesh refinement*. This turns out to be a generic feature of such constrained minimization problems, and thus applies equally well to elasticity imaging, diffuse optical tomography, electrical impedance tomography, or any inverse problem which is formulated as a constrained minimization problem.

Stabilization methods exist to yield convergent numerical formulations of constrained optimization problems. In this project, we are exploring the application of Multiscale (MS), Galerkin Least Squares (GLS), and similar stabilization strategies to inverse elasticity imaging problems. In this context, the elasticity imaging (EI) problem serves as both a practical and important inverse problem in its own right, and also as a model problem for other inverse problems with elliptic forward operators (e.g. DOT, EIT, time-harmonic acoustics, electromagnetics, etc.)

Stabilization methods have the objective of enhancing stability of the discrete formulation. These are to be contrasted with *regularization*, which has the objective of yielding a mathematically well-posed problem. Given a mathematically well-posed problem, one still needs a stable, convergent numerical method to solve it. Stabilization can be used to construct such a method.

The problem we are solving in this project arises in the field of elastography that is an emerging image modality that have several clinical applications. We are investigating the necessity and behavior of some stabilization strategies in our formulation.

In this poster we define a well-posed plane-stress EI problem to serve as a model. We demonstrate that standard discretization strategies (finite elements - FEM) applied to the inverse optimization problem yields a nonconvergent numerical method. This manifests itself in the appearance of spurious FEM

solutions that do not exist in the continuous case. We show that the addition of a GLS term stabilizes the numerical results in the examples considered.

## Background and Motivation

- Elasticity imaging or elastography is a technique which uses imaging modalities to measure and image relative displacements within tissue volumes. Its main goal is to map the elastic modulus (stiffness) of soft tissues. Tissue stiffness or elastic modulus distribution is believed to be clinically significant and may be correlated to tumor and other tissue pathologies[4, 1].

- Stabilization techniques as [2, 5] have been developed to overcome numerical problems that pollute the solutions when some appropriate stability conditions are violated. The main idea of these methods is to add products of suitable perturbation terms and the residuals, thereby maintaining consistency. The application of these approaches in our formulation enable us to overcome the difficulty of fulfilling rigorous mixed stability conditions, such as the Babuska-Brezzi or the K-ellipticity condition[2]. One computational advantage in considering stabilization is that we can employ equal-order interpolation function in our discretization that are easier to implement.

## State of the Art

- We continue to lead the field of elasticity imaging in terms of quantitative modulus reconstructions in 2D and 3D, mathematical analysis of the inverse problem, and extensions to hyperelasticity and multiphase phenomena.
- To our knowledge, nobody has studied the convergence with mesh refinement of numerical methods to solve inverse problems. This is completely fresh territory, within the field of elasticity imaging and beyond.

## Formulation of the Inverse problem

### Constrained minimization problem

We formulated the inverse problem by minimizing a functional subject to an elasticity constraint:

given  $\bar{u}(x)$ , find  $u(x)$ ,  $\mu(x)$  and  $\lambda(x)$  that minimizes:

$$\Pi(u, \mu, \lambda) = \frac{1}{2} \|u - \bar{u}\|^2 - a(\lambda, u; \mu)$$

Here,  $u \in \mathcal{S}$ ,  $\lambda \in \mathcal{V}$ ,  $\mu \in \mathcal{M}$  where:

$$\mathcal{S} = \{u | u|_{\Gamma} = \bar{u}|_{\Gamma}\}$$

$$\mathcal{V} = \{\lambda | \lambda|_{\Gamma} = 0\}$$

$$\mathcal{M} = \{\mu | \mu_C = \bar{\mu}\}$$

where  $\mathcal{C}$  is some manifold on which  $\mu$  is known. In this formulation  $\bar{u}$  is the measured displacement field and  $u$  is the predicted displacement field.  $\lambda$  is the Lagrange multiplier and  $\mu$  is the shear modulus.  $a(\lambda, u; \mu)$  is the weak form of the governing partial differential equation; it defines the constraint that the variables have to satisfy and depends on the assumed mathematical model.

## First Variationals

We take the variations of the functional  $\Pi$  with respect to each variable to calculate the first order necessary conditions:

$$D_u \Pi.v = (v, u - \bar{u}) - a(\lambda, v; \mu) = 0$$

$$D_\lambda \Pi.w = a(w, u; \mu) = 0$$

$$D_\mu \Pi.\gamma = a(\lambda, u; \gamma) = 0$$

where  $v, w$ , and  $\gamma$  are the variational of  $u, \lambda$  and  $\mu$  respectively. These equations represent a system of 3 non-linear PDEs.

## Least Squares Stabilization

In order to improve stability we add a stabilization term to our initial functional. So the new functional to minimize is:

$$\Pi_s(u, \mu, \lambda) = \frac{1}{2} \|u - \bar{u}\|^2 - a(\lambda, u; \mu) + \frac{\tau}{2} (L(u), L(u))_{\Omega'}$$

where  $(w, L(u)) = a(w, u; \mu)$  is the differential operator that depends on the mathematical model employed and  $\tau$  is the stabilization parameter that weights the added perturbation term.  $\Omega'$  is the union of finite element interiors. It is important to indicate that when we discretize our equations with the FEM this stability contribution is done only in the element interior, so problems with jumps and Dirac delta functions in the boundaries of the elements are avoided.

## Results

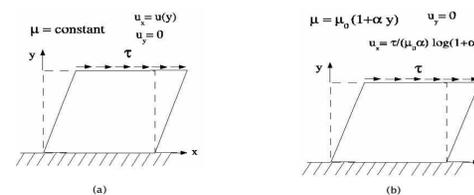


Figure 1: Pure shear stress Test  
(a) case 1: For an homogeneous  $\mu$  (b) case 2: For a linear distribution of  $\mu$

- Two computational experiments were tested in order to evaluate the formulation of the inverse problem. In the first one, we solved the forward problem using the FEM for an homogeneous and isotropic material considering only pure shear stress in the top of its domain as shown in Figure 1a. Then we used the calculated displacement field as the measured displacement to solve the inverse problem. In the second test, we solve analytically the elasticity equation considering a linear distribution of  $\mu$  as shown in Figure 1b. Then, we used the analytic solution of the displacement field as the measured data to solve the inverse problem. In both cases, the measured displacement can be considered perfect in the sense that they satisfy the elasticity equations. we considered a plane-stress incompressible material for the mathematical model.

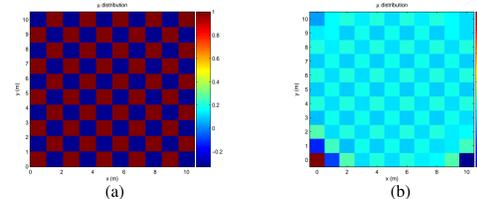


Figure 2: Shear modulus reconstruction without stabilization  
(a) For an homogeneous  $\mu$  (b) For a linear distribution of  $\mu$

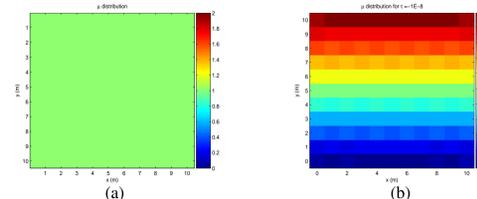


Figure 3: Shear Modulus reconstruction considering stabilization  
(a) For an homogeneous  $\mu$  (b) For a linear distribution of  $\mu$

- We discretized our equations using the Galerkin approximation of the FEM and then we applied the Newton method to solve the nonlinear system of equations that was generated after the discretization. In doing that, we employed bilinear shape functions for the three variables  $(u, \lambda, \mu)$ . This let us work with the stabilization term that contains first derivative for  $\mu$ .

- We solved the inverse problem for the two test problems first without stabilization and then considering stabilization. In both cases the initial guess for  $u$  was the exact answer. For  $\lambda$  and  $\mu$  the initial guess was a homogeneous distribution of a small number and ones respectively.

## Discussion

- The elastic modulus reconstructions shown in Figure 2a,b for the case 1 and 2, respectively, are the solutions for  $\mu$  when any stabilization term is included. We can see that these are not the right solutions. we call them spurious solutions because they also minimize our functional, but they do not have physical meaning and they do not satisfy the original continuous equations. These spurious solutions are excited usually when a bad initial guess is used to begin our iterative solution.

- Figure 3a,b show the elastic modulus reconstruction for the two cases when stabilization is considered. we can see that now the solutions are much better and stable with no spurious solutions. These solution were obtained assuming a homogeneous distribution for  $\mu$  as the initial guess.

- In both test problems case 1 and case 2, we got accurate solutions for the displacement fields and Lagrange multipliers. These solutions were independent of any stabilization term used. It means that the spurious solutions appeared only for  $\mu$ .

## Future Work

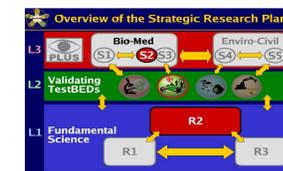
- We plan to analyze and compare some stabilization strategies (GLS, SUPG, etc) and try to prove convergence and stability analytically. Then to apply these stabilization techniques to the plane strain and 3D elasticity models.
- Further analysis will be done to determine a good approximation for the stabilization parameter  $\tau$  such that we guarantee optimal performance. also, we can work with experimental data to solve the inverse problem.

## Opportunities for technology transfer

- In the context of elastography, by imaging in vivo distributions of the biomechanical properties of soft tissues, we expect this research to enable clinical applications in the diagnosis and treatment of breast, prostate cancer and other soft tissue pathologies. In the context of computational methods, we will develop an accurate, efficient and stable method for solving the inverse elasticity problem.

## Acknowledgments

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Three-Level Diagram.

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