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Ratio Scales, Category Scales, and Variability in the Production of Loudness and Softness

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Several studies have shown that category scales are nonlinearly related to ratio scales of subjective magnitude. A variability model has been proposed previously to account for this departure from linearity. This article examines the model in the light of the empirical relations that enter into it: the ratio scale of subjective magnitude, the corresponding category scale, and the variability of judgments in both physical and psychological units. These relations are determined, through repeated measurement with a single observer, for the psychological continuum, loudness, and its inverse, softness. The ratio scales are shown to be reciprocals, and the category scales complements. The category scale of softness is more concave downward, relative to its magnitude scale, than is the category scale of loudness. This outcome is also derived mathematically from the empirical equations relating the four scales to physical magnitude. Variability is found to increase with increasing stimulus magnitude at the same rate for both loudness and softness productions, expressed either in physical units or in psychological units. Hence, the variability model is found not to accord with the observed difference in concavity between softness and loudness category scales relative to their respective psychological magnitude scales.

INTRODUCTION

A NUMBER of psychophysical studies have shown that category scales are nonlinearly related to ratio scales of subjective magnitude on prothetic continua.[1] Stevens and Guirao[2] provide the most recent account of this observed departure from linearity in terms of a variability model. This model starts from the assumption that ratio-scaling procedures (magnitude estimation, magnitude production, and crossmodality matching) provide valid measures of the psychological continuum, and then seeks to explain the nonlinear relation between category and ratio scales in terms of the variability associated with judgments along the continuum. According to Stevens and Guirao, variability is not constant along the psychological continuum, but increases with increasing magnitude. Consequently, when an observer attempts to assign stimuli to categories equally spaced in psychological units, he makes fewer errors at low psychological magnitudes, where he can easily discriminate two points that are separated by a given distance, and he makes more errors at higher magnitudes, where two points, although separated by the same distance, are not so easily discriminated and, hence, are assigned more often to the same category. The graphic outcome is that the category scale appears concave downward when plotted against the magnitude scale.[3]

This article examines the variability model in the light of an empirical example of the relations that enter into it: the ratio scale of subjective magnitude, the corresponding category scale, and the variability of judgments in both physical and psychological units. These relations are determined, through repeated measurement with a single observer, for the psychological continuum, loudness, and its inverse, softness. The fit of the variability model to findings provided by the technique of inverse scaling is assessed.

[1] J. C. Stevens and J. D. Mack, "Scales of Apparent Force," J. Exptl. Psychol. 58, 405 (1959); S. S. Stevens and E. H. Gallanter, "Ratio Scales and Category Scales for a Dozen Perceptual Continua," *ibid.* 54, 377 (1957); S. S. Stevens and G. Stone, "Finger Span: Ratio Scale, Category Scale, and jnd Scale," *ibid.* 57, 91 (1959); W. S. Torgerson, "Quantitative Judgment Scales," in *Psychological Scaling: Theory and Practice* (John Wiley & Sons, Inc., New York, 1960), pp. 21–31.

[2] S. S. Stevens and M. Guirao, "Loudness, Reciprocality, and Partition Scales." J. Acoust. Soc. Am. 34, 1466 (1962).

[3] Stevens and Gallanter, Ref. 1.

METHOD

Procedure

Our subject, who began as an untrained observer, gave judgments of sensory magnitude according to four procedures: (1) magnitude production of loudness, (2) category production of loudness, (3) magnitude production of softness, and (4) category production of softness. The corresponding sets of instructions, read to the subject before each session, were as follows (the words in the parentheses were used when appropriate):

Magnitude production. "This is an experiment to see how you perceive the loudness (softness) of sounds. When the experiment begins, I will present a standard sound to you. Call the loudness (softness) of this sound '10.' I will then present a series of numbers, one at a time. Your task will be to adjust the noise so that its loudness (softness) stands in the same proportion to the loudness (softness) of the standard noise as the number I have given you stands to 10. For example, if I give you the number 20, you should adjust the noise so that it is twice as loud (soft) as the standard. If I give you the number 3.3, you should adjust the noise so that it is one-third as loud (soft) as the standard. Always adjust the noise so that its loudness (softness) stands in the same proportion to the standard as the number I have given you stands to 10."

Category production. "In this part of the experiment we are again interested in seeing how you perceive the loudness (softness) of sounds. Before the experiment begins, I will present two noises in succession over the earphones. The loudness (softness) of the first will be called 'one,' and the loudness (softness) of the second will be called 'five.' I will then present a series of numbers from one to five in irregular order. You should regard these numbers as marking off equal distances in loudness (softness). Your task is to adjust the noise so that its loudness (softness) corresponds to the number I have given you."

For magnitude production, the standard loudness had a sound pressure of 1.78 μbar (79 dB, sound-pressure level); it was presented and identified at the start of the stimulus series and again after every 25th determination. One hundred forty determinations were obtained for each criterion value (2.5, 5, 10, 20, 40) in irregular order; they were collected in five sessions, lasting approximately two hours each.

For category production, the sound pressures defining the end categories were the geometric means of the sound pressures assigned to the lowest and highest values in the corresponding magnitude-production experiments (0.31 and 17.6 μbar for loudness; 10.5 and 0.275 μbar for softness). These levels were also presented and identified at the outset and after every 25th determination. Four sessions were employed.

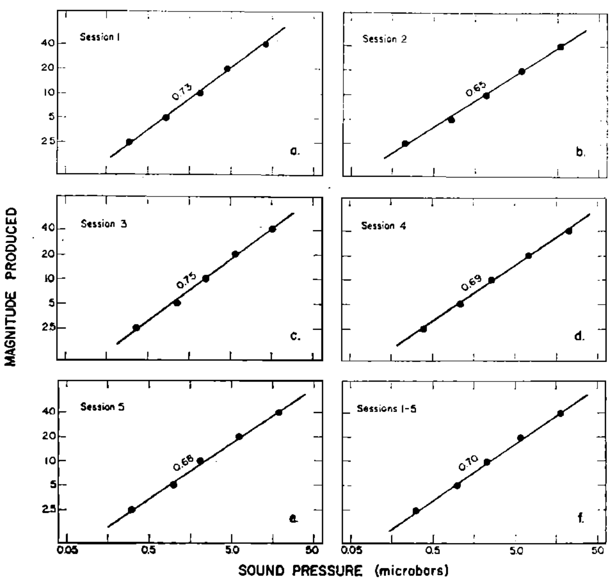

FIG. 1. Magnitude productions of the loudness of broad-band noise for a single observer. Each point in the five sessions (a)–(e) is the geometric mean of 25–40 determinations. The final figure (f) is a composite of the five sessions; each point is the geometric mean of 140 determinations. The slopes of the straight lines were determined by the method of least squares.

Apparatus

The subject was seated in a sound-treated room; he wore a binaural headset with calibrated TDH-39 earphones, matched in frequency response and mounted in sponge-neoprene cushions. The stimuli were produced by a noise generator (Grason–Stadler 901A) whose spectrum was flat up to 1000 cps and fell at 12 dB/octave thereafter. The signal was sent to a linear potentiometer, adjusted by the subject. In order to minimize bias in these productions arising from the position of the potentiometer on prior trials, an attenuator (Hewlett–Packard 350D) was interposed between the potentiometer output and the earphones and the experimenter varied the starting level of the noise from trial to trial. Furthermore, the subject was required to set the potentiometer to the zero position after each trial. The subject could adjust the level of the noise for the appropriate magnitude or category production over the following ranges: 0–75.4, 0–30.4, 0–8.1, 0–3.4, and 0–1.7 μbar. After each production, the subject pressed a button that terminated the noise. The voltage produced at the earphones was measured by a vacuum-tube voltmeter (Hewlett–Packard 400D) and later converted to sound pressure by means of earphone calibrations.

RESULTS AND DISCUSSION

Magnitude Scales

Figure 1 shows the geometric means of the sound pressures produced to each of the criterion values for magnitude production of loudness. The determinations in sessions 1 through 5 are well-fit in each case by a

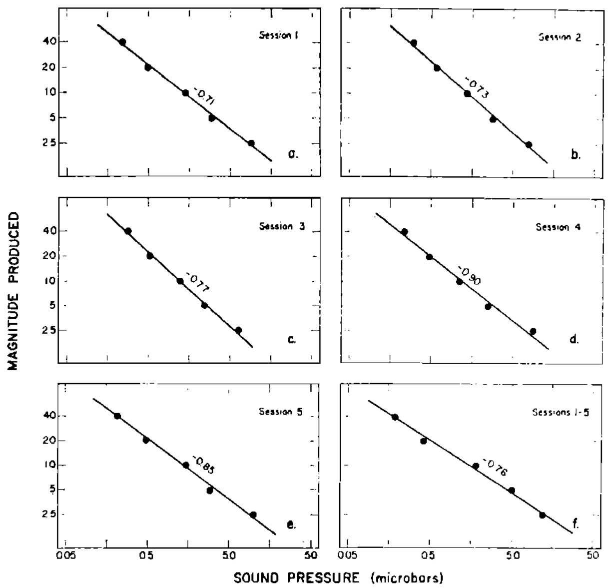

FIG. 2. Magnitude productions of the softness of white noise for a single observer. Each point in the five sessions (a)–(e) is the geometric mean of 25–40 determinations. The fifth figure (f) is a composite of the four sessions; each point is the geometric mean of 140 determinations.

straight line in log–log coordinates, in other words, a power function. The slope (exponent) of this function, determined by the method of least squares, ranges from 0.65 to 0.75. The value of the slope for sessions 1–5 combined, 0.70, agrees with that obtained by Stevens and Guirao[2] for two determinations on each of ten subjects. There is no evidence in the present study of a systematic change in the slope or intercept of the loudness function over the course of the five sessions.

Figure 2 shows that magnitude production of softness is also well-described by a power law. The range of exponents (slopes) of this function is, however, greater than that for loudness: −0.71 to −0.90. The softness function becomes somewhat steeper in the first four sessions, but this trend is reversed in session 5. Similarly, the intercept increases in sessions 1–4, reverts in session 5. For the combined sessions, the softness function [Fig. 2(f)], with exponent −0.76, is very nearly the reciprocal of the loudness function [Fig. 1(f)], with exponent +0.70.

Category Scales

Category production scales have been shown to fall somewhere between a logarithmic function and a power function of stimulus intensity.[4] This finding is confirmed by the results of category production of loudness and softness, shown in Figs. 3 and 4, respectively. For loudness, the curves are slightly concave upward; for softness, the curves are slightly concave downward; in either case, the category scale does not describe a straight line in the semilog coordinates.

Equations of the following form describe the obtained category scales reasonably well:

$$G_L = b_1 \log_{10}(\phi + c) + a_1,$$

$$G_S = b_2 \log_{10}(\phi + c) + a_2,$$

where G_L is the category loudness, G_S is the category softness, ϕ is the sound pressure (μbar), c a transformation constant, and a_1, a_2, b_1, b_2 are empirical constants. When these functions were fit to the mean loudness- and softness-category productions by the method of least squares (with $c = 0.4427\ \mu$bar in both cases), the slope constants (b_1, b_2) turned out to be 3.1 and −3.8, respectively. In other words, the category scale for loudness is approximately the complement of the category scale for softness. The fit of the equations to the obtained points (plotted in log ϕ coordinates) is shown for sessions 1–4 combined in Fig. 3(e) (loudness) and Fig. 4(e) (softness). These equations also describe the category scales of loudness and softness of a 1000-cps tone, obtained by Stevens and Guirao[2] with a group of observers (Fig. 5). The curves' fit to their data was obtained in the manner described above, after adding

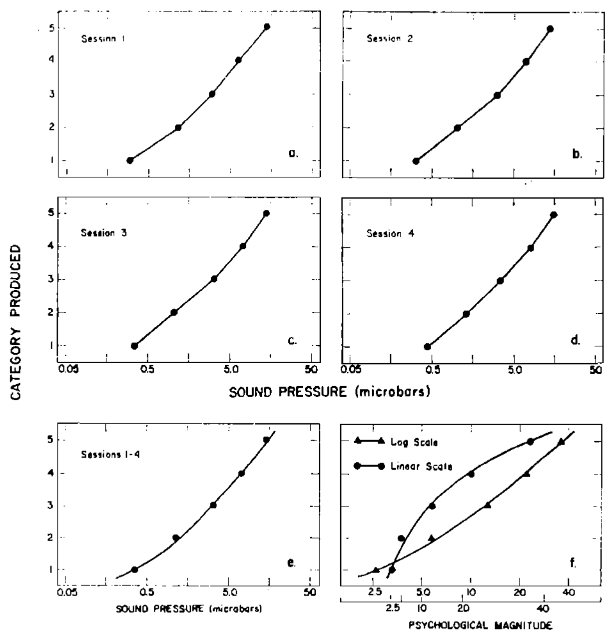

FIG. 3. Category productions of the loudness of white noise for a single observer. Each point in the four sessions (a)–(d) is the geometric mean of 25–45 determinations. The fifth figure (e) is a composite of the four sessions; each point is the geometric mean of 140 determinations. The smooth curve is the function of best fit to the obtained data. The final figure (f) presents the category scale of loudness as a function of the magnitude scale (circles) and the logarithm of the magnitude scale (triangles). The smooth curves were determined from the functions relating the category and magnitude scales to sound pressure (see text).

[4] S. S. Stevens, "On the New Psychophysics," Scand. J. Psychol. 1, 27 (1960).

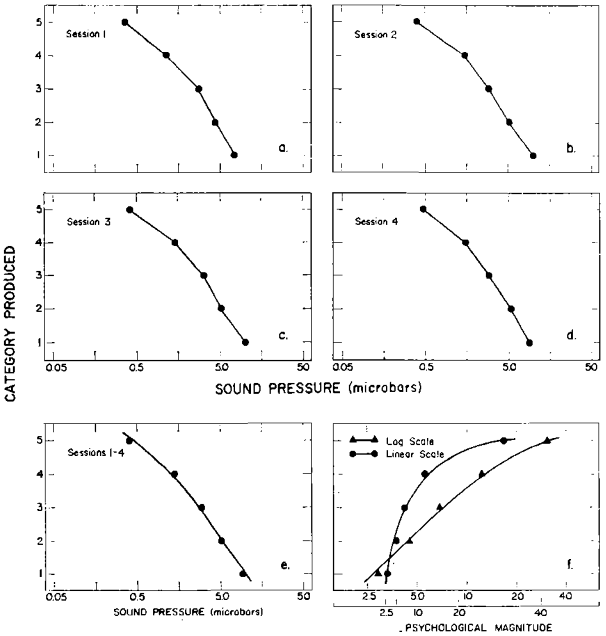

FIG. 4. Category production of the softness of white noise for a single observer. Each point in the four sessions (a)–(d) is the geometric mean of 25–45 determinations. The fifth figure (e) is a composite of the four sessions; each point is the geometric mean of 140 determinations. The smooth curve is the function of best fit to the obtained data. The final figure (f) presents the category scale of softness as a function of the magnitude scale (circles) and the logarithm of the magnitude scale (triangles). The smooth curves were determined from the functions relating the category and magnitude scales to sound pressure (see text).

the constant $c=0.09$ to each of the mean sound pressures. The two functions are nearly complementary; the slope for loudness equals 4.2 and the slope for softness, -3.8.

The findings discussed so far provide the following description of ratio scales and category scales for loudness and softness. Magnitude productions of loudness and softness yield reciprocal scales that grow as a power function of sound pressure (ϕ). Category production of loudness and softness yield complementary scales that grow as a logarithmic function of sound pressure plus a constant ($\phi+c$). In mathematical form and with the parameters obtained in this experiment,

$$M_L=5.5\phi^{0.70},$$
$$M_S=15.8\phi^{-0.76},$$
$$G_L=3.1\log(\phi+0.4427)+1.3,$$
$$G_S=-3.8\log(\phi+0.4427)+4.9,$$

where M_L is the loudness magnitude and M_S is the softness magnitude.

Category Scales vs Magnitude Scales

When the category scale is plotted as a function of the magnitude scale for loudness [Fig. 3(f)] and for softness [Fig. 4(f)], a curious result is obtained. In linear coordinates, both scales are concave downward; however, in semilog coordinates, the scale for loudness is concave upward, that for softness concave downward. It follows that the degree of concavity, in linear coordinates, is greater for the softness function than it is for the loudness function. The logarithmic transformation on the abscissa overcorrects the downward concavity of the loudness scale and leaves the function slightly concave upward. This same transformation on the softness scale does not straighten the function, which still remains slightly concave downward.

The differing concavities of the softness and loudness category scales, relative to their corresponding ratio scales, turn out to be a necessary consequence of (may be derived mathematically from) the empirical form of these four scales, which were described in the previous section. Using the equations stated earlier, we may derive an expression for the category scale as a function of psychological magnitude. Specifically:

$$M_L=k_1\phi^n,$$
$$\log M_L=n\log\phi+\log k_1,$$
$$\log\phi=(1/n)\log(M_L/k_1),$$

and $$\phi=(M_L/k_1)^{1/n}.$$

Similarly, $$\phi=(M_S/k_2)^{-1/n}.$$

It was shown earlier that $G_L=b\log(\phi+c)+a_1$, and $G_S=-b\log(\phi+c)+a_2$. Let $k_1'=(1/k_1)^{1/n}$, $k_2'=(1/k_2)^{-1/n}$. Substituting for ϕ in G_L, G_S: $G_L=b\log\times(k_1'M^{1/n}+c)+a_1$ and $G_S=-b\log(k_2'M^{-1/n}+c)+a_2$. In terms of the parameters of the category and ratio scales shown earlier,

$$G_L=3.1\log(0.11M^{1/0.70}+0.4427)+1.3;$$
$$G_S=3.8\log(38.29M^{-1/0.76}+0.4427)+4.88.$$

These functions, relating category to ratio scales of loudness and softness, are shown in Figs. 3(f) and 4(f),

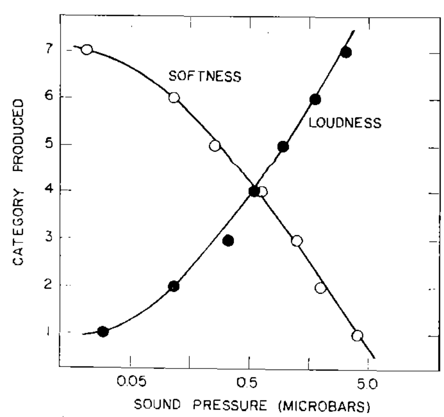

FIG. 5. Category production of loudness (filled circles) and softness (unfilled circles) of a tone of 1000 cps. Each point is the geometric mean of 2 settings by each of 10 observers. The smooth curves are the functions of best fit to the obtained data. [After Stevens and Guirao.[2]]

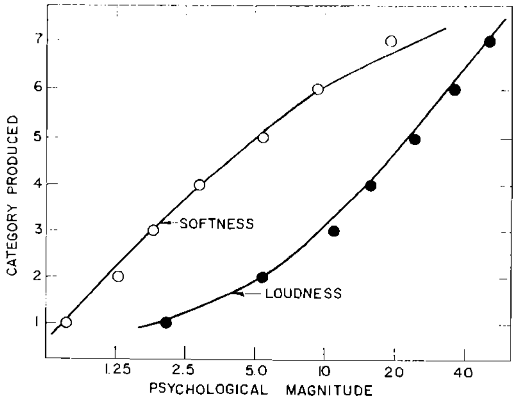

FIG. 6. Category scales of loudness (filled circles) and softness (unfilled circles) as a function of psychological magnitude. The smooth curves were determined from the functions relating the category and magnitude scales to sound pressure (see text). [After Stevens & Guirao.[2]]

along with the obtained data points. It can be shown (Appendix A) that G_L is always concave upward and G_S concave downward (for $c>0$) in logarithmic coordinates. In linear coordinates, G_S must be more concave downward than G_L; that is, the rate at which its sense of concavity changes must be greater (Appendix B).

(In the special case when $c=0$, the category scales are logarithmic functions of the corresponding physical continua. Although this is not a likely empirical finding, we should note that, in this case,

$$G_L = b\log(M^{1/n}) + a_1 = b/n \log M + a_1,$$
$$G_S = -b\log(M^{-1/n}) + a_2 = b/n \log M + a_2,$$

and the functions do not differ in concavity.)

When the expressions relating category and magnitude scales of softness and loudness are applied to the group data collected by Stevens and Guirao (Fig. 6), we find an equally good fit and confirmation of the conclusion that the concavities of these two functions necessarily differ.

Eisler[5] has reported findings that appear, at first, to contradict this conclusion. In three series of experiments in which subjects gave magnitude and category estimates of the loudness and softness of noise, the functions relating category scales to log magnitude were found to be concave upward for *both* loudness and softness. However, an examination of the functions relating psychological magnitude (magnitude estimation) to sound pressure (Fig. 7) reveals a rather curious anomaly. Although a power function describes the growth of estimates of loudness, it is evident that a reciprocal function does not describe the growth of softness. This outcome may be contrasted with the reciprocal power functions found in the present experiment (Figs. 1 and 2) and by Stevens and Guirao. In fact, the estimates of softness collected by Eisler do not follow a power law at all. It is not surprising, therefore, that the predicted difference in concavity did not obtain in his experiment. Whenever the ratio scales of loudness and softness are reciprocal power functions and the category scales are complementary logarithmic functions (of $\phi+c$), the functions relating category to magnitude scales will differ in concavity.

[5] H. Eisler, "Empirical Test of a Model Relating Magnitude and Category Scales," Scand. J. Psychol. 3, 88 (1962).

Variability Model

The variability model relates the degree of concavity of the category scale (when plotted against psychological magnitude) to the increase in variability with increasing psychological magnitude. The arguments presented above show that it is necessarily true that this concavity is greater for softness than it is for loudness. If the variability hypothesis is to be in accord with the necessary outcome, then one or both of two conditions must exist: (1) psychological variability is greater for softness than it is for loudness; (2) psychological variability grows at a faster rate for softness than for loudness. In this section, we examine variability in the magnitude and category production of loudness and of softness.

Variability in Physical Units

Figure 8 presents the distribution of magnitude or category productions (in physical units) observed at each criterion value in the scaling of both loudness and softness. The 550 determinations under each procedure are shown; the data for the initial sessions were omitted in order to exclude atypical measures of variability. The distributions are slightly skewed toward higher sound pressures in all four experimental conditions. Variability increases with increasing sound pressure for both loudness and softness productions. Since loudness also increases as a function of stimulus magnitude, this means that the variability of loudness productions,

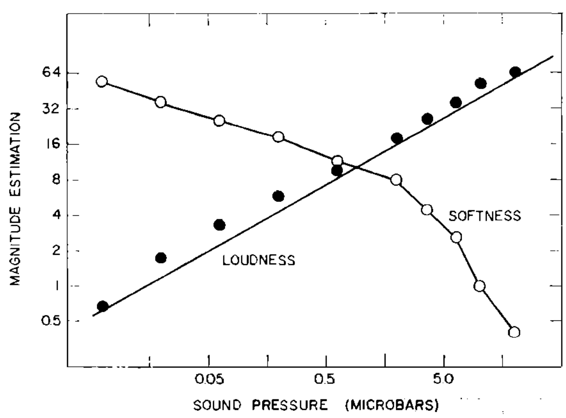

FIG. 7. Magnitude estimation of loudness (filled circles) and softness (unfilled circles) of white noise. Each point is the geometric mean of 4 estimations by each of 12 observers. [After Eisler.[5]]

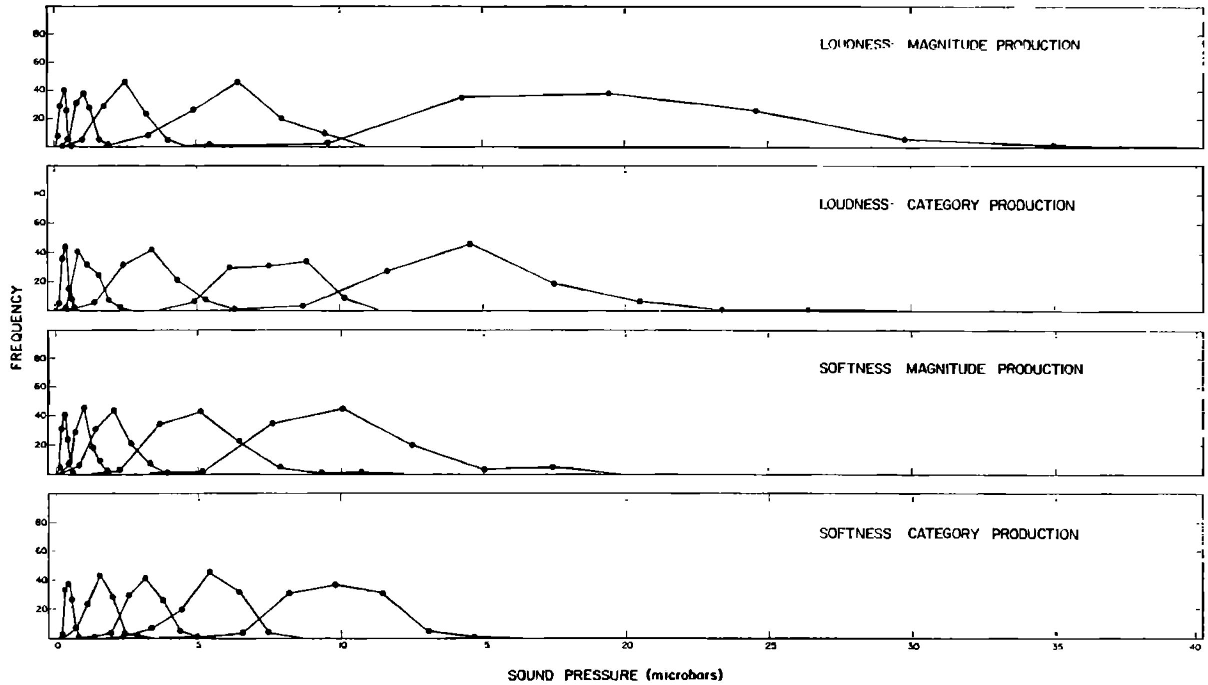

FIG. 8. Variability, in physical units, for magnitude and category productions of loudness and softness. Each distribution is based on 110 productions. The midpoints of adjacent class intervals are separated by one standard deviation.

measured in physical units, increases as a function of loudness. However, softness *decreases* as a function of sound pressure, while the variability of softness productions, measured in physical units, increases as a function of sound pressure. Hence, the variability of softness productions, *measured in physical units*, decreases as a function of softness. We cannot, then, attribute the difference in the degree of concavity of the loudness and softness scales [Figs. 3(f), 4(f)] to the way in which variability, measured in physical units, grows as a function of psychological magnitude—in one case, it increases; in the other, it decreases.

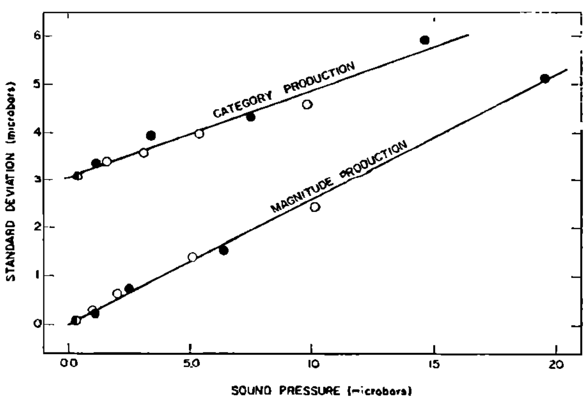

FIG. 9. The standard deviation (physical units) of magnitude and category productions of loudness (filled circles) and softness (unfilled circles) as a function of sound pressure. The straight lines were determined by the method of least squares. For clarity, the category production function has been raised three units on the *y*-axis.

Furthermore, the rate at which variability changes as a function of physical magnitude is the same for both the loudness and softness scales. Figure 9 shows the standard deviations of the distributions for loudness (filled circles) and softness (unfilled circles) plotted against the physical magnitude of their means. It will be seen that the standard deviation is approximately the same linear function of sound pressure for both the loudness and softness scales, obtained by category or by magnitude production. Since variability grows, in physical units, at the same rate for both loudness and softness, we may conclude that, on this second count as well, the findings for variability in physical units do not explain the different concavities of the loudness and softness category scales relative to their magnitude scales.

Psychological Variability

Figure 10 presents the "psychological variability" associated with the production of a particular magnitude or category. To obtain these distributions, each production was transformed into psychological units based on the reciprocal scales obtained from the magnitude-production experiments. A best estimate of the absolute value of the exponent for the softness and loudness scales may be 10.73, which is intermediate between the observed exponents of 10.70 (loudness) and 10.76 (softness). Accordingly, each loudness production was transformed to psychological units by $M_L = 5.5\phi^{0.73}$ and each softness production by $M_S = 15.8\phi^{-0.73}$. (The initial sessions again were omitted.)

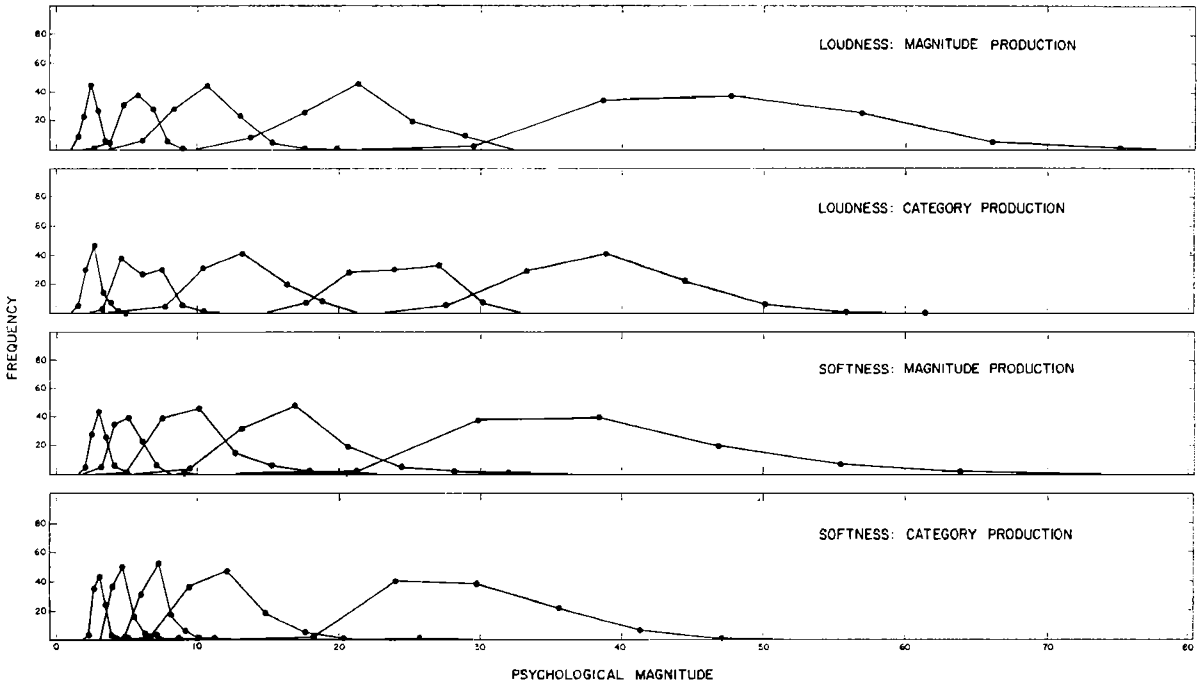

FIG. 10. Variability, in psychological units, for magnitude and category production of loudness and softness. Each distribution is based on 110 productions. The midpoints of adjacent class intervals are separated by one standard deviation.

When variability is expressed in psychological units, it increases with increasing magnitude for both loudness and softness. The frequency distributions are, once again, slightly skewed toward higher intensities. If the variability hypothesis is to account for the differing concavities of the loudness and softness scales, it is necessary that variability be either greater or grow more rapidly for softness than for loudness. Figure 11 presents the standard deviation of magnitude productions in psychological units as a function of the mean psychological magnitude for both scales. When straight lines were fit to the loudness and softness points separately (by the method of least squares), a small difference in the slopes of the lines, 0.03, suggested that indeed the variability of softness productions grew more rapidly, however slightly, than the variability of loudness productions. This inference was not borne out by an F test of the ratio of the variances at several corresponding points along the least-squares lines. In fact, the line of best fit to the softness data at no point fell within the region that permitted rejection ($\alpha=0.01$) of the null hypothesis that the softness and loudness variances were sampled from the same population. Since the variability of loudness and of softness productions grows at approximately the same rate as a function of psychological magnitude, the variability model fails to account for the difference in concavity between the loudness and softness category scales relative to their magnitude scales.

This conclusion does more than curtail slightly the explanatory power of the variability model as it applies to the relations among psychophysical scales. The observed difference in concavity of the loudness and softness category scales, relative to their magnitude scales, is not an epiphenomenon; it is deduced mathematically from the "basic" psychophysical scales: magnitude and category scales of loudness and softness as a function of sound pressure. Predictions from the variability model did not accord with this deduction (and the obtained data) in this experiment. Therefore, the variability model was found to be incompatible with the basic psychophysical functions, to whose relations that it applies.

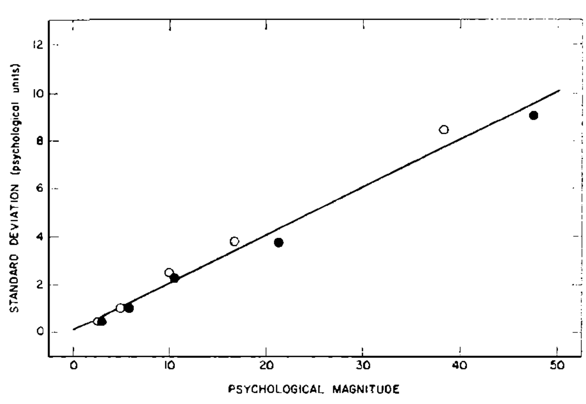

FIG. 11. The standard deviation (psychological units) of magnitude productions of loudness (filled circles) and softness (unfilled circles) as a function of psychological magnitude. The straight line was determined by the method of least squares.

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APPENDIX A. DIRECTION OF CONCAVITY OF THE CATEGORY SCALES

The category scales of loudness and softness are related to their respective magnitude scales by the equations

$$G_L=f_L(M_L)=b\log(k_1'M_L^{1/n}+c)+a_1,$$
$$G_S=f_S(M_S)=-b\log(k_2'M_S^{-1/n}+c)+a_2,$$

where G_L is the category scale—loudness; G_S the category scale—softness; M is the psychological magnitude, and a_1, a_2, b, n, k_1', k_2', c are constants with b, n, k_1', k_2', $c>0$.

In order to test these functions for their sense of concavity in logarithmic coordinates, it is necessary to express G_L, G_S as functions g_L, g_S of $\log M$. Let $z=\log M$; then, $M=e^z$. Substituting e^z for M in G_L, G_S,

$$G_L=g_L(z)=b\log(k_1'e^{z/n}+c)+a_1;$$
$$G_S=g_S(z)=-b\log(k_2'e^{-z/n}+c)+a_2.$$

We test the sense of concavity of these functions by examining the sign of their second derivatives.

The second derivative of G_L is

$$\frac{d^2G_L}{dz^2}=\frac{(b)(1/n)^2(k_1')(c)(\log_{10}e)e^{z/n}}{(k_1'e^{z/n}+c)^2}.$$

Since $e^{z/n}$ is always positive and $\log_{10}e$, n, b, c, and k_1' are positive, the second derivative is positive and the category scale of loudness is concave upward when plotted against the log of psychological magnitude.

The second derivative of G_S is

$$\frac{d^2G_S}{dz^2}=\frac{-(b)(1/n)^2(k_2')(c)(\log_{10}e)e^{-z/n}}{(k_2'e^{-z/n}+c)^2}.$$

Since $e^{-z/n}$ is always positive, and $\log_{10}e$, n, b, c, and k_2' are positive, d^2G_S/dz^2 is negative; therefore, the category scale of softness is concave downward when plotted against the log of psychological magnitude.

In the special case where $c=0$, the second derivatives of G_L, G_S are zero, and G_L, G_S are linear functions of the log of psychological magnitude.

APPENDIX B. DEGREE OF CONCAVITY OF THE CATEGORY SCALE

The category scales of loudness and softness as a function of psychological magnitude [Figs. 3(f), 4(f)] are described by the following equations:

$$G_L=b\log(k_1'M^{1/n}+c)+a_1;$$
$$G_S=-b\log(k_2'M^{-1/n}+c)+a_2.$$

When G_L and G_S are evaluated with respect to M (linear coordinates), the second derivative of G_S is negative over the entire range of psychological values and that of G_L is also negative, except for very small values of M (not considered here).

To determine which of these functions has the greater concavity, we solve for the third derivative of each function, giving the rate of change of concavity. If

$$\frac{d^3G_S}{dM^3}>\frac{d^3G_L}{dM^3},\quad (c>0),$$

then the softness function has the greater degree of concavity; i.e., it will appear to curve downward more rapidly than the loudness function. The third derivatives of G_L and G_S are

$$\frac{d^3G_L}{dM^3}=\frac{b(\log_{10}e)(1/n)(k_1')M^{1/n}[2(k_1')^2M^{2/n}-c(1/n+4)(1/n-1)k_1'M^{1/n}+c^2(1/n-1)(1/n-2)]}{M^3(k_1'M^{1/n}+c)^3};$$

$$\frac{d^3G_S}{dM^3}=\frac{-b(\log_{10}e)(-1/n)(k_2')M^{-1/n}[2(k_2')^2M^{-2/n}-c(-1/n+4)(-1/n-1)k_2'M^{-1/n}+c^2(1/n+1)(1/n+2)]}{M^3(k_2'M^{-1/n}+c)^3}.$$

Let us assume that the converse holds; that is, that

$$\frac{d^3G_L}{dM^3}\geq\frac{d^3G_S}{dM^3},\quad (c>0).$$

Multiplying both terms of the inequality by $M^3/b(\log_{10}e)(1/n)$, we have

$$\frac{(k_1')M^{1/n}[2(k_1')^2M^{2/n}-c(1/n+4)(1/n-1)k_1'M^{1/n}+c^2(1/n-1)(1/n-2)]}{(k_1'M^{1/n}+c)^3}$$
$$\geq\frac{(k_2')M^{-1/n}[2(k_2')^2M^{-2/n}-c(-1/n+4)(-1/n-1)k_2'M^{-1/n}+c^2(1/n+1)(1/n+2)]}{(k_2'M^{-1/n}+c)^3}.$$

The variable M, psychological magnitude, in the above inequality is related to physical magnitude ϕ by

$$\phi = k_1' M^{1/n} \text{ and } \phi = k_2' M^{-1/n}.$$

Making these substitutions in the above inequality, we have

$$\frac{\phi[2\phi^2 - c(1/n+4)(1/n-1)\phi + c^2(1/n-1)(1/n-2)]}{(\phi+c)^3} \geq \frac{\phi[2\phi^2 - c(-1/n+4)(-1/n-1)\phi + c^2(1/n+1)(1/n+2)]}{(\phi+c)^3}.$$

Multiplying both terms by $(\phi+c)^3/\phi$ and expanding the parentheses, we have

$$2\phi^2 - c\phi/n^2 - 3c\phi/n + 4c\phi + c^2/n^2 - 3c^2/n + 2c^2 \geq \times 2\phi^2 - c\phi/n^2 + 3c\phi/n + 4c\phi + c^2/n^2 + 3c^2/n + 2c^2.$$

Cancelling like terms, the inequality reduces to

$$-3c\phi/n - 3c^2/n \geq 3c\phi/n + 3c^2/n.$$

Multiplying both terms by $n/3c$ $(c>0)$, we have

$$-(\phi+c) \geq (\phi+c).$$

But this is a contradiction, since $c>0$, and ϕ, a physical magnitude, can never be less than zero. It follows that the converse is true: namely, that

$$\frac{d^3G_S}{dM^3} > \frac{d^3G_L}{dM^3}, \quad (c>0).$$

Therefore, the category scale of softness is necessarily more concave downward than the category scale of loudness. Only in the case where $c=0$ are the degrees of concavities the same. For then,

$$\frac{d^3G_L}{dM^3} = \frac{d^3G_S}{dM^3} = \frac{2(b)(\log_{10}e)(1/n)}{M^3}.$$