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Standard bases for coordinate rings of cotangent varieties

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STANDARD BASES
FOR
COORDINATE RINGS OF COTANGENT VARIETIES

A dissertation presented

by

Marcus Lloyd Fries

to

The Department of Mathematics

In partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in the field of

Mathematics

Northeastern University

Boston, Massachusetts

April, 2010

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FOR
COORDINATE RINGS OF COTANGENT VARIETIES

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Marcus Lloyd Fries

ABSTRACT OF DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate School of Arts and Sciences of
Northeastern University, April, 2010

Abstract

We extend the results of the Standard Monomial Theory to the cotangent variety of the Grassmannian, $T^*\mathrm{Grass}(k,n)$. We exhibit a standard basis in terms of triples of tableaux. We are also able to show that $T^*\mathrm{Grass}(k,n)$ is arithmetically Cohen-Macaulay and normal in all characteristics. Results on Cohen-Macaulayness and normality are shown for other algebraic groups and cominuscule parabolic subgroups, but only in characteristic zero.

Acknowledgment

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Introduction

Let G be a reductive algebraic group and P a parabolic subgroup. The coordinate rings of G/P and their Schubert varieties are well understood through the Standard Monomial Theory. Well known results include the aritmetic Cohen-Macaulayness and arithemetic normality.

The goal of this paper is to extend these results to the coordinate rings of cones over the cotangent varieties of G/P, $B(G,P) = K[\widehat{T^*G/P}]$. We achieve success when P is a cominscule parabolic. Our approach is based on Grosshans deformation theory, see [**7**], the Kempf Vanishing theorem, and good filtrations, see [**8**]. The proof of Cohen-Macaulayness for $B(G,P)$ still depends on characteristic zero because we use the notion of rational singularities. For type A_n and D_n we prove Cohen-Macaulayness by using results of [**6**] and [**1**].

In the case A_n we give an explicit standard basis and the defining ideal in terms of stacked tableaux.

In [**12**], E. Strickland exhibits an explicit basis and the defining ideals for the conormal varieties to the determenantal varieties. We extend these results to the conormal varieties of symmetric and skew-symmetric matrices. In the case of conormals to rank varieties of skew-symmetric matrices we are able to show that the coordinate ring is Cohen-Macaulay.

This paper is written in three major parts: A short introduction to representations of reductive algebraic groups; Construction of standard bases for the conormal

varieties to rank varieties of symmetric and skew-symmetric matrices; Coordinate rings for the cone over the cotangent variety to G/P for P a cominuscule parabolic. We now give a brief description of the results in each of these three chapters.

1. G varieties and representations

In the second section of this chapter, Schur Functors, we give a presentation of modules for $GL(K^n)$ containing a unique irreducible module. We also give explicit decompositions of the modules $Sym(\bigwedge^2 K^n)$ and $Sym(S_2K^n)$ in terms of pfaffians and minors, respectively.

The third section is focused on the analogue to the Schur functors for arbitrary reductive groups. Let G be a reductive algebraic group with Borel subgroup B and maximal torus T. To each character λ of T we associate a line bundle $\mathcal{L}(\lambda)$ on G/B. The cohomology of this line bundle $H^0(G/B; \mathcal{L}(\lambda))$ is a module containing a unique irreducible G-module.

The last section, Deformation of G Varieties, is concerned with constructing flat deformations of G-modules. Given a G-module A with a G invariant filtration $A_0 \subset A_1 \subset \cdots \subset A$ we build the module $D(A) = \oplus_n A_n x^n$. It is shown in [**7**] that $D(A)$ is a flat deformation of the G-module A with special fiber $gr(A)$.

2. Conormal Varieties to Rank Varieties

In the second chapter we give explicit standard bases to conormal varieties to the rank varieties of symmetric and skew-symmetric matrices. For both cases we exhibit a standard basis in terms of stacked tableaux. In the case of skew-symmetric symmetric matrices we show that the poset of monomials is wonderful.

3. Cominuscule G/P

Let $B(G, P)$ be as above. We show in section 1 the existence of a good filtration for $B(G, P)$. For the cases E_6/P_{ω_6} and E_7/P_{ω_7} we are able to prove a good filtration only in characteristic zero. For A_n and D_n the existence of a good filtration for $B(G, P)$ is characteristic free.

We the use the existence of a good filtration and the results of chapter 1 section 4 to show Cohen-Macaulayness and normality in the above cases.

In the A_n case we are able to give a standard basis in terms of stacked triples of tableaux.

CHAPTER 1

G varieties and representations

1. The Grassmannian

Let K be an algebraically closed field of arbitrary characteristic. Let $\mathrm{Grass}(k, K^n)$ be the set of all k-dimensional subspaces of K^n.

1.1. The Plücker Embedding.

Definition 1.1. The Plücker embedding $\pi\colon \mathrm{Grass}(k, K^n) \hookrightarrow \mathbb{P}(\bigwedge^k K^n)$ is given by sending the k-subspace V to the tensor $v_1 \wedge \ldots \wedge v_k$ in $\bigwedge^k K^n$, where $\{v_1, \ldots, v_k\}$ is any basis of V.

Let us define a set of coordinates on $\mathbb{P}(\bigwedge^k K^n)$ called Plücker coordinates indexed by the set

$$I_{k,n} = \{\{s_1, \ldots, s_k\} \mid 1 \leq s_1 < s_2 < \cdots < s_k \leq n\}. \tag{1.1}$$

Let $I \in I_{k,n}$, $I = \{i_1, \ldots, i_k\}$ define a coordinate p_I on $\bigwedge^k K^n$ by

$$p_I(e_{j_1} \wedge \ldots \wedge e_{j_k}) = \begin{cases} 1 & \text{if } \{j_1, \ldots, j_k\} = I \\ 0 & \text{else.} \end{cases} \tag{1.2}$$

The p_I's define a set of coordinates on $\mathbb{P}(\bigwedge^k K^n)$. We extend the definition of the Plücker coordinates to all ordered subsets $J \subset \{1, \ldots, n\}$, $J = \{j_1, \ldots, j_k\}$ by letting p_I be anti-symmetric in the indices I. Thus for any $\sigma \in \Sigma_k$ we have $p_{i_1 \ldots i_k} = sgn(\sigma) p_{i_{\sigma(1)} \ldots i_{\sigma(k)}}$.

The evaluation of the p_I's on a matrix of rank k will be important to our defining a variety structure on $\mathrm{Grass}(k, K^n)$. Using a basis $\{e_1, \ldots, e_n\}$ of K^n, a k-dimensional subspace of K^n can be identified with a $k \times n$ matrix of maximal rank. Let $V \subset K^n$ be a k-dimensional subspace. Pick a basis for V say $\{v_1, \ldots, v_k\}$. Now if we expand the v_i's in terms of the basis for K^n we obtain a set of equalities

$$\begin{aligned} v_1 &= a_{11}e_1 + a_{12}e_2 + \cdots + a_{1n}e_n \\ v_2 &= a_{21}e_1 + a_{22}e_2 + \cdots + a_{2n}e_n \\ &\vdots \\ v_k &= a_{k1}e_1 + a_{k2}e_2 + \cdots + a_{kn}e_n. \end{aligned} \tag{1.3}$$

We may then express V as a $k \times n$ matrix

$$M_V = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}. \tag{1.4}$$

Note that this identification is not unique. This will not pose a problem for us. The value $p_I(V)$ is then given by $V \mapsto p_I(M_V)$ where

$$p_I(M_V) = \begin{vmatrix} a_{1,i_1} & \cdots & a_{1,i_k} \\ \vdots & \ddots & \vdots \\ a_{k,i_1} & \cdots & a_{k,i_k} \end{vmatrix}. \tag{1.5}$$

Let M'_V be another matrix representative for the subspace V. The matrices M_V and M'_V are related by a change of basis in V. In other words an element of $GL(V)$

acting on the left, $gM_V = M'_V$. Let us consider the value $p_I(M'_V) = p_I(gM_V) = det(g)p_I(M_V)$ for all $I \in I_{k,n}$. Thus the point $[p_I(V)]_{I \in I_{k,n}}$ in $\mathbb{P}(\bigwedge^2 K^n)$ does not depend on the choice of matrix M_V.

REMARK 1.2. The *Plücker embedding* $\pi\colon \mathrm{Grass}(k, K^n) \hookrightarrow \mathbb{P}(\bigwedge^k K^n)$ is given in coordinates by sending the subspace V to the point with homogeneous coordinates $[p_I(V)]_{I \in I_{k,n}}$.

PROPOSITION 1.3. *The map* $\pi\colon \mathrm{Grass}(k, K^n) \hookrightarrow \mathbb{P}(\bigwedge^k K^n)$ *is an embedding of* $\mathrm{Grass}(k, K^n)$ *onto a closed algebraic subset of* $\mathbb{P}(\bigwedge^k K^n)$.

THEOREM 1.4 ([**6**], p.90). *The Grassmannian* $\mathrm{Grass}(k, K^n)$ *is identified with a subvariety of* $\mathbb{P}(\bigwedge^k K^n)$ *by the vanishing of the following quadratic polynomials,*

$$\sum_{t=1}^{k+1} (-1)^t p_{i_1,i_2,\ldots,i_{k-1},j_t} p_{j_1,\ldots,j_{t-1},\hat{j}_t,j_{t+1},\ldots,j_k+1} \tag{1.6}$$

where $I = \{i_1, \ldots, i_{k-1}\}$ *is an ordered subset of* $\{1, \ldots, n\}$ *of* $k-1$ *elements and* $J = \{j_1, \ldots, j_{k+1}\}$ *is an ordered subset of* $\{1, \ldots, n\}$ *of* $k+1$ *elements.*

1.2. Open Sets. Let $I = \{i_1, \ldots, i_k \mid 1 \leq i_1 < \cdots < i_k \leq n\}$. Let us consider the set $\mathcal{U}_I \subset \mathrm{Grass}(k, K^n)$ such that $p_I \neq 0$. Since the Grassmannian is projective we may assume that $p_I = 1$. As a matrix we obtain

(1.7)

$$\mathcal{U}_I = \begin{pmatrix} x_{1,1} & \cdots & x_{1,i_1-1} & 1 & x_{1,i_1+1} & \cdots & x_{1,i_2-1} & 0 & \cdots & x_{1,i_k-1} & 0 & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,i_1-1} & 0 & x_{2,i_1+1} & \cdots & x_{2,i_2-1} & 1 & \cdots & x_{2,i_k-1} & 0 & \cdots & x_{2,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{k,1} & \cdots & x_{k,i_1-1} & 0 & x_{k,i_1+1} & \cdots & x_{k,i_2-1} & 0 & \cdots & x_{k,i_k-1} & 1 & \cdots & x_{k,n} \end{pmatrix}.$$

REMARK 1.5. Consider the set $\mathcal{U}_{\{1,2,\ldots,k\}}$. Let $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$. Split I into two parts $I = I_1 \cup I_2$ by setting I_1 to be all entries of I less than or equal to k and I_2 all entries strictly larger than k. Let us consider the evaluation of $p_I(\mathcal{U}_{\{1,\ldots,k\}})$. From the description in (1.7) and the definition of the Plücker coordinate p_I we see that up to sign $p_I(\mathcal{U}_{\{1,\ldots,k\}})$ is the minor of $\mathcal{U}_{\{1,\ldots,k\}}$ with columns given by I_2 and rows $\{1,\ldots,k\} \setminus I_1$.

If $V \in \mathcal{U}_I$ then V has a unique basis and therefore $\mathcal{U}_I \cong \mathbb{A}_K^{k(n-k)}$.

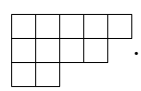
In each open set $\mathcal{U}_I$ there is a unique point corresponding to $\mathbf{0}$ in $\mathbb{A}_K^{k(n-k)}$. Denote this point by x_I. In matrix form it is $M_{x_I} = (x_{i,j})$ where

$$x_{i,j} = \begin{cases} 1 \text{ if } i = s, j = i_s \in I \\ 0 \text{ otherwise.} \end{cases} \tag{1.8}$$

2. Schur Functors

DEFINITION 2.1. Let $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq \cdots \geq 0\}$ be a sequence of integers such that $\sum \lambda_i = m$. We say that λ is a *partition* of m. Let λ be a partition of m. If $\lambda_r > 0$ and $\lambda_{r+1} = 0$ we say that λ *has r parts*.

2.1. Young Tableaux. We may express partitions graphically through the use of a Young Frame. Given a partition λ the *Young Frame* of λ denoted $D(\lambda)$ is a series of boxes drawn in the fourth quadrant with λ_i boxes in the ith row.

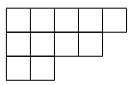
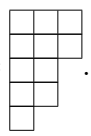
EXAMPLE 2.2. If $\lambda = (5, 4, 2)$ the corresponding Young frame is  .

REMARK 2.3. As an abuse of notation we will use partition and Young frame interchangeably.

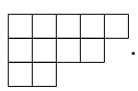
NOTATION 2.4. *Let k and d be positive integers. By (k^d) we mean the partition $(k, k, \ldots, k)$ with k appearing d times. As a Young frame this is the $d \times k$ rectangle.*

DEFINITION 2.5. Let λ be a partition. Define a new partition λ' called the *conjugate* partition by

$$\lambda'_i = \mathrm{card}\{t \mid \lambda_t \geq i\}.$$

EXAMPLE 2.6. Let $\lambda = (5, 4, 2)$ then $\lambda' = (3, 3, 2, 2, 1)$. The corresponding Young frames are $\lambda =$  and $\lambda' =$  .

DEFINITION 2.7. Let λ be a partition. By a *tableau T of shape λ* we mean a filling of the boxes of λ with positive integers. We will say a tableau T is *standard* if the entries of T are strictly increasing along the rows and non-decreasing along the columns.

EXAMPLE 2.8. Continuing with $\lambda =$  . Then $T =$ $\begin{array}{|c|c|c|c|c|}\hline 3&3&2&5&1\\\hline 2&2&2&2\\\cline{1-4} 1&1\\\cline{1-2}\end{array}$ is a tableau of shape λ but T is not a standard tableau. The tableau $S =$ $\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 2&3&5&6\\\cline{1-4} 2&3\\\cline{1-2}\end{array}$ is standard.

2.2. Definition of the Schur Functors. We need to set up some notation in order to give a presentation of the Schur functor L_λ.

DEFINITION 2.9. Let E be a vector space of dimension n over a field K. We define a multiplication map

$$\mu\colon \bigwedge^{i} E \otimes \bigwedge^{j} E \to \bigwedge^{i+j} E \tag{2.1}$$

$$\mu((v_1 \wedge \ldots \wedge v_i) \otimes (w_1 \wedge \ldots \wedge w_j)) = v_1 \wedge \ldots \wedge v_i \wedge w_1 \wedge \ldots \wedge w_j. \tag{2.2}$$

The comultiplication map will be called diagonalization

$$\Delta\colon \bigwedge^{i+j} E \to \bigwedge^{i} E \otimes \bigwedge^{j} E \tag{2.3}$$

$$\Delta(v_1 \wedge \ldots \wedge v_{i+j}) = \sum_{\substack{I \subset \{1,\ldots,i+j\} \\ |I|=i}} \operatorname{sign}(I, I') v_I \otimes v_{I'}. \tag{2.4}$$

Where if $I = \{m_1, \ldots, m_i\}$, and $m_1 < m_2 < \cdots < m_i$ we set $v_I = v_{m_1} \wedge \ldots \wedge v_{m_i}$.

These maps are $GL(E)$-equivariant.

Example 2.10.

$$\mu(v_1 \wedge v_2 \otimes v_3 \wedge v_4) = v_1 \wedge v_2 \wedge v_3 \wedge v_4.$$

$$\begin{aligned}\Delta(v_1 \wedge v_2 \wedge v_3 \wedge v_4) = {} & v_1 \wedge v_2 \otimes v_3 \wedge v_4 - v_1 \wedge v_3 \otimes v_2 \wedge v_4 \\ & + v_1 \wedge v_4 \otimes v_2 \wedge v_3 + v_2 \wedge v_3 \otimes v_1 \wedge v_4 \\ & - v_2 \wedge v_4 \otimes v_1 \wedge v_3 + v_3 \wedge v_4 \otimes v_1 \wedge v_2.\end{aligned}$$

$$\mu(\Delta(v_1 \wedge v_2 \wedge v_3 \wedge v_4)) = 6(v_1 \wedge v_2 \wedge v_3 \wedge v_4).$$

Definition 2.11. Let λ be a partition. We define the *Schur functor* L_λ associated to λ by

$$L_\lambda(E) := (\bigwedge^{\lambda_1} E \otimes \cdots \otimes \bigwedge^{\lambda_m} E)/R(\lambda, E). \tag{2.5}$$

Where $R(\lambda, E)$ is generated by the sum of all submodules

$$\bigwedge^{\lambda_1} E \otimes \cdots \otimes \bigwedge^{\lambda_{a-1}} E \otimes R_{a,a+1} \otimes \bigwedge^{\lambda_{a+2}} E \otimes \cdots \otimes \bigwedge^{\lambda_m} E. \tag{2.6}$$

Where the modules $R_{a,a+1}$ are generated by all images of the form

$$\begin{array}{c} \bigwedge^u E \otimes \bigwedge^{\lambda_a - u + \lambda_{a+1} - v} E \otimes \bigwedge^v E \\ \Big\downarrow {\scriptstyle 1\otimes\Delta\otimes 1} \\ \bigwedge^u E \otimes \bigwedge^{\lambda_a - u} E \otimes \bigwedge^{\lambda_{a+1} - v} E \otimes \bigwedge^v E \\ \Big\downarrow {\scriptstyle \mu_{12}\otimes\mu_{34}} \\ \bigwedge^{\lambda_a} E \otimes \bigwedge^{\lambda_{a+1}} E. \end{array} \tag{2.7}$$

Where we sum over all a, u, v with $u + v < \lambda_{a+1}$. This is a functor because all the defining compositions are $GL(E)$-equivariant maps.

Let T be a tableau of shape λ and let $\{e_1, \ldots, e_n\}$ be a basis for E. We associate to T a vector coset in $L_\lambda E$ by

$$T = e_{T(1,1)} \wedge \ldots \wedge e_{T(1,\lambda_1)} \otimes e_{T(2,1)} \wedge \ldots \wedge e_{T(2,\lambda_2)} \otimes \cdots \otimes e_{T(m,1)} \wedge \ldots \wedge e_{T(m,\lambda_m)}. \tag{2.8}$$

The relations in $L_\lambda E$ can be expressed graphically in terms of shuffles of tableaux. Let u and v be whole numbers such that $u + v < \lambda_{a+1}$. Consider the picture below showing the rows a and $a+1$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & \ldots & & * & \ldots & * & \ldots & \ldots & * \\ \hline * & \ldots & * & \ldots & * & & \ldots & \\ \cline{1-7} \end{array} \quad .$$

The upper left has u unmarked boxes and the lower right has v unmarked boxes. We leave the entries corresponding to the unmarked boxes alone and shuffle the entries corresponding to the $*$'s, with appropriate signs.

Let T be a tableau of shape λ. We associate to T, u, v, a a tensor in $\bigwedge^{\lambda_1} E \otimes \cdots \otimes \bigwedge^u E \otimes \bigwedge^{\lambda_a - u + \lambda_{a+1} - v} E \otimes \bigwedge^v E \otimes \cdots \otimes \bigwedge^{\lambda_l} E$ by

$$\begin{aligned}\iota(a,u,v,T) = e_{T(1,1)} \wedge \ldots \wedge e_{T(1,\lambda_1)} \otimes \cdots \\ \otimes\, e_{T(a,1)} \wedge \ldots \wedge e_{T(a,u)} \\ \otimes\, e_{T(a,u+1)} \wedge \ldots \wedge e_{T(a,\lambda_a)} \wedge e_{T(a+1,1)} \wedge \ldots \wedge e_{T(a+1,\lambda_{a+1}-v)} \\ \otimes\, e_{T(a+1,\lambda_{a+1}-v+1)} \wedge \ldots \wedge e_{T(a+1,\lambda_{a+1})} \otimes \cdots \\ \otimes\, e_{T(m,1)} \wedge \ldots \wedge e_{T(m,\lambda_m)}.\end{aligned}$$

Then we denote the image of $\iota(a,u,v,T)$ under the compositions from (2.7) by $\theta(a,u,v,T)$. For the relations $R_{a,a+1}$ it is enough to express the shuffles between neighboring rows. Thus let us consider a partition of only two parts $\lambda = (\lambda_1, \lambda_2)$. We consider the young frame associated to λ and pick values u, v with $u + v < \lambda_2$.

We begin with an example

Example 2.12. Let us consider the tableau $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline\end{array}$ with $u = v = 1$. The overlap occurs in the second column, thus we have the sequence $\{2,3,4,5\}$. We begin with a vector in $E \otimes \bigwedge^4 E \otimes E$ and want to compute the image in $\bigwedge^3 E \otimes \bigwedge^3 E$. The vector associated to T with $u = v = 1$ in $E \otimes \bigwedge^4 E \otimes E$ is $\iota(1,1,1,T) = e_1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5 \otimes e_6$. We want the image of v under the maps $1 \otimes \Delta \otimes 1$ and $m_{12} \otimes m_{34}$. To find $(1 \otimes \Delta \otimes 1)(v)$ it is enough to consider $\Delta(e_2 \wedge e_3 \wedge e_4 \wedge e_5)$ which we did in Example 2.10. Thus we find

$$\begin{aligned}1 \otimes \Delta \otimes 1(e_1 \otimes e_2 \wedge e_3 \wedge e_4 \wedge e_5 \otimes e_6) \\ = e_1 \otimes (e_2 \wedge e_3) \otimes (e_4 \wedge e_5) \otimes e_6 - e_1 \otimes (e_2 \wedge e_4) \otimes (e_3 \wedge e_5) \otimes e_6 \\ + e_1 \otimes (e_2 \wedge e_5) \otimes (e_3 \wedge e_4) \otimes e_6 + e_1 \otimes (e_3 \wedge e_4) \otimes (e_2 \wedge e_5) \otimes e_6\end{aligned}$$

$$- e_1 \otimes (e_3 \wedge e_5) \otimes (e_2 \wedge e_4) \otimes e_6 + e_1 \otimes (e_4 \wedge e_5) \otimes (e_2 \wedge e_3) \otimes e_6.$$

Applying $\mu_{12} \otimes \mu_{34}$ we obtain

$$\begin{aligned} & e_1 \wedge e_2 \wedge e_3 \otimes e_4 \wedge e_5 \wedge e_6 - e_1 \wedge e_2 \wedge e_4 \otimes e_3 \wedge e_5 \wedge e_6 \\ & + e_1 \wedge e_2 \wedge e_5 \otimes e_3 \wedge e_4 \wedge e_6 + e_1 \wedge e_3 \wedge e_4 \otimes e_2 \wedge e_5 \wedge e_6 \\ & - e_1 \wedge e_3 \wedge e_5 \otimes e_2 \wedge e_4 \wedge e_6 + e_1 \wedge e_4 \wedge e_5 \otimes e_2 \wedge e_3 \wedge e_6. \end{aligned}$$

In terms of tableaux

$$\theta(1,1,1,T) = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array} - \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array} - \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array} + \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2&3&6\\\hline\end{array}. \tag{2.9}$$

We considered the last $\lambda_1 - u$ entries of row one and the first $\lambda_2 - v$ entries of row two. We then shuffle these entries among the boxes in the same manner as given by Δ. So in our example 2.12 we shuffle the entries $\{2,3,4,5\}$, taking two entries a_1, a_2 for the first row and the remainder for the second row. We then multiply by the sign of the permutation given by $(a_1, a_2, \ldots, a_{\lambda_1+\lambda_2-u-v})$.

Lemma 2.13. *The relations $R_{a,a+1}$ are generated by all images $\theta(a, u-1, \lambda_{a+1} - u, T)$ where $0 \leq u < \lambda_{a+1}$*

We will need an order on tableaux of the same shape. To do this we use the following order on $\mathbb{N} \times \mathbb{N}$ given by $(i', j') \leq (i, j)$ if $i' < i$ or $i' = i$ and $j' \leq j$. This is the lexicographical order on $\mathbb{N} \times \mathbb{N}$.

Definition 2.14. Let T and U be tableaux on the shape λ. Let us assume that the rows of T and U are such that the entries are in strictly increasing order. We say $T \preceq U$ if $T = U$ or for the smallest pair $(i, j) \in \mathbb{N} \times \mathbb{N}$ such that $T(j, i) \neq U(j, i)$ we have $T(j, i) < U(j, i)$.

Recall that a tableau T is standard if the entries of T are increasing along the rows and non-decreasing on the columns. We have the following theorem.

THEOREM 2.15 ([**14**], p.36). *Let $\{e_1, \ldots, e_n\}$ be a basis of E. Then the standard tableaux of shape λ with entries from $\{1, \ldots, n\}$ form a basis for $L_\lambda(E)$.*

REMARK 2.16. The main point in the proof is that given a non-standard tableau T of shape μ we may apply relations arising from $R(\lambda, E)$ to obtain a sum of standard tableaux. First note that since our elements are inside a quotient of a sum of wedge products and the rows of T correspond to individual wedge products thus we may place the entries in the rows in increasing order. Thus we assume that our tableau T is standard along the rows. We need to show that we can apply relations to a tableau T which is not standard along some column. Pick the smallest pair $(i-1, j)$ such that $T(i-1, j) > T(i, j)$. We have a strictly increasing sequence of integers

$$T_{(i,1)} < T_{(i,2)} < \cdots < T_{(i,j)} < T_{(i-1,j)} < T_{(i-1,j+1)} < \cdots < T_{(i-1,\mu_{i-1})}.$$

We perform shuffles between the rows μ_{i-1}, μ_i with overlapping square in the jth column. The punch line is that any such shuffle of T will be a sum of earlier tableaux and have the property that it will be not standard for $(i', j') > (i, j)$. Thus we apply the reduction to these tableaux. This process eventually terminates due to the fact that our tableau has finitely many boxes.

REMARK 2.17. It is worth noting that the above algorithm, sometimes called a straightening law, converges very slowly.

2.3. Filtrations and Cauchy Formulas.

Definition 2.18. Let Λ be a poset, R a commutative ring and M an R-module. A *filtration* $\mathcal{F} = \{F_{\leq\lambda}\}_{\lambda\in\Lambda}$ of M is a sequence of submodules $F_{\leq\lambda}$, $\lambda \in \Lambda$ of M satisfying

(i) $\bigcup_{\lambda\in\Lambda} F_{\leq\lambda} = M$.
(ii) $F_{\leq\lambda} \subset F_{\leq\mu}$ for all $\lambda, \mu \in \Lambda$ with $\lambda \leq \mu$.

We define $F_{<\lambda} := \sum_{\mu<\lambda} F_{\leq\mu}$. Given a filtration $\mathcal{F}$ of M we may form the *associated graded R-module* $gr(\mathcal{F})$ defined by

$$gr(\mathcal{F}) := \bigoplus_{\mu\in\Lambda} F_{\leq\mu}/F_{<\mu}. \tag{2.10}$$

Definition 2.19. Define a total order on the set of partitions $\mathcal{P}$ by setting $\mu \leq_{rev} \lambda$ if and only if there is a j such that $\mu_i = \lambda_i$ for $1 \leq i < j$ and $\mu_j > \lambda_j$. This order will be referred to as the *reverse lexicographical order*.

Definition 2.20. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition. By $\bigwedge^\lambda E$ we mean

$$\bigwedge^\lambda E := \bigwedge^{\lambda_1} E \otimes \cdots \otimes \bigwedge^{\lambda_l} E. \tag{2.11}$$

Definition 2.21. Let E and F be K vector spaces. Let $\mu = (l)$ be a partition of one part. Let $\{e_1, \ldots, e_a\}$ and $\{f_1, \ldots, f_b\}$ be bases of E and F respectively. Define a map $\rho_{(l)}\colon \bigwedge^l E \otimes \bigwedge^l F \to S_l(E \otimes F)$ such that on basis elements we have

$$\rho_{(l)}(e_{s_1} \wedge \ldots \wedge e_{s_l} \otimes f_{t_1} \wedge \ldots \wedge f_{t_l}) = \sum_{\sigma\in\Sigma_l} sgn(\sigma) \prod_{i=1}^{l} e_{s_i} \otimes f_{t_{\sigma(i)}}. \tag{2.12}$$

Then extend $\rho_{(l)}$ linearly.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of r parts. Define a map $\rho_\lambda\colon \bigwedge^\lambda E \otimes \bigwedge^\lambda F \to S_{|\lambda|}(E \otimes F)$ by $\rho_\lambda := \rho_{\lambda_1} \cdots \rho_{\lambda_r}$.

The image of the map ρ_λ is best viewed as mapping tableaux to products of minors of a matrix. Setting $x_{ij} = e_i \otimes f_j$ we obtain a matrix (x_{ij}). In this way we make the identification $Sym(E \otimes F) \cong K[x_{i,j}]$. The image of $\rho_{(l)}(e_{s_1} \wedge \ldots \wedge e_{s_l} \otimes f_{t_1} \wedge \ldots \wedge f_{t_l})$, where S, T are tableaux on a partition of one part, is the minor of the matrix (x_{ij}) with rows given by the s_i's and columns given by t_j's.

Example 2.22.

(a) $\rho_{(2,1)}((e_1 \wedge e_2 \otimes e_3) \otimes (f_1 \wedge f_2 \otimes f_3)) = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} x_{33} = (x_{11}x_{22} - x_{12}x_{21})x_{33}$

(b) The action of ρ_λ on shuffles will be important for further discussion. Let us consider the pair of tableaux $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$. We will shuffle the first tableau with the only overlap. Doing this we obtain

$$S = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Let us apply the map $\rho_{(2,1)}$ and consider the image

$$\begin{aligned}
\rho_{(2,1)}(S) &= \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} x_{33} - \begin{vmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{vmatrix} x_{23} + \begin{vmatrix} x_{21} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} x_{13} \\
&= \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \\
&= \rho_{(3)}\Big(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\Big)
\end{aligned}$$

The principal observation here is the relation

$$\rho_{(21)}(S) = \rho_{(3)}\Big(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\Big) \tag{2.13}$$

with $\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix} \geq_{rev} \square\square\square$.

Definition 2.23. Let λ be a partition. By the *canonical tableau of shape* λ we mean the tableau Υ_λ with entries $\Upsilon_\lambda(i,j) = i$.

Lemma 2.24 ([**2**], Thm. 3.3). *Let E, F be K vector spaces. Choose a unipotent subgroup $U \subset GL(E)$ of strictly upper triangular matrices with 1's on the diagonal. Then given the $GL(E)$ representation $L_\lambda E$ there is a unique (upto scalar) U invariant given by the canonical tableau Υ_λ. The weight of this U invariant is λ'.*

Theorem 2.25 ([**14**],p.59). *There is a natural filtration on $S_m(E \otimes F)$ whose associated graded object is*

$$\bigoplus_{|\lambda|=m} L_\lambda E \otimes L_\lambda F.$$

2.4. Symmetric and Anti-symmetric Tensors. We may identify $Sym(\bigwedge^2 E^*)$ with the polynomial ring $K[x_{i,j}]$ with $x_{i,j}$ antisymmetric in the indices i, j. Choose a basis $\{e_1, \ldots, e_n\}$ for E. Then a tensor $v \in \bigwedge^2 E$ is of the form $v = \sum_{1 \leq i < j \leq n} a_{i,j} e_i \wedge e_j$. Let $x_{i,j}(v) = a_{i,j}$. Then $Sym(\bigwedge^2 E^*) \cong K[x_{i,j}]$. In a similar manner we have $Sym(S_2 E^*) \cong K[x_{i,j}]$, with the $x_{i,j}$ symmetric in indices i, j.

Definition 2.26. Let T be a tableau of shape (t,t) with entries $\{i_1, \ldots, i_t, i'_1, \ldots, i'_t\}$. Define a map $\rho^S_{(t,t)} \colon S_2(\bigwedge^t E) \to S_t(S_2 E)$ by

$$\rho^S_{(t,t)}(T) := \sum_{\sigma \in \Sigma_t} sgn(\sigma) x_{i_1, i'_{\sigma(1)}} \cdots x_{i_t, i'_{\sigma(t)}}. \tag{2.14}$$

If $(a_{i,j})$ is a symmetric matrix of elements from K we have $\rho^S_{(t,t)}(T)(a_{i,j})$ is the minor of $(a_{i,j})$ with rows $i_1, \ldots, i_t$ and columns $i'_1, \ldots, i'_t$.

THEOREM 2.27 ([**14**] p.63). *There exists a filtration on $S_m(S_2(E))$ given by $F_{\leq\lambda} = \sum_{\substack{\mu\leq_{rev}\lambda \\ \mu \text{ has even columns}}} \mathrm{Im}(\rho^S_\mu)$ with associated graded object*

$$\bigoplus_{\substack{|\lambda|=2m \\ \lambda \text{ has even columns}}} L_\lambda E.$$

COROLLARY 2.28.

$$\mathrm{gr}(Sym(S_2(E))) = \bigoplus_{\lambda,\ \lambda \text{ has even columns}} L_\lambda E \tag{2.15}$$

DEFINITION 2.29. Let T be a tableau of shape $(2s)$ with entries $\{i_1, \ldots, i_{2s}\}$. Define a map $\varrho^A_{(2s)}\colon \bigwedge^{2s} E \to S_s(\bigwedge^2 E)$ by setting

$$\varrho^A_{(2s)}(T) := \sum_{\substack{i_1<i_2<\cdots<i_s \\ i_1<j_1, i_2<j_2, \ldots, i_s<j_s}} sgn(IJ) x_{i_1,j_1} \cdots x_{i_s,j_s}, \tag{2.16}$$

the *Pfaffian indexed by* T of the skew symmetric matrix $(x_{i,j})$.

Let $\mu = (\mu_1, \ldots, \mu_r)$ be a partition with all μ_i even. Define the map ϱ^A_μ by

$$\varrho^A_\mu :\ = \varrho^A_{\mu_1} \cdots \varrho^A_{\mu_r}. \tag{2.17}$$

THEOREM 2.30 ([**14**] p.63). *There exists a filtration on $S_m(\bigwedge^2 E)$ given by $F_{\leq\lambda} = \sum_{\substack{\mu\leq_{rev}\lambda \\ \mu \text{ has even rows}}} \mathrm{Im}(\varrho_\mu)$ whose associated graded object is*

$$\bigoplus_{\substack{|\lambda|=2m \\ \lambda \text{ has even rows}}} L_\lambda E.$$

COROLLARY 2.31.

$$\mathrm{gr}(Sym(\bigwedge^2 E)) = \bigoplus_{\lambda,\ \lambda \text{ has even rows}} L_\lambda E. \tag{2.18}$$

3. Schur Modules for Algebraic Groups

Let G be an algebraic group, $B \subset G$ a Borel subgroup and $T \subset B$ a maximal torus. Let $P \subset G$ be a parabolic subgroup.

Fix a vector space F with non-degenerate inner product $\langle \cdot, \cdot \rangle$.

Let $X(T) = \mathrm{Hom}(T, K^*) \subset F$ be the set of weights of F. Let E be a representation of G. Then as a T-module E is a direct sum of weight spaces $E = \bigoplus_\chi E_\chi$ where $E_\chi = \{v \in E \mid t(v) = \chi(t)v\}$. We say that χ is a *weight of* E if $E_\chi \neq 0$ and that $v \in E_\chi$, $v \neq 0$ is a *weight vector*. Let $\Phi(T, E)$ be the set of non-zero weights in E. We call $\Phi(T, E)$ the *roots of* T *in* E. If we consider the adjoint action of G on $Lie(G)$ then $R\colon = \Phi(T, Lie(G))$ is called the *root system of* G. The subset $R^+\colon = \Phi(T, Lie(B))$ of $\Phi(T, Lie(G))$ is the set of positive roots relative to B. The set of indecomposable elements of R^+ are called the *simple roots*, denoted by S.

Let $S = \{\alpha_1, \ldots, \alpha_l\} \subset F$ be a set of simple roots. Then the space of weights has a corresponding dual basis, in F, $\{\omega_1, \ldots, \omega_l\}$ defined by $\dfrac{2\langle \omega_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j}$. We say that the set $\{\omega_1, \ldots, \omega_l\}$ are the *fundamental weights* of G. We say that a weight $\omega = \sum_{i=1}^{l} c_i \omega_i$ is *dominant* if $c_i \geq 0$ for $1 \leq i \leq l$. Denote the set of dominant weights of T by $X(T)^+$.

Definition 3.1. Let W be a P-module. Define a vector bundle over G/P by

$$\mathcal{E}(W) = G \times_P W = \{(g, w) \in G \times W \mid (gp, w) = (g, pw)\}, \tag{3.1}$$

with map $\pi\colon \mathcal{E}(W) \to G/P$ given by $\pi(g, w) = gP$.

Definition 3.2. Let ω be a fundamental weight of G. We say that ω is a *minuscule weight* if for all roots α we have $\left| \dfrac{2\langle \omega, \alpha \rangle}{\langle \alpha, \alpha \rangle} \right| \leq 1$.

Let us assume that P is a parabolic subgroup associated to a cominuscule fundamental weight. The cominuscule weights for all Dynkin diagrams are classified. We list only the cominuscule weights for simply laced Dynkin diagrams, as those are the cases covered in this paper.

Dynkin diagram	cominuscule weights
A_n	all fundamental weights ω_k
D_n	$\omega_1, \omega_{n-1}, \omega_n$
E_6	ω_1, ω_6
E_7	ω_7.

Let $P = L_P U_P$ be the *Levi decomposition* of P. Let us then consider the cotangent bundle T^*G/P.

Proposition 3.3. *Let P be a parabolic associated to a cominuscule weight. Then U_P acts trivially on T^*G/P. Hence there exists a L_P-module W_P such that*

$$T^*G/P = \mathcal{E}(W_P).$$

3.0.1. *Weights and Line Bundles.*

Definition 3.4. Let $\lambda \in X(G)$. Define an action of B on $K_\lambda = K$ by $b \cdot k = \lambda(b)k$. Define a line bundle $\mathcal{L}(\lambda)$ over G/B by

$$\mathcal{L}(\lambda) = (G \times K)/((gb, b \cdot k) = (g, k)).$$

The structure map $p\colon \mathcal{L}(\lambda) \to G/B$ is given by $p(g, k) = p(gb, b \cdot k) = gB$.

Remark 3.5. When it is necessary to specify the group G we will write $\mathcal{L}^G(\lambda)$.

The line bundles $\mathcal{L}(\lambda)$ play an important role in the theory of G representations. Recall that a weight λ is *dominiant* if it is a non-negative sum of fundamental weights.

Definition 3.6. Let $\lambda = a_1\omega_1 + \cdots + a_n\omega_n$, $a_i \geq 0$ be a dominant weight. Define a module L_λ^G by

$$L_\lambda^G := H^0(G/B, \mathcal{L}(\lambda)).$$

We call L_λ^G the *Schur module* of weight λ.

Remark 3.7.

i) Our notation here differs from that previously. Here L_λ^G is the module of weight λ where $L_\lambda(E)$ is the module of weight λ'.

ii) The definition of $\mathcal{L}(\lambda)$ is a special case of equation 3.1.

Let U be the unipotent radical of G contained in our chosen B.

Theorem 3.8 ([**8**],p.176-7). *Let λ be a dominant weight. Then*

i) $\dim(L_\lambda^G)^U = 1$*;*

ii) Every G-submodule of L_λ^G contains a scalar multiple of the U-invariant;

iii) L_λ^G contains a unique irreducible G-module.

Lemma 3.9. *Let M be a rational G-module. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ be a filtration such that $M_i/M_{i-1} \cong L_{\lambda_i}^G$. Assume that μ is a weight such that $\mu \neq \lambda_i$ for all i. Then M has no U invariant of weight μ.*

Proof. Recall that L_λ has a U invariant of weight μ if and only if $\lambda = \mu$. We proceed by induction on i. The case $i = 1$ is $M_1 \cong L_{\lambda_1}$ and since $\mu \neq \lambda_1$ there is no invariant of weight μ'. Now assume that through some j we have that M_j has no U

invariants of weight μ'. The piece of the filtration M_{j+1} fits into a sequence

$$M_j \longrightarrow M_{j+1} \longrightarrow L^G_{\lambda_{j+1}} E \longrightarrow 0 \ . \tag{3.2}$$

We know that $L^G_{\lambda_{j+1}}$ has no invariants of weight μ' and by induction M_j has no invariants of weight μ' hence M_{j+1} has no invariants of weight μ'. $\diamond$

Remark 3.10. The above lemma holds for λ a partition. But in the case when λ is a partition the U invariant will have weight λ'.

3.0.2. *Description of T^*G/P for cominuscule weights.* Now that we have a description of G-modules we can describe the cotangent bundles T^*G/P. The coordinate ring of T^*G/P_ω is given by the sections of $Sym_{\mathcal{O}_{G/P_\omega}}((T^*G/P_\omega)^*)$, where

$$\mathcal{O}_{G/P_\omega} = \bigoplus_n \mathcal{L}(n\omega).$$

We will use this formula in computing sections of $Sym_{\mathcal{O}_{G/P_\omega}}((T^*G/P_\omega)^*)$.

Remark 3.11. For the remainder of the paper we will use Sym instead of $Sym_{\mathcal{O}_{G/P_\omega}}$.

In the following we write P for P_ω. The Levi factors and modules W_P are known for cominuscule weights. We list them in the following table

Group	weight	Levi factor	W_P
Sl_n	ω_k	$Sl_k \times Sl_{n-k}$	$L^{Sl_k}_{\omega_{k-1}} \otimes L^{Sl_{n-k}}_{\omega_1}$
$Spin_{2n}$	ω_{n-1}, ω_n	Sl_n	$L^{Sl_n}_{\omega_{n-1}}$
E_6	ω_1, ω_6	$Spin_{10}$	$L^{Spin_{10}}_{\omega_5}$
E_7	ω_7	E_6	$L^{E_6}_{\omega_6}$.

The next step in the description of the coordinate rings of T^*G/P is to compute $Sym(T^*G/P)$.

THEOREM 3.12 ([**11**]). *The sheaves $Sym(T^*G/P)$ for a parabolic associated to a cominuscule weight have filtrations giving associated graded objects as follows:*

- A_{n-1}, ω_k

$$gr(Sym(T^*Sl_n/P_k)) = \bigoplus_{a_1,\dots,a_k} \mathcal{L}^{Sl_k}(a_1\omega'_k + \cdots + a_k\omega'_1) \otimes \mathcal{L}^{Sl_{n-k}}(a_1\omega''_1 + \cdots + a_k\omega''_k)$$

  where ω'_i are the fundamental weights of Sl_k and ω''_j are the fundamental weights of Sl_{n-k}, with $k \leq n-k$. The Levi factor in this case can be viewed as $Gl_k \times Sl_{n-k}$. Hence the weights ω'_k and ω''_k are the determinant.
- $D_n, \omega_{n-1}, \omega_n$ *First note that $Spin_{2n}/P_{n-1} \cong Spin_{2n}/P_n$.*

$$gr(Sym(T^*Spin_{2n}/P_n)) = \bigoplus_{a_1,\dots,a_s} \mathcal{L}^{Sl_n}(a_1\omega'_2 + \cdots + a_s\omega'_{2s})$$

  where ω'_i are the fundamental weights of Sl_n and $s = \lfloor (n-1)/2 \rfloor$.
- E_6, ω_1, ω_6 *Note that $E_6/P_1 \cong E_6/P_6$. In characteristic zero we have*

$$gr(Sym(T^*E_6/P_6)) = \bigoplus_{l,m\geq 0} \mathcal{L}^{Spin_{10}}(l\omega'_4 + m\omega'_2),$$

  where ω'_2 and ω'_4 are fundamental weights of $Spin_{10}$. Further since we are in characteristic zero we may write

$$S_d(T^*E_6/P_6) = \bigoplus_{l+2m=d} \mathcal{L}^{Spin_{10}}(l\omega'_4 + m\omega'_2).$$

- E_7, ω_7 *In characteristic zero we have*

$$gr(Sym(T^*E_7/P_7)) = \bigoplus_{k,l,m} \mathcal{L}^{E_6}(k\omega'_6 + l\omega'_1) \otimes W^{\otimes m},$$

where ω_1' and ω_6' are fundamental weights of E_6 and W is an invariant of E_6. Further since we are in characteristic zero we may write

$$S_d(T^*E_7/P_7) = \bigoplus_{k+2l+3m=d} \mathcal{L}^{E_6}(k\omega_6' + l\omega_1') \otimes W^{\otimes m}.$$

We will give an explicit description of the A_{n-1}, ω_k case later in the paper.

4. Deformation of G Varieties

We recall some results from F. Grosshans's book [**7**]. Let G be an algebraic group. Let A be a commutative k-algebra on which G acts rationally. Let $A_0 \subset A_1 \subset \cdots \subset A$ be a G invariant filtration such that $A_m \cdot A_n \subset A_{m+n}$. Let $U \subset G$ be a unipotent subgroup and $T \subset G$ a maximal torus. Let $A[x]$ be the polynomial algebra over A.

Definition 4.1. Define a ring $D(A)$ by

$$D(A) = \bigoplus_{n \geq 0} x^n A_n.$$

Lemma 4.2 ([**7**], 15.8, p92). *The algebra $D(A)/xD(A)$ is isomorphic to* $\mathrm{gr}(A)$ *and this isomorphism is G-equivariant.*

Theorem 4.3 ([**7**], 15.14, p. 93). *Let i be the inclusion of $k[x]$ in $D(A)$. Then $D(A)$ is flat over $K[x]$. If $M = (x) \subset K[x]$ then the fiber of i over M is* $\mathrm{Spec}(\mathrm{gr}(A))$ *and if $M = (x-a) \subset K[x]$ with $a \neq 0$ then the fiber over M is* $\mathrm{Spec}(A)$.

Definition 4.4. In the above case we say that the action of G on A *contracts* to the action of G on $\mathrm{gr}(A)$ and that the action of G on A is a *deformation* of the action of G on $\mathrm{gr}(A)$.

Definition 4.5. Let A have a G-invariant filtration $A_0 \subset A_1 \subset \cdots \subset A$. We say that this is a *good filtration* if $\mathrm{gr}(A) \cong \bigoplus_\lambda L_\lambda^G$.

LEMMA 4.6 ([**7**], 15.3, p.88). *Let A be as above but now assume that G is reductive. If A has a good filtration, then A^U is isomorphic to $(\mathrm{gr}(A))^U$ and this isomorphism is T invariant.*

THEOREM 4.7 ([**7**], 18.4, p. 103). *If A is normal, then A^U is normal. If A^U is normal, A is finitely generated over k, and A has a good filtration, then A is normal.*

PROPOSITION 4.8. *Let G be a connected group. Let A be a commutative integral K-algebra on which G acts rationally. Then A^G is normal.*

PROOF. We need to show that if $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ with $a_i \in A^G$ has a solution x in the quotient field of A^G, then $x \in A^G$. Let x satisfy the above. Let $g \in G$ and consider $g(x^n + a_{n-1}x^{n-1} + \cdots + a_0) = g(x)^n + a_{n-1}g(x)^{n-1} + \cdots + a_0 = 0$. Thus $g(x)$ is also a root. Hence G acts on the roots of the polynomial by permutation. But since G is connected the only homomorphism from G to a symmetric group is the trivial map. Thus $g(x) = x$ and hence $x \in A^G$. $\square$

THEOREM 4.9 ([**10**], p.188, 24.5). *Let A be a Noetherian ring, B a finitely generated A-algebra, and M a finite B-module. Set $\mathcal{U} = \{P \in \mathrm{Spec}(B) \mid M_P$ is Cohen-Macaulay over $A\}$. Then $\mathcal{U}$ is open in $\mathrm{Spec}(B)$.*

We will apply the above theorem to the family given by $D(A)$. It is then sufficient to show that the fiber over (x) is Cohen-Macaulay to show that the original ring A is Cohen-Macaulay.

CHAPTER 2

Conormal Varieties to Rank Varieties for Skew-symmetric and Symmetric Matrices

1. Conormal Varieties to Rank Varieties for Skew-symmetric Matrices

Let E be a K vector space of dimension n. Let $Y = \bigwedge^2 E$ be the space of anti-symmetric bilinear forms. Let us consider the rank variety $Y_{2r} \subset Y$ where Y_{2r} consists of those forms with rank less than or equal to $2r$. The variety Y_{2r} is given by the vanishing of $(2r+2) \times (2r+2)$ Pfaffians.

The purpose of this section is to compute the coordinate rings of the conormal varieties $T^*_{Y_{2r}} \bigwedge^2 E$.

Set

$$2s + 2 := \min_t \{2t > n - 2r\}. \tag{1.1}$$

The conormal space to the rank varieties sits inside $T^*(\bigwedge^2 E)$. We make the identification $T^*(\bigwedge^2 E) \cong \bigwedge^2 E \oplus \bigwedge^2 E^*$.

Let A be the coordinate ring of $\bigwedge^2 E \oplus \bigwedge^2 E^*$, $A := Sym(\bigwedge^2 E^*) \otimes Sym(\bigwedge^2 E)$. By the Cauchy formula we have a filtration of A giving an associated graded object

$$gr(A) = gr(Sym\left(\bigwedge^2 E^*\right) \otimes Sym\left(\bigwedge^2 E\right)) = \bigoplus_{\lambda,\mu \text{ with even rows}} L_\lambda E^* \otimes L_\mu E. \tag{1.2}$$

Let $M = (x_{i,j})$ and $N = (w_{j,i})$ be matrices of variables in $K[x_{i,j}, w_{k,l}]$,with $x_{i,j}$ and $w_{k,l}$ anti-symmetric in i, j and k, l. We want to express the entries of $M \cdot N$ in terms of tableaux. To this end we introduce some new notation.

Definition 1.1. Let $\mu = (\mu_1, \ldots, \mu_l)$ be a partition. Let n be a positive integer. Define a new partition $\hat{\mu}$ by $\hat{\mu}_i = n - \mu_{l-i+1}$.

Let T be a tableaux of shape μ filled with numbers from $\{1, \ldots, n\}$ with $\mu_1 \leq 2s$. Assume that μ has l rows. We define the *dual tableau* $\hat{T}$ by defining $\hat{T}_i$ to be the complement of T_{l-i+1} in $\{1, \ldots, n\}$ taken in increasing order. Note that $\hat{T}$ is a tableau of shape $\hat{\mu}$.

Definition 1.2. Let S and T be tableaux of shapes λ and μ with entries from the set $\{1, \ldots, n\}$. Let us also assume that μ has l rows and $\lambda_1 \leq n - \mu_1$ or equivalently $\lambda_1 \leq \hat{\mu}_l$. Then we may form a new tableau $\begin{smallmatrix}\hat{T}\\ S\end{smallmatrix}$ called the *stacked tableau* of T and S.

Example 1.3. (i) Let $2r = 2$, $n = 6$, and $2s = 4$. Let us consider the tableau $T = \boxed{1}\boxed{2}$. Then the dual tableau is $\hat{T} = \boxed{3}\boxed{4}\boxed{5}\boxed{6}$. If we then consider the pair of tableaux $(S, T) = (\boxed{1}\boxed{2}, \boxed{1}\boxed{2})$. We form a stacked tableau $\begin{smallmatrix}\hat{T}\\ S\end{smallmatrix} = \begin{array}{|c|c|c|c|}\hline 3&4&5&6\\ \hline 1&2\\ \cline{1-2}\end{array}$.

(ii) Let us again take $n = 6$ and this time consider the tableaux $T = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 1&5\\ \cline{1-2}\end{array}$ and $S = \begin{array}{|c|c|c|}\hline 2&3&4\\ \hline 3&6\\ \cline{1-2}\end{array}$. The dual tableau to T is then $\hat{T} = \begin{array}{|c|c|c|c|}\hline 2&3&4&6\\ \hline 4&5&6\\ \cline{1-3}\end{array}$ and the stacked tableau becomes

$$\begin{smallmatrix}\hat{T}\\ S\end{smallmatrix} = \begin{array}{|c|c|c|c|}\hline 2&3&4&6\\ \hline 4&5&6\\ \cline{1-3} 1&2&3\\ \hline 1&5\\ \cline{1-2}\end{array}.$$

Example 1.4. Take the pair of tableaux $(\boxed{1}\boxed{2}, \boxed{2}\boxed{1}) = (S, T)$ with $n = 6$. Let us form the tableau $\begin{smallmatrix}\hat{T}\\ S\end{smallmatrix} = \begin{array}{|c|c|c|c|}\hline 3&4&5&6\\ \hline 1&2\\ \cline{1-2}\end{array}$. Now we consider the shuffles between the top row and

the first box of the second row. Doing this shuffle we obtain

$$\begin{array}{|c|c|c|c|}\hline 3&4&5&6\\\hline 1&2\\\cline{1-2}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&4&5&6\\\hline 2&3\\\cline{1-2}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4\\\cline{1-2}\end{array} - \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5\\\cline{1-2}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6\\\cline{1-2}\end{array}.$$

Going back to pairs of tableau we obtain the expression

$$(\boxed{2}\boxed{1}, \boxed{1}\boxed{2}) - (\boxed{2}\boxed{3}, \boxed{3}\boxed{2}) + (\boxed{2}\boxed{4}, \boxed{4}\boxed{2}) - (\boxed{2}\boxed{5}, \boxed{5}\boxed{2}) + (\boxed{2}\boxed{6}, \boxed{6}\boxed{2}).$$

This is the entry (2,2) in $M \cdot N$.

Notice that if $\hat{\substack{T\\S}}$ is defined then $\hat{\substack{S\\T}}$ is defined. This follows from the equivalence of inequalities $\lambda_1 \leq n - \mu_1$ if and only if $\mu_1 \leq n - \lambda_1$.

Lemma 1.5. *Let S and T be a pair of tableaux of shapes λ and μ such that $\hat{\substack{T\\S}}$ is defined. Then $\hat{\substack{T\\S}}$ is standard if and only if $\hat{\substack{S\\T}}$ is standard.*

Proof. We only need to check $\hat{\substack{T\\S}}$ standard implies $\hat{\substack{S\\T}}$ standard. Then apply the same proof in the other direction. Since standardness is checked pairwise by row, it is enough to prove the lemma for two tableaux of one row each. Denote the entries (or content) of S by $\{i_1 < i_2 < \cdots < i_s\}$ and the content of $\hat{T}$ by $\{j_1 < j_2 < \cdots < j_t\}$. The assumption that $\hat{\substack{T\\S}}$ is standard means we have the inequalities

$$\begin{array}{ccccccccccc} i_1 & < & i_2 & < & \cdots & < & i_t & < & \cdots & < & i_s \\ \wedge| & & \wedge| & & & & \wedge| & & & & \\ j_1 & < & j_2 & < & \cdots & < & j_t & & & & \end{array}$$

Denote the elements of the complementary sets by $\{\hat{i}_1 < \cdots < \hat{i}_{n-s}\}$ and $\{\hat{j}_1 < \cdots < \hat{j}_{n-t}\}$. We want to show that the relations $\hat{j}_l \leq \hat{i}_l$ for $1 \leq l \leq n - s$.

To construct the dual tableau $\hat{S}$ to S we fill the first $i_1 - 1$ boxes with the numbers $1, 2, \ldots, i_1 - 1$, then we fill the next $i_2 - i_1 - 1$ boxes with $i_1 + 1, \ldots, i_2 - 1$. We

continue until we fill the last $n-i_t-1$ boxes with the numbers $i_t+1,\dots,n$. Similarly to construct T from $\hat{T}$ we fill the first j_1-1 boxes with $1,\dots,j_1-1$ and the last boxes $n-j_s-1$ boxes with $j_s+1,\dots,n$. Comparing elements of T and $\hat{S}$ we see that for the first i_1-1 boxes of both we have that the entries of T are equal to the entries of $\hat{S}$. Now for the next boxes until we reach the j_1-1th box we have a strict inequality, this is because we omit i_1 from $\hat{S}$ and since $i_1 \leq j_1$, we know the entries in $\hat{S}$ are larger. Once we reach the j_1th box the entries of T jump and we have an inequality until we reach the i_2th box. Continuing this process until we reach the $n-s$th box we arrive at our result. □

NOTATION 1.6. *Let M be a matrix with entries from a ring R. Denote the ideal of R generated by all $2l \times 2l$ Pfaffians of M by $I^{alt}_{2l}(M)$.*

DEFINITION 1.7. Let $A_n = K[x_{i,j}, w_{i,j}]$ be a polynomial ring in generic skew symmetric variables $x_{i,j} = -x_{j,i}$ and $w_{i,j} = -w_{j,i}$ with $1 \leq i,j \leq n$. Define an ideal $J_n = (\sum_{i=1}^n x_{k,i}w_{i,j})$ over all j,k. Let M and N be skew-symmetric matrices of variables from A, $M = (x_{i,j})$ and $N = (w_{i,j})$. We define the ideals $\mathcal{I}_r$ in A_n by

$$\mathcal{I}_r := ((I^{alt}_{2r+2}(M) + I^{alt}_{2s+2}(N) + J_n) \text{ for all } 1 \leq i,j \leq n \tag{1.3}$$

DEFINITION 1.8. Define the rings $B_{n,r}$ by

$$B_{n,r} := A/\mathcal{I}_r. \tag{1.4}$$

DEFINITION 1.9. Let μ and λ be partitions with even length rows. Define a map

$$\varrho^A_{\lambda,\mu} :\ = \varrho^A_\lambda \cdot \hat{\varrho}^A_\mu. \tag{1.5}$$

Here $\hat{\varrho}^A_\mu$ is a map $\hat{\varrho}^A_\mu\colon \bigwedge^\mu E^* \to S_{\mu_1/2}(\bigwedge^2 E^*) \otimes \cdots \otimes S_{\mu_l/2}(\bigwedge^2 E^*)$.

DEFINITION 1.10. We define an order on pairs of partitions (λ, μ) by $(\lambda, \mu) \geq' (\tau, \upsilon)$ if $\lambda \geq_{rev} \tau$ or $\lambda = \tau$ and $\mu \geq_{rev} \upsilon$ where we use the order from Definition 2.19. Let us define a filtration $\mathcal{F}_{n,r} = \{F_{\leq'(\lambda,\mu)}\}$ by setting

$$F_{\leq'(\lambda,\mu)} := \sum_{\substack{(\tau,\upsilon)\leq'(\lambda,\mu),\\ |\tau|=2m,|\upsilon|=2n \text{ both with even rows}}} \mathrm{Im}(\varrho_{\tau,\upsilon}). \tag{1.6}$$

1.1. Standard Bases for the Coordinate Ring of $T^*_{Y_{2r}} \bigwedge^2 \mathbb{C}^n$.

THEOREM 1.11.
(i) The ring $B_{n,r}$ has a basis given by the stacked standard tableaux of shapes $\begin{smallmatrix}\hat{\mu}\\ \lambda\end{smallmatrix}$ for all λ, μ with even rows and satisfying $\lambda_1 \leq 2r$ and $\mu_1 \leq 2s$.
(ii) The ring $B_{n,r}$ is the coordinate ring of the conormal variety to the rank variety of skew symmetric matrices.

PROOF. We proceed in three major steps.

1) We will show that $\mathcal{I}$ vanishes on $T^*_{Y_{2r}}(\bigwedge^2 E)$.

2) We will show that the stacked standard tableaux generate $B_{n,r}$.

3) We show that the stacked standard tableaux are linearly independent.

1) Vanishing: Choose a basis $\{v_1, \ldots, v_n\}$ for E. The variety $T^* \bigwedge^2 E$ can be identified with pairs of anti-symmetric matrices with coordinates $((x_{i,j}), (w_{j,i}))$ where $x_{i,j}$ is the coefficient of $v_i \wedge v_j$ in some vector v and $w_{j,i}$ is the coefficient of $v_j^* \wedge v_i^*$. Let us restrict our view to pairs of matrices (M, N) with M of rank less than or equal to $2r$. The matrix D_{2r} defined in our basis by

$$D_{2r} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{1.7}$$

has the property that the closure of the $GL(E)$ action on D_{2r} is Y_{2r}.

We want to understand the tangent space to Y_{2r} at D_{2r}. Let us consider an element g of $GL(E)$. In our basis let $g = (a_{i,j})$. The action of g on D_{2r} is given in the (k, l) entry by

$$(g \circ D_{2r})_{k,l} = \sum_{i=1}^{r} a_{2i-1,k}a_{2i,l} - a_{2i-1,l}a_{2i,k} = \sum_{i=1}^{r} g^{k,l}_{2i-1,2i}, \tag{1.8}$$

where $g^{k,l}_{2i-1,2i}$ is the minor of g with rows $2i-1, 2i$ and columns k, l. We use the action given in equation (1.8) to analyze the tangent space to Y_{2r} at D_{2r}.

Lemma 1.12. *The tangent space to Y_{2r} at D_{2r} is given by a set of skew symmetric matrices*

$$T_{D_{2r}} = \left(\begin{array}{c|c} M_1 & M_2 \\ \hline -M_2^T & \mathbf{0} \end{array}\right) \tag{1.9}$$

where M_1 is skew-symmetric of size $2r \times 2r$ and M_2 is of size $(n-2r) \times 2r$.

From this we see that the space of normal vectors is given in matrix form as

$$N_{D_{2r}} = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & N_{n-2r} \end{array}\right) \tag{1.10}$$

where N_{n-2r} is skew symmetric of size $(n-2r)\times(n-2r)$.

Define a subset of $\bigwedge^2 E \otimes \bigwedge^2 E^*$ by

$$Z_{D_{2r}} = \left(\left(\begin{array}{c|c} D_{2r} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right), \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & N_{n-2r} \end{array} \right) \right). \tag{1.11}$$

We refer to $Z_{D_{2r}}$ as our *distinguished set.*

Let us proceed to the proof of vanishing of shuffles of tableaux. Since a shuffle involves only two rows of a tableaux it will be enough to consider a general shuffle on the scheme

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & \ldots & & * & \ldots & \ldots & * & \ldots & * \\ \hline * & \ldots & * & \ldots & * & & \ldots & & \\ \hline \end{array}.$$

Denote this sum by W with individual terms W_i, i.e. $W = \sum_i W_i$. From the Cauchy formula applied to $\bigwedge^2 E$ or $\bigwedge^2 E^*$ we know that if our shuffle involves two rows of S or two rows of T then it is a sum of tableaux of earlier shape, in the order on tableaux given in definition 1.10. Hence it vanishes in $gr(\mathcal{F}_{n,r})$. Thus we are left considering shuffles between the first row of S and the last row of $\hat{T}$.

It will be enough for us to show that the shuffle involving the first row of S and the last row of $\hat{T}$ vanishes on $Z_{D_{2r}}$. We see from the construction of $Z_{D_{2r}}$ that if for all i we never find a term $W_i = \begin{smallmatrix} \hat{T}_l \\ S_1 \end{smallmatrix}$ such that S_1 has all entries less than r we are done. For in this case the evaluation of W on $Z_{D_{2r}}$ will be zero. If in W we cannot find a term W_i such that $\hat{T}_l$ contains $1, 2, ..., 2r$ we are also done. This would mean that there is a j with $1 \leq j \leq 2r$ not in $\hat{T}_l$ and hence T_1 would contain j. Thus the evaluation of T_1 on $Z_{D_{2r}}$ would be zero.

Let us now assume that we are able to find a term $W_i = \begin{smallmatrix} \hat{T}_l \\ S_1 \end{smallmatrix}$ in W such that $\hat{T}_l$ contains $1, 2, \ldots, 2r$ and the entries of S_1 are less than or equal to $2r$. Thus we are

shuffling at most $2r$ distinct entries amongst at least $2r+2$ boxes. Hence some entry is repeated. Since the shuffle is anti-symmetric in the b's we see the shuffle is zero.

(2) Generation:

Let λ and μ be partitions with even rows and $|\lambda| = 2m$, $|\mu| = 2n$.

We have shown that $\mathcal{I}_{2r}$ contains all images $\theta(l, u-1, |S_1| - u, \overset{\hat{T}}{S})$. This implies the following remark:

Remark 1.13. The module $F_{\leq'(\lambda,\mu)}/F_{<'(\lambda,\mu)}$ is a factor of $L_{\overset{\hat{\mu}}{\lambda}}E$. Thus there is a surjection

$$L_{\overset{\hat{\mu}}{\lambda}}E \to F_{\leq'(\lambda,\mu)}/F_{<'(\lambda,\mu)} \to 0. \tag{1.12}$$

Hence the stacked standard tableaux generate $B_{n,r}$.

(3) Linear Independence: To show independence of the stacked standard tableaux we will need the following lemma.

Let $\Upsilon_{\overset{\hat{\mu}}{\lambda}}$ be the U invariant of weight $\overset{\hat{\mu}'}{\lambda}$ in $F_{\leq'(\lambda,\mu)}$. Set $\tau = \overset{\hat{\mu}}{\lambda}$. As a stacked tableau $\Upsilon_{\overset{\hat{\mu}}{\lambda}}$ is filled in each row with $1, 2, \ldots, \tau_i$. When we return to pairs of tableaux $\Upsilon_{\overset{\hat{\mu}}{\lambda}}$ has the form (S, T) where S is filled with $1, 2, \ldots, \lambda_i$ and T is filled with $n, n-1, \ldots, n-\mu_j+1$. We see $\varrho^A_{\lambda,\mu}(S,T)(Z_{D_{2r}}) \not\equiv 0$. Hence the U invariants are non-vanishing on $T^*_{Y_{2r}} \bigwedge^2 E$.

To show independence of the stacked standard tableaux we use the filtration $\{F_{\leq'(\lambda,\mu)}\}$ of $B_{n,r}$ with graded component $F_{\leq'(\lambda,\mu)}/F_{<'(\lambda,\mu)}$. Let us proceed by induction on the pair (λ, μ). The base case is when $\lambda = (2r)$ and $\mu = (2s)$. In this case we see that $F_{\leq'((2r),(2s))} \cong L_{\overset{\hat{(2s)}}{(2r)}}E$. To see this note that for each tableaux pair (S, T) of shapes $((2r), (2s))$ there is a point of $T^*_{Y_{2r}} \bigwedge^2 E$ on which the tableaux do not vanish. Let us then assume that the module $F_{<'(\lambda,\mu)}$ is filtered as in Lemma 3.9. Note that we use $<'$ and not $\leq'$, this does not pose a problem because we are using

a total ordering of the partitions. Consider the piece of the filtration $F_{\leq'(\lambda,\mu)}$. We know by our inductive hypothesis and 3.9 that $F_{<'(\lambda,\mu)}$ has no U invariants of weight $\overset{\hat{\mu}}{\lambda}'$. In the graded piece $F_{\leq'(\lambda,\mu)}/F_{<'(\lambda,\mu)}$ there is a U invariant of weight $\overset{\hat{\mu}}{\lambda}'$. We have shown there is a surjection $L_{\overset{\hat{\mu}}{\lambda}}E \to F_{\leq'(\lambda,\mu)}/F_{<'(\lambda,\mu)}$. Let $N(\lambda,\mu)$ be the kernel of this surjection. The kernel $N(\lambda,\mu)$ is a $GL(E)$ submodule of $L_{\overset{\hat{\mu}}{\lambda}}$ hence $N(\lambda,\mu)$ must contain a multiple of the highest weight vector $\Upsilon_{\overset{\hat{\mu}}{\lambda}}$. The module $L_{\overset{\hat{\mu}}{\lambda}}E$ has a unique U invariant of weight $\overset{\hat{\mu}}{\lambda}'$ as does $F_{\leq'(\lambda,\mu)}/F_{<'(\lambda,\mu)}$. Thus by Lemma 3.9 $N(\lambda,\mu)$ does not. Since every submodule of $L_{\overset{\hat{\mu}}{\lambda}}$ contains a multiple of $\Upsilon_{\overset{\hat{\mu}}{\lambda}}$ we are able to conclude $N(\lambda,\mu)=0$. From which it follows that the stacked standard tableaux of shape $\overset{\hat{\mu}}{\lambda}$ are linearly independent in $F_{\leq'(\lambda,\mu)}/F_{<'(\lambda,\mu)}$ and thus independent in $B_{n,r}$.

□

1.2. Properties of the Conormal Variety to Y_{2r}.

Definition 1.14. Let $\underline{i} = \boxed{i_1|i_2|\ldots|i_{2t}}$ and $\underline{j} = \boxed{j_1|j_2|\ldots|j_{2u}}$ with $2t \leq 2r$ and $2u \leq 2s$, be a pair of tableaux defining monomials $(\underline{i},\emptyset)$ and $(\emptyset,\underline{j})$ on $T^*_{Y_{2r}}(\bigwedge^2 E^n)$. Let $H_{2r,2s}$ be the set of all pairs of tableaux of shapes $((2t),0)$ or $(0,(2u))$ with $2t \leq 2r$ and $2u \leq 2s$. Define a partial order on $H_{2r,2s}$ by

$$
\begin{aligned}
&(\underline{i},\emptyset) \leq (\underline{i}',\emptyset) && \text{if} \quad \tfrac{\underline{i}}{\underline{i}'} \text{ is standard} \\
&(\emptyset,\underline{j}) \leq (\emptyset,\underline{j}') && \text{if} \quad \tfrac{\underline{j}'}{\underline{j}} \text{ is standard} \\
&(\underline{i},\emptyset) \not\leq (\emptyset,\underline{j}) && \\
&(\emptyset,\underline{j}) \leq (\underline{i},\emptyset) && \text{if} \quad \tfrac{\hat{\underline{j}}}{\underline{i}} \text{ is standard.}
\end{aligned}
\tag{1.13}
$$

Proposition 1.15. *The ring $B_{n,r}$ is a graded Hodge algebra on $H_{2r,2s}$.*

Proof. That $B_{n,r}$ is generated as a free K-module and that there is a straightening law follows from Theorem 1.11. □

DEFINITION 1.16. Let P be a poset.

1) We say that x is a *cover* of y if $x > y$ and there are no elements z such that $x > z > y$.

2)We say that P is a *wonderful poset* if the following condition holds on $H \cup \{-\infty, \infty\}$. For all y_1, y_2 covering x with $z > y_1$, $z > y_2$ there exists an element $y \leq z$ covering y_1 and y_2.

PROPOSITION 1.17. *The poset $H_{2r,2s}$ is wonderful.*

PROOF. The cases when y_1, y_2 and x are all elements of the form $(\underline{i}, \emptyset)$ or $(\emptyset, \underline{j})$ are proved in [**3**] where it is shown that the set of $\underline{i}$ forms a wonderul poset. If $x = (\underline{i}, \emptyset)$ by the definition of the order on $H_{2r,2s}$ we have $y_1 = (\underline{i}', \emptyset)$ and $y_2 = (\underline{i}'', \emptyset)$, which is covered in the previous case. Thus the only case requiring discussion is when $x = (\emptyset, \underline{j})$ and $y_1 = (\underline{i}, \emptyset)$.

LEMMA 1.18 (Lemma 2.7 in [**12**]). *Let $x = (\emptyset, \boxed{j_1}\boxed{j_2}\boxed{\dots}\boxed{j_{2u}})$. Assume that there is some $(\underline{i}, \emptyset) \geq x$. Define a new tableau $\check{x} = (\boxed{i_1}\boxed{i_2}\boxed{\dots}\boxed{i_{2r}}, \emptyset)$ by letting $i_1, \dots, i_{2r}$ be the smallest $2r$ numbers in $\{1, \dots, n\} \setminus \{j_1, \dots, j_{2u}\}$.*

Then $(\underline{i}, \emptyset) \geq \check{x} \geq x$.

LEMMA 1.19 (Lemma 2.8 in [**12**]). *If x and y cover z, then $\check{x} = \check{z}$ (resp. $\check{y} = \check{z}$) or $\check{x}$ and $\check{y}$ cover $\check{z}$ and $\check{x} \neq \check{y}$. If z and $\check{x}$ cover x, then $\check{z}$ covers z and $\check{x}$.*

Continuing with the proof Proposition 1.17. If $x = (\emptyset, \underline{j})$, $y_1 = (\emptyset, \underline{j}_1)$, $y_2 = (\emptyset, \underline{j}_2)$ and $z = (x, \emptyset)$. We have that a cover y of y_1 and y_2 is of the form $y = (\emptyset, \underline{j}_3)$. By 1.18 we see that $z \geq \check{y} \geq y$. Hence wonderfulness follows in this case. The last case is when $x = (\emptyset, \underline{j})$ and $y_1 = (\underline{i}, \emptyset)$. From Lemma 1.18 we have $y_1 = \check{x}$ and if $z \geq y_1, y_2$ then z is of type $(\underline{i}', \emptyset)$. So $z \geq \check{y}_2 \geq y_1 = \check{y}$ by Lemmas 1.18 and 1.19. Further by

Lemma 1.19 $\check{y}_2$ covers both y_2 and y_1. Thus we have found a required cover and our poset is wonderful. □

PROPOSITION 1.20 ([**3**]). *Let B be a Hodge algebra on a poset H. If H is a wonderful poset then B is a Cohen-Macaulay ring.*

THEOREM 1.21. *The ring $B_{n,r}$ is Cohen-Macaulay.*

1.3. Grass$(2, K^n)$**.** The cone on Grass$(2, K^n)$ is the variety Y_2 given above. We may express an element of Grass$(2, K^n)$ as a decomposable element of $\mathbb{P}(\bigwedge^2 K^n)$. Given a decomposable tensor $w = v_1 \wedge v_2$ we may associate to w a skew-symmetric matrix M_w in the following manner: express w in our chosen basis, $w = a_{1,2}e_1 \wedge e_2 + a_{1,3}e_1 \wedge e_3 + \cdots + a_{n-1,n}e_{n-1} \wedge e_n$ then define M_w to be the matrix

$$M_w = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ -a_{1,2} & 0 & a_{2,3} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1,n} & -a_{2,n} & -a_{3,n} & \cdots & 0 \end{pmatrix}. \tag{1.14}$$

That M_w is an element of Grass$(2, K^n)$ is equivalent to the condition that the 4×4 Pfaffians of M_w vanish. The equations derived above are in fact the Plücker relations on the coordinates $p_{i,j}$.

We now wish to find equations for $T^*_{\mathrm{Grass}(2,K^n)}\mathbb{P}\bigwedge^2 K^n$. Let $\pi\colon T^*_{\mathrm{Grass}(2,K^n)}\mathbb{P}\bigwedge^2 K^n \to \mathrm{Grass}(2, K^n)$ be the projection map.

Let us consider two anti-symmetric matrices A and B of size n. Assume that we have $AB = \mathbf{0}$. Then we may view A as a point on Grass$(2, K^n)$ and B as a point in $\pi^{-1}(A)$. We see that our first set of equations is that we need the 4×4 Pfaffians of

A to vanish and the traces of coordinates $x_{i,j}$ and $w_{j,k}$ to vanish,

$$\sum_j x_{i,j} w_{j,k} = 0 \qquad \text{for all } i, k. \tag{1.15}$$

We need to find equations in the w's which vanish.

THEOREM 1.22. *The ring $B_{n,r}$ is normal.*

PROOF. In the proof of Theorem 1.11 we showed that the associated graded object $\mathrm{gr}(B_{n,r})$ is a direct sum of Schur functors. Using the flat deformation from section 4, chapter 1 we see that $B_{n,r}$ has a good filtration. Hence by Theorem 4.7 and Proposition 4.8 we find that $B_{n,r}$ is normal. □

THEOREM 1.23. *A and B as above. The ideal defining $T^*_{\mathrm{Grass}(2,K^n)}\mathbb{A}^n_k$ in $k[x_{1,2}, x_{1,3} \ldots, x_{n-1,n}, w_{1,2}, \ldots, w_{n-1,n}]$ is generated by the following elements:*

i) 4×4 Pfaffians of A;

ii) The sums $\sum_{l=1}^n x_{i,l} y_{l,j}$;

iii) $m \times m$ Pfaffians of B. Where m is the largest even number less than n.

2. Conormal Varieties to Rank Varieties for Symmetric Matrices

Let M be a matrix with entries in a ring R. Set $I_r(M)$ to be the ideal of R generated by the $r \times r$ minors of M.

Let E be an n-dimensional K vector space. Identify $A_n = Sym(S_2E) \otimes Sym(S_2E^*)$ with the polynomial ring $K[x_{i,j}, w_{i,j}]$ with x and w symmetric in i, j. Let $M = (x_{i,j})$ and $N = (w_{i,j})$ be matrices with entries from A. Let $\mathcal{J}_r$ be the ideal of A generated by $I_{r+1}(M)$, plus $I_{s+1}(N)$, and all entries of $M \cdot N$, i.e. all sums of the form $\sum_{l=1}^n x_{i,l} y_{l,j}$.

DEFINITION 2.1. Define a new ring $B_{n,r}$ by

$$B_{n,r} := A_n / \mathcal{J}_r. \tag{2.1}$$

DEFINITION 2.2. Let λ and μ be partitions with even columns. Define a map $\rho^S_{\lambda,\mu}$ by

$$\rho^S_{\lambda,\mu} := \rho^S_\lambda \cdot \hat{\rho}^S_\mu. \tag{2.2}$$

Here $\hat{\rho}^S_\mu$ is a map from $\bigwedge^{\mu_1}(S_2E^*) \otimes \bigwedge^{\mu_3}(S_2E^*) \otimes \cdots \otimes \bigwedge^{\mu_{n-1}}(S_2E^*)$ to $S_{\mu_1}(S_2E^*) \otimes S_{\mu_3}(S_2E^*) \otimes \cdots \otimes S_{\mu_{l-1}}(S_2E^*)$.

We use the order defined previously on pairs of partitions (λ, μ) by $(\lambda, \mu) \geq' (\tau, \upsilon)$ if $\lambda \geq_{rev} \tau$ or $\lambda = \tau$ and $\mu \geq_{rev} \upsilon$ where we use the order from Definition 2.19. Let us define a filtration of $B_{n,r}$, $\mathcal{F}_{n,r} = \{F_{\leq'(\lambda,\mu)}\}$ by setting

$$F_{\leq'(\lambda,\mu)} := \sum_{\substack{(\tau,\upsilon)\leq'(\lambda,\mu),\\ |\tau|=2m,|\upsilon|=2n| \text{ both with even columns}}} \mathrm{Im}(\rho_{\tau,\upsilon}). \tag{2.3}$$

2.1. Standard Bases for Coordinate Rings to $T^*_{Y_r}S_2(E)$.

THEOREM 2.3.

(i) $B_{n,r}$ has a basis given by the standard tableaux of shapes $\begin{smallmatrix}\hat{\mu}\\ \lambda\end{smallmatrix}$ with λ, μ having even columns and $\lambda_1 \leq r$, $\mu_1 \leq n-r$.

(ii) $B_{n,r}$ is the coordinate ring of $T^*_{Y_r}S_2(E)$.

PROOF. We again proceed in three steps:

1) The ideal $\mathcal{J}_r$ vanished on $T^*_{Y_r}S_2(E)$;

2) The stacked standard tableaux of shapes $\begin{smallmatrix}\hat{\mu}\\ \lambda\end{smallmatrix}$, with μ and λ having even length columns, generates $B_{n,r}$;

3) The stacked standard tableaux of shapes $\begin{smallmatrix}\hat{\mu}\\ \lambda\end{smallmatrix}$, with μ and λ having even length columns are linearly independent in $B_{n,r}$.

1) Vanishing: The space of cotangent vectors $T^*S_2(E)$ can be identified with pairs of symmetric matrices $((x_{ij}), (w_{ij}))$. Where the x_{ij} is the coefficient of v_iv_j and w_{ij}

is the coordinate of $v_i^* v_j^*$. Here $\{v_i\}$ is a basis for E and $\{v_j^*\}$ a basis for E^*, dual to the $v's$. The conormal space to the symmetric matrices of rank $\leq r$ is a subvariety of $T^*S_2(E)$. Consider the tensor $D_r = \sum_{i=1}^r e_i^2$. This will be our called our distinguished point in $S_2(E)$. The variety Y_r is the $GL(E)$ orbit closure of D_r. As a matrix D_r has the form

$$D_r = \left(\begin{array}{c|c} Id_r & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right). \tag{2.4}$$

LEMMA 2.4. *The tangent space to Y_r at D_r can be identified with the symmetric matrices of the form*

$$T_{D_r} = \left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_2^T & \mathbf{0} \end{array}\right). \tag{2.5}$$

Where M_1 is an $r \times r$ matrix and M_2 is $r \times (n-r)$. The space of conormal vectors to Y_r at D_r can be identified with the symmetric matrix

$$N_{D_r} = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & N_{n-r} \end{array}\right), \tag{2.6}$$

where N_{n-r} is of size $(n-r) \times (n-r)$.

This gives us a a distinguished set

$$Z_{D_r} = \left(\left(\begin{array}{c|c} Id_r & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right), \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & N_{n-r} \end{array}\right) \right). \tag{2.7}$$

We know from the Cauchy formula applied to $S_2(E)$ or $S_2(E^*)$ that a shuffle involving only rows from S or rows from T is a sum of earlier tableaux. Hence the

shuffle vanishes in $gr(\mathcal{F}_{n,r})$. Let T have l parts. Then we are left with showing that a shuffle involving S_1 and $\hat{T}_l$ vanishes.

Let us consider a generic shuffle

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline & \dots & & * & \dots & \dots & * & \dots & * \\ \hline * & \dots & * & \dots & * & & \dots & & \\ \hline \end{array}\ .$$

Let us denote the individual terms in the shuffle by W_i and the sum of the W_i as W. Let us consider each of the W_i first. For W_i to be nonvanishign on Z_{D_r} we need the row corresponding to $\hat{T}_l$ to contain $1, 2, \dots, r$ and the row corresponding to S_1 to have all entries $\leq r$. If no such W_i can be found in W we have that W is zero on Z_{D_r}. If we do find at least one such W_i we see that we are shuffling at most r distinct entries among at least $r+1$ boxes. Hence some entry must be repeated and since shuffles are anti-symmetric in the $*$'s we have that W is zero.

2)Generation:

Remark 2.5. Thus far we have shown there is a surjection

$$L_{\substack{\hat{\mu}\\ \lambda}}(E) \to F_{\leq'(\lambda,\mu)}/F_{<'(\lambda,\mu)} \to 0.$$

3)Linear Independence:

As in Theorem 1.11 we show that the U invariant in each filtered piece is non-vanishing.

Let $\Upsilon_{\substack{\hat{\mu}\\ \lambda}}$ be the U invariant of weight $\substack{\hat{\mu}'\\ \lambda}$. The stacked tableau corresponding to $\Upsilon_{\substack{\hat{\mu}\\ \lambda}}$ has entries $1, 2, \dots, \hat{\mu}_i$, or $1, 2, \dots, \lambda_j$ in each row. Thus the corresponding pair of tableaux (S, T) of shape (λ, μ) has entries $1, 2, \dots, \lambda_i$ in S and $n, n-1, \dots, \mu_j$ in T.

Let us now consider $\rho_{\lambda,\mu}(S, T)(Z_D)$. This is not identically zero because it does not vanish on the set

$$\left(\left(\begin{array}{c|c} Id_r & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right), \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & Id_s \end{array}\right)\right).$$

Hence the U invariant of shape $\overset{\hat{\mu}}{\lambda}$ is non-vanishing.

From this point the remainder is analogous to Theorem 1.11. □

THEOREM 2.6. *The ring $B_{n,r}$ is normal.*

CHAPTER 3

Cominuscule G/P

1. Existence of Good Filtrations

Definition 1.1. Let G, B, and T be as before. Let ω be a cominuscule weight and P the associated parabolic. Define a sheaf $\mathcal{B}(G,P)$ by

$$\mathcal{B}(G,P) = Sym(T^*G/P) \otimes Sym(\mathcal{L}(\omega)).$$

Define a ring $B(G,P)$ by

$$B(G,P) = H^0(G/B; \mathcal{B}(G,P)).$$

Remark 1.2. The ring $B(G,P)$ be thought of as the coordinate ring of the cone over the cotangent variety of G/P. Indeed, denoting $\pi\colon T^*G/P \to G/P$ the structure map, $\mathcal{O}_{T^*G/P} \cong Sym_{\mathcal{O}_{G/P}}(TG/P)$, so its sections give regular functions on the cone over T^*G/P.

Definition 1.3. Define an ordering on the dominant weights $X(T)^+$ by $a_1\omega_1 + \cdots + a_n\omega_n >_{tot} b_1\omega_1 + \cdots + b_n\omega_n$ if $\sum a_i > \sum b_i$ or if $\sum a_i = \sum b_i$ then for the smallest i such that $a_i \neq b_i$ we have $a_i > b_i$. We will call this the *total ordering* on dominant weights.

Recall that in Chapter 1, Theorem 3.12, we listed the graded pieces of $Sym(T^*G/P)$ for the different cominuscule weights. Let $F(G,P)_{\leq_{tot}\lambda}$ be the associated filtration, with $\lambda \in X(T)^+$. We need to remember the assumptions from

Theorem 3.12, mainly that the result is proved for E_6 and E_7 only in characteristic zero.

Our goal in this section is to show that the ring $B(G,P)$ has a good filtration.

We will make use of the following results.

THEOREM 1.4 (Kempf Vanishing Theorem, [**9**]). *Let λ be a dominant weight. Then for $i > 0$ we have*

$$H^i(G/B, \mathcal{L}(\lambda)) = 0.$$

PROPOSITION 1.5 ([**4**], p.683). *Suppose that $\mathscr{A}$ and $\mathscr{B}$ are categories of modules over some rings and $\mathscr{C}$ is a category of abelian groups. Let $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ and $\mathscr{G} : \mathscr{B} \to \mathscr{C}$ be left exact functors. Let $A \in Obj\mathscr{A}$, suppose that A has a resolution by $\mathscr{F}$-acyclic objects that are carried by $\mathscr{F}$ to $\mathscr{G}$-acyclic objects. Then there is an E_2-spectral sequence*

$$E_2^{p,q} = R^p\mathscr{G}(R^q\mathscr{F}(A)) \implies E_\infty^{p,q} = R^{p+q}\mathscr{G}\mathscr{F}(A).$$

Throughout the following proofs we write $F_{\leq_{tot}\lambda} = F_{\leq_{tot}\lambda}(G,P)$.

PROPOSITION 1.6.

$$H^i(G/P, F_{\leq_{tot}\lambda}(G,P) \otimes Sym(\mathcal{L}(\omega))) = 0 \qquad \text{for } i > 0.$$

PROOF. From the fact that we have a total ordering on the weights, we know given a λ there is a μ such that $F_{\leq_{tot}\mu} = F_{<_{tot}\lambda}$.

These terms then fit into an exact sequence

$$H^1(G/P, F_{<_{tot}\mu} \otimes Sym(\mathcal{L}(\omega))) \longrightarrow H^1(G/P, F_{<_{tot}\lambda} \otimes Sym(\mathcal{L}(\omega))) \longrightarrow \quad .$$

$$\longrightarrow H^1(G/P, F_{\leq_{tot}\lambda}/F_{<_{tot}\mu} \otimes Sym(\mathcal{L}(\omega)))$$

Let us now consider the composition of maps

$$G/B \xrightarrow{f} G/P \xrightarrow{g} \{pt\} \ .$$

Then by theorem 1.5 there is a spectral sequence with E_2 term given by

$$E_2^{p,q} = R^p g_* R^q f_* F_{\leq_{tot}\lambda}/F_{<_{tot}\lambda} \otimes Sym(\mathcal{L}(\omega))$$

converging to

$$E_\infty^{p,q} = R^{p+q}(g \circ f)_* F_{\leq_{tot}\lambda}/F_{<_{tot}\lambda} \otimes Sym(\mathcal{L}(\omega)).$$

This last term can be rewritten as

$$E_\infty^{p,q} = H^{p+q}(G/B; F_{\leq_{tot}\lambda}/F_{<_{tot}\lambda} \otimes Sym(\mathcal{L}(\omega)).$$

The map f is a fibration with fiber a product of groups associated to smaller root systems. Hence we induct over the size of the root system and the weight λ. For the trivial root system A_1 the result is clear.

By induction on the size of the root system and the weight we see that

$$E_2^{p,q} = 0 \qquad \text{for } q \geq 1.$$

Now since the E_∞ term is non-zero only for $p = q = 0$ we find that $E_2^{p,q}$ is zero except when $p = q = 0$.

From this our result follows.

□

THEOREM 1.7. *The ring $B(G, P)$ has a good filtration.*

Proof. We have a short exact sequence

$$0 \longrightarrow F_{<_{tot}\lambda}(G,P) \otimes Sym(\mathcal{L}(\omega)) \longrightarrow F_{\leq_{tot}\lambda}(G,P) \otimes Sym(\mathcal{L}(\omega)) \longrightarrow \tag{1.1}$$

$$\longrightarrow \frac{F_{\leq_{tot}\lambda}(G,P)}{F_{<_{tot}\lambda}(G,P)} \otimes Sym(\mathcal{L}(\omega)) \longrightarrow 0.$$

This gives rise to a long exact sequence in cohomology

$$0 \longrightarrow H^0(G/P, F_{<_{tot}\lambda} \otimes Sym(\mathcal{L}(\omega)) \longrightarrow H^0(G/P, F_{\leq_{tot}\lambda} \otimes Sym(\mathcal{L}(\omega))) \longrightarrow$$

$$\longrightarrow H^0(G/P, \frac{F_{\leq_{tot}\lambda}}{F_{<_{tot}\lambda}} \otimes Sym(\mathcal{L}(\omega))) \longrightarrow H^1(G/P_{\omega_n}, F_{<_{tot}\lambda}) \otimes Sym(\mathcal{L}(\omega)) \longrightarrow \cdot$$

By Proposition 1.6 we know this is in fact a short exact sequence

$$0 \longrightarrow H^0(G/P, F_{<_{tot}\lambda} \otimes Sym(\mathcal{L}(\omega)) \longrightarrow H^0(G/P, F_{\leq_{tot}\lambda} \otimes Sym(\mathcal{L}(\omega))) \longrightarrow$$

$$\longrightarrow H^0(G/P, \frac{F_{\leq_{tot}\lambda}}{F_{<_{tot}\lambda}} \otimes Sym(\mathcal{L}(\omega))) \longrightarrow 0.$$

By the definition of L^G_λ our result follows. □

Corollary 1.8. *The following is a complete list of associated graded objects for $B(G,P)$, with P cominuscule, in characteristic zero:*

- A_{n-1}, ω_k

$$gr(B(Sl_n, P_k)) = \bigoplus_{d \geq 0} H^0(Sym^d_{\mathcal{O}_{Sl_n/P_k}}(TSl_n/P_k)),$$

with

$$H^0(Sym^d_{\mathcal{O}_{Sl_n/P_k}}(TSl_n/P_k)) = \bigoplus_{\substack{a_1+2a_2+\cdots+ka_k=d \\ b}} L^{Sl_n}_{a_1(\omega_1+\omega_{n-1})+\cdots+a_k(\omega_k+\omega_{n-k})+b\omega_k}.$$

- $D_n, \omega_{n-1}, \omega_n$ *First note that* $Spin_{2n}/P_{n-1} \cong Spin_{2n}/P_n$.

$$gr(B(Spin_{2n}, P_n)) = \bigoplus_{d\geq 0} H^0(Sym^d_{\mathcal{O}_{Spin(2n)/P_n}}(TSpin(2n)/P_n)),$$

with

$$H^0(Sym^d_{\mathcal{O}_{Spin(2n)/P_n}}(TSpin(2n)/P_n)) = \bigoplus_{\substack{a_1+2a_2+\cdots+sa_s=d \\ b}} L^{Spin_{2n}}_{a_1\omega_{n-2}+a_2\omega_{n-4}\cdots+a_s\omega_{n-2s}+a_n\omega_n}.$$

- E_6, ω_1, ω_6 *Note that* $E_6/P_1 \cong E_6/P_6$.

$$gr(B(E_6, P_6)) = \bigoplus_{d\geq 0} H^0(Sym^d_{\mathcal{O}_{E_6/P_6}}(TE_6/P_6)),$$

with

$$H^0(Sym^d_{\mathcal{O}_{E_6/P_6}}(TE_6/P_6)) = \bigoplus_{\substack{l+2m=d \\ n}} L^{E_6}_{l\omega_4+m\omega_2+n\omega_6}.$$

- E_7, ω_7

$$gr(B(E_7, P_7)) = \bigoplus_{d\geq 0} H^0(Sym^d_{\mathcal{O}_{E_7/P_7}}(TE_7/P_7)),$$

with

$$H^0(Sym^d_{\mathcal{O}_{E_7/P_7}}(TE_7/P_7)) = \bigoplus_{\substack{k+2l+3m=d \\ n}} L^{E_7}_{k\omega_6+l\omega_1+n\omega_7} \otimes W^{\otimes m},$$

where W *is an invariant of* E_6.

THEOREM 1.9.

a) $B(G,P)$ is normal.

b) If the characteristic of K is zero, $B(G,P)$ has rational singularities, i.e. it is Cohen-Macaulay.

Proof.

a) This follows from section 4, chapter 1.

b) The ring $B(G,P)$ has a flat deformation to $\mathrm{gr}(B(G,P))$, which in characteristic zero is a sum of Schur modules. By [**5**] having rational singularities is an open condition on the total space of a flat deformation. Since in each case our coordinate ring deforms to a sum of Schur modules over the lattice span of some fundamental weights we know that T^*G/P deforms to a cone over G'/P' with the weights of P' given in the deformation. We showed in Proposition 1.6 that the higher cohomology over a G'/P' vanishes hence the cone over G'/P' has rational singularities. By openness of rational singularities along a flat family we know that $\widehat{T^*G/P}$ has rational singulartities, hence $B(G,P)$ is Cohen-Macaulay. □

2. Cotangent Bundles of the Cone over the Grassmannians

In this section we make the filtration given in 1.7 explicit for the case A_n. This allows us to prove Cohen-Macaulayness of $B(G,P)$ in arbitrary characteristic. For this section let E be a K vector space of dimension n. We give a presentation of the coordinate ring $K[T^*\mathrm{Grass}(k,E)]$ as a quotient of $K[\mathrm{Grass}(k,E)] \otimes Sym((E\otimes E)^*)$. Picking a basis $\{e_1,\dots,e_n\}$ for E allows us to make the identification $Sym(E\otimes E) \cong K[y_{i,j}]$ by letting $y_{i,j}$ be the coefficient of $e_i \otimes e_j$. On $K[\mathrm{Grass}(k,E)]$ we will use the Plücker coordinates p_I from Section 1.

There is an isomorphism $\delta\colon \mathrm{Grass}(k,K^n) \to \mathrm{Grass}(n-k,(K^n)^*)$ given by $\delta(V) = \{\eta \in (K^n)^* \mid \eta_{|_V} \equiv 0\}$. This is entirely functorial and does not depend upon choices

of basis for K^n or V. If we pick a non-degenerate innerproduct $\langle\cdot,\cdot\rangle\colon E\times E\to K$ we obtain an isomorphism $\delta'\colon \mathrm{Grass}(k,E)\to\mathrm{Grass}(n-k,E)$ defined by $\delta'(V)=\{w\in K^n \mid \langle w,\cdot\rangle_{|V}\equiv 0\}$. This allows us to make the assumption that $k\le n/2$ for the remainder of the section.

Define a vector bundle $\mathcal{R}$ on $\mathrm{Grass}(k,E)$ called the *tautological subbundle* by

$$\mathcal{R}=\{(V,v)\in\mathrm{Grass}(k,E)\times E \mid v\in V\subset E\}. \tag{2.1}$$

Let $\mathcal{E}=\mathrm{Grass}(k,E)\times E$ be the trivial bundle with fiber E. Consider the bundle $\mathcal{Q}$ defined by the sequence

$$0\longrightarrow\mathcal{R}\longrightarrow\mathcal{E}\longrightarrow\mathcal{Q}\longrightarrow 0\ . \tag{2.2}$$

Let $M_V=(a_{i,j})$ be as in equation 1.4. Let $I\subset I_{k,n}$. Set $M_V^I=(a_{i_t,j})_{i_t\in I}$. Notice that $p_I(M_V)=\det(M_V^I)$ and hence if $p_I(M_V)\neq 0$ then M_V^I is invertible.

We will exhibit transition functions for the bundle $\mathcal{R}$.

Let M_V^t be the t-th column of M_V. To M_V, I, and a k vector $(b_1,\dots,b_k)$ we associate a point $(M_F,v)\in\mathcal{R}|_{U_I}$ where v is the vector $v=\sum_{t=1}^k b_t(M_VM_I^{-1})^t$.

Proposition 2.1. *Let $M_V=(a_{i,j})$ be a $n\times k$ matrix. Let $I,J\in I_{k,n}$ and assume that $p_I(M_V)\neq 0$ and $p_J(M_V)\neq 0$. Then the transition function from $U_I\cap U_J\times K^k\to U_J\cap U_I\times K^k$ is given by $M_V^I(M_V^J)^{-1}$, acting on the left.*

Proof. The vector v defined by $M_V(M_V^I)^{-1}$ and $(b_i)_{i=1}^k$ is the product

$$v=M_V(M_V^I)^{-1}\begin{pmatrix} b_1\\ \vdots\\ b_k\end{pmatrix}.$$

To transition from expressing v as a point over U_I to a point over U_J we consider the product

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = M_V^I (M_V^J)^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}.$$

We have

$$M_V (M_V^I)^{-1} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = M_V (M_V^I)^{-1} M_V^I (M_V^J)^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} = M_V (M_V^J)^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}.$$

This is a point in $\mathcal{R}|_{U_J}$. □

We will now exhibit transition functions for the bundle Q^*.

Given a $n \times k$ matrix $M = (a_{i,j})$ let N be a $k \times n$ matrix $N = (b_{j,i})$ such that

$$N \cdot M = (0) \quad \text{and} \quad \mathrm{rk}(N) = k$$

Define N^J to be the $k \times k$ matrix $(b_{j_t,i})_{j_t \in J}$.

Let M_V be a $n \times k$ matrix defining a point V of $\mathrm{Grass}(k, K^n)$. Pick a matrix N_V such that $N_V \cdot M_V = (0)$ and $N^{I'}$ is invertible. Let $C = (c_i)_{i=1}^k$ be a k-tuple of elements from K. Associate to the pair (M_V, C) a point in $\mathcal{Q}^*|_{U_I}$ by $(M_V (M_V^I)^{-1}, C(N_V^{I'})^{-1} N_V)$.

Proposition 2.2. *Let M_V be a $n \times k$ matrix such that $p_I(M_V) \neq 0$ and $p_J(M_V) \neq 0$. Then the transition function from $U_I \cap U_J \times (K^k)^* \to U_J \cap U_I \times (K^k)^*$ is given by $N_V^{J'} (N_V^{I'})^{-1}$ acting on C from the right.*

PROPOSITION 2.3 ([**14**] Proposition 3.3.5). *Let $\mathcal{R}$ and $\mathcal{Q}$ be as above. There are isomorphisms*

$$T\mathrm{Grass}(k,E) \cong \mathrm{Hom}(\mathcal{R},\mathcal{Q}) \cong \mathcal{R}^* \otimes \mathcal{Q}, \tag{2.3}$$

and

$$\Omega = T^*\mathrm{Grass}(k,E) \cong \mathrm{Hom}(\mathcal{Q},\mathcal{R}) \cong \mathcal{Q}^* \otimes \mathcal{R}. \tag{2.4}$$

COROLLARY 2.4. *Let $V \in \mathrm{Grass}(k,E)$ be a point of the Grassmannian. Then*

$$(T\mathrm{Grass}(k,E))_V \cong V^* \otimes (E/V), \tag{2.5}$$

and

$$(T^*\mathrm{Grass}(k,E))_V \cong (E/V)^* \otimes V. \tag{2.6}$$

Let $I = \{1 \le i_1 < i_2 < \cdots < i_k \le n\}$ and consider the open $\mathcal{U}_I \subset \mathrm{Grass}(k,E)$. We want to give a description of $(T^*\mathrm{Grass}(k,E))_{\mathcal{U}_I}$ in terms of affine coordinates. In Section 1 we discussed the affine presentation of $\mathcal{U}_I$ as a matrix with entries $a_{i,j}$ with the conditions $a_{s,i_s} = 1$ and $a_{t,i_s} = 0$ for $t \neq s$.

Let M_V and N_V be as above. Let A and C be two k-vectors. Let I be such that $p_I(M_V) \neq 0$. To M_V, N_V, A, C we associate a point of $(Q^* \otimes R)|_{U_I}$ by $(M_V(M_V^I)^{-1}, (M_V(M_V^I)^{-1}A) \otimes (C(N_V^{I'})^{-1}N_V))$. Taking a transpose, so that C is a column vector we have the point

$$(M, A \otimes C) \mapsto (M_V(M_V^I)^{-1}, (M_V(M_V^I)^{-1}A) \otimes ({}^tN_V^t(N_V^{I'})^{-1}C)).$$

PROPOSITION 2.5. *The transition function from* $(Q^* \otimes R)|_{\mathcal{U}_I \cap \mathcal{U}_J} \to (Q^* \otimes R)|_{\mathcal{U}_J \cap \mathcal{U}_I}$ *is given by* $M_V^I (M_V^J)^{-1} \otimes^t N_V^{I't} (N_V^{J'})^{-1}$

2.1. Affine Chart. To discuss the locally affine structure of $T^*\mathrm{Grass}(k, E)$ note that $\mathcal{Q}^* \otimes \mathcal{R} \subset \mathcal{E}^* \otimes \mathcal{E}$. Choose a basis for E, say $\{e_1, \ldots, e_n\}$ and a dual basis for $\mathcal{E}^*, \{f_1, \ldots, f_n\}$. Define a matrix $(b_{i,j})$ by letting $b_{i,j}$ be the coordinate of $f_i \otimes e_j$ in $E^* \otimes E$. We may then identify $\mathcal{E}^* \otimes \mathcal{E} \cong \mathrm{Grass}(k, E) \times (b_{i,j})$ as $\mathcal{E} = \mathrm{Grass}(k, E) \times E$ is trivial. Let $p\colon T^*\mathrm{Grass}(k, E) \to \mathrm{Grass}(k, E)$ be the projection. Using this identification we give a description of $(T^*\mathrm{Grass}(k, E))_{X_I} := p^{-1}(X_I)$.

Let us describe the fiber $p^{-1}(x_I)$ over the point x_I defined in equation 1.8. To describe $p^{-1}(x_I)$ we need to give a set of matrices mapping $E \to E$ such that E/V is mapped to V. These matrices are not unique, but we can choose the 0 coset in E/V to arrive at a unique set of matrices. With these choices we find a set of matrices $(y_{i,j})$ where $y_{i,j} = 0$ unless $i \in I$ and $j \notin I$.

EXAMPLE 2.6. Let $\dim E = 4$ and consider $\mathrm{Grass}(2, E)$. Let $I = (1, 2)$. The fiber over x_I in $T^*\mathrm{Grass}(2, E)$ is given as a pair of matrices

$$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & y_{1,3} & y_{1,4} \\ 0 & 0 & y_{2,3} & y_{2,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right).$$

If we instead chose $I = (1, 4)$ we would have

$$\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & y_{1,2} & y_{1,3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y_{4,2} & y_{4,3} & 0 \end{pmatrix}\right).$$

The connecting homomorphism in this case is a permutation matrix taking $e_1 \to e_1, e_2 \to e_4, e_3 \to e_3, e_4 \to e_2$.

The the matrices we have identified with the fibers $N = (y_{i,j})$ have the property that $N^2 = (0)$.

2.2. Projectors. Let $M = (x_{i,j})$ be an $n \times n$ matrix over K. For a matrix M we denote the characteristic polynomial $p_M(z) = \sum_{i=0}^{n}(-1)^i s_i(M) z^{n-i}$. Denote the entry in position i, j of M^2 by $p_{i,j} = \sum_{l=1}^{n} x_{i,l} x_{l,j}$.

Let Y_h be the set of $n \times n$ matrices A over K of rank h such that $A^2 = 0$.

THEOREM 2.7 ([**13**] Theorem 1.2). *Let M be the matrix of variables $(x_{i,j})$ from $K[x_{i,j}], 1 \le i, j \le n$. The defining ideal J_h of Y_h in $K[x_{i,j}]$ is generated by the following polynomials:*

(i) *the determinants of the $(h+1)$-order minors of M;*

(ii) *$p_{i,j}$;*

(iii) *$s_i(M)$, $i \le h$.*

Further $K[x_{i,j}]/J_h$ has a basis given by the standard tableaux of shapes $\overset{\hat{\lambda}}{\lambda}$ with $\lambda_1 \le h$.

DEFINITION 2.8. Define a map

$$\xi_{((k^d),\lambda)} \colon \bigwedge^{(k^d)} E \otimes \bigwedge^{\lambda} E \otimes \bigwedge^{\lambda} E^* \to S_d\left(\bigwedge^{k} E\right) \otimes S_{|\lambda|}(E \otimes E^*) \tag{2.7}$$

as $\xi_{(k^d),\lambda} := p_{(k^d)} \otimes \rho_\lambda$. Here $p_{(k^d)}$ is the map of minors of size k and ρ_λ is the map from Definition 2.21.

2.3. Standard Bases for $K[T^*\widehat{\mathrm{Grass}(k,n)}]$. Let $\mathcal{B}_{k,n}$ be the sheaf of rings on $\mathrm{Grass}(k, K^n)$ defined by

$$\mathcal{B}_{k,n} := Sym(\mathcal{Q} \otimes \mathcal{R}^*) \otimes Sym(\bigwedge^k \mathcal{R}^*) \tag{2.8}$$

Let $B(k,n)$ be the ring

$$B(k,n) := H^0(\mathrm{Grass}(k, K^n), \mathcal{B}_{k,n}). \tag{2.9}$$

We will show that $B(k,n)$ is the coordinate ring of the cone over $T^*\mathrm{Grass}(k, K^n)$.

Lemma 2.9. *The ring $\mathcal{B}(k,n)$ has a natural filtration with associated graded object*

$$\bigoplus_{\substack{\lambda,d \\ \lambda_1 \leq k}} L_\lambda \mathcal{Q} \otimes L_\lambda \mathcal{R}^* \otimes L_{(k^d)} \mathcal{R}^*. \tag{2.10}$$

Definition 2.10. Let T and U be tableaux of shape $\lambda = (\lambda_1, \ldots, \lambda_l)$ and S a tableau of shape (k^d) with $\lambda_1 \leq k$. We form a new tableau $\begin{smallmatrix} \hat{T} \\ S \\ U \end{smallmatrix}$ of shape $\begin{smallmatrix} \hat{\lambda} \\ (k^d) \\ \lambda \end{smallmatrix}$ called the *stacked triple tableau* of (S, T, U).

Remark 2.11. Let R be a tableau of shape $\begin{smallmatrix} \hat{\lambda} \\ (k^d) \\ \lambda \end{smallmatrix}$. In order to view R as a monomial on $T^*\mathrm{Grass}(k, E)$ we need to keep track of d. This will tell us how to split up R as a product of Plücker coordinates and minors of $(y_{i,j})$. The reason we need to keep track of d is that we can have partitions $\lambda \neq \mu$ and integers $d \neq d'$ such that the Young frames $\begin{smallmatrix} \hat{\lambda} \\ (k^d) \\ \lambda \end{smallmatrix}$ and $\begin{smallmatrix} \hat{\mu} \\ (k^{d'}) \\ \mu \end{smallmatrix}$ are the same.

Definition 2.12. Define an order on triples of shapes $((k^d), \lambda, \lambda) >_{part} ((k^e), \mu, \mu)$ if $d > e$ or if $d = e$ then either $\sum \lambda_i > \sum \mu_j$ or $\lambda <_{lex} \mu$.

DEFINITION 2.13. Let (S, T, U) be a stacked triple of tableaux of shape $((k^d), \lambda, \lambda)$ and (S', T', U') a stacked triple of tableaux of shape $((k^e), \mu, \mu)$. Define an order $\leq_{trip}$ on the set of stacked triples of tableaux (S, T, U) by $(S', T', U') <_{trip} (S, T, U)$ if $((k^e), \mu, \mu) <_{part} ((k^d), \lambda, \lambda)$ or $((k^e), \mu, \mu) = ((k^d), \lambda, \lambda)$ and $\begin{smallmatrix} \hat{T}' \\ S' \\ U' \end{smallmatrix} \leq \begin{smallmatrix} \hat{T} \\ S \\ U \end{smallmatrix}$ as tableaux.

DEFINITION 2.14. Define a filtration on $\mathcal{F} = \{F_{\leq_{part}((k^d),\lambda)}\}$ of $B(k, n)$ with d a positive integer and λ a partition with $\lambda_1 \leq k$ by

$$F_{\leq_{part}((k^d),\lambda)} = \sum_{(k^e,\mu) \leq_{part} (k^d,\lambda)} \operatorname{Im} \xi_{((k^e),\mu)}. \tag{2.11}$$

THEOREM 2.15.

(i) The ring $B(k, n)$ has a standard basis given by stacked tableaux of shapes $\begin{smallmatrix} \hat{\lambda} \\ (k^d) \\ \lambda \end{smallmatrix}$.

(ii) $B(k, n)$ is the coordinate ring of $T^*\mathrm{Grass}(k, E)$.

PROOF. We do the proof in three parts:

1) Vanishing of shuffles of stacked triples of tableaux;

2) Generation by stacked triples;

3) Linear independence of stacked triple tableaux.

By an abuse of notation we will identify triples of tableaux (S, T, U) with the corresponding monomials. Given a point $(\underline{p}, \eta) \in T^*\mathrm{Grass}(k, E)$ by $(S, T, U)(\underline{p}, \eta)$ we mean the minors given by S on $\underline{p}$ times the minor of η given by $\rho_\lambda(U, T)(\eta)$.

Let us consider the fiber over the point $(Id_k \mid \mathbf{0}) \in \mathrm{Grass}(k, E)$. In matrix form we have the pair of matrices

(2.12)

$$(\underline{p}, H) = \left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 & y_{1,k+1} & \cdots & y_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & y_{k,k+1} & \cdots & y_{k+1,n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \right)$$

By $GL(E)$ equivariance it is enough to show that all shuffles vanish on the set $(\underline{p}, H)$.

Let (S, T, U) be a triple of tableaux of shape $((k^d), \lambda, \lambda)$. Let us assume that λ has l rows.

Let us consider a shuffle

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & \ldots & & * & \ldots & \ldots & * & \ldots & * \\ \hline * & \ldots & * & \ldots & * & & \ldots & & \\ \hline \end{array} .$$

The cases when the shuffle involves only rows from T or rows from U are covered by 2.7.

Hence it is enough to show vanishing when our shuffle involves either $\hat{T}_l$ and S_1 or S_d and U_1. We will show $\theta(l, u-1, k-u, \begin{smallmatrix} \hat{T} \\ S \\ U \end{smallmatrix}) = 0$ and $\theta(l+d, u-1, \lambda_1 - u, \begin{smallmatrix} \hat{T} \\ S \\ U \end{smallmatrix}) = 0$ for all u.

Let $W = \sum_i W_i$ be a shuffle of the form $\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & \ldots & & * & \ldots & \ldots & * & \ldots & * \\ \hline * & \ldots & * & \ldots & * & & \ldots & & \\ \hline \end{array}$ with individual terms $W_i = \begin{smallmatrix} \hat{T}_l^i \\ S_1^i \end{smallmatrix}$. If for all i we see that W_i has either some entry of S_1^i larger than d or there is some $1 \leq t \leq d$ with $t \notin \hat{T}_l^i$, then W is zero on $(\underline{p}, H)$. Let us then assume that there is some W_i such that all values $1, 2, \ldots, d$ occur in $\hat{T}_l^i$ and all entries of S_1^i are less than or equal to d. Then there must be some repeated entry among the $*$'s. Hence W is zero.

Let us now assume that we are considering a shuffle between S_d and U_1. For $S_d^i(\underline{p})$ to be not zero we need that $S_d^i = \boxed{1}\boxed{2}\boxed{..}\boxed{d}$ and all entries of U_1 must be less than or equal to d. Hence if no $W_i = \begin{smallmatrix} S_d^i \\ U_1^i \end{smallmatrix}$ satisfies these conditions we see that W vanishes. If some W_i does satisfy these conditions we notice that we are shuffling at most d distinct entries among at least $d+1$ boxes. Hence by antisymmetry of the shuffle we see that W is zero.

2) Generation: Thus far we have shown there is a surjection

$$L_{\substack{\hat{\lambda} \\ (k^d) \\ \lambda}} \to F_{\leq_{part}((k^d),\lambda)}/F_{<_{part}((k^d),\lambda)} \to 0.$$

Hence the stacked standard triples of talbleaux generate $B(k,n)$.

3) Indpenendence: The independence of the standard triples of tableaux follows in a manner similar to Theorems 1.11 and 2.3.

Let $\Upsilon_{\substack{\hat{\lambda} \\ (k^d) \\ \lambda}}$ be the U invariant of weight $\begin{smallmatrix} \hat{\lambda} \\ (k^d) \\ \lambda \end{smallmatrix}'$. Let $\tau = \begin{smallmatrix} \hat{\lambda} \\ (k^d) \\ \lambda \end{smallmatrix}$. The corresponding tableau has entries $1, 2, \ldots, \tau_i$. We see then that the corresponding triple of tableaux (S,T,U) with entries in S and U being $1, 2, \ldots, k$ or $1, 2 \ldots, \lambda_i$ in each row and the entries of T are $n, n-1, \ldots, n-\lambda_i+1$. Consider the evaluation of (S,T,U) on the point

$$\left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}\right).$$

The U invariant $\Upsilon_{\substack{\hat{\lambda} \\ (k^d) \\ \lambda}}$ takes the value 1 on the above set. Hence $\Upsilon_{\substack{\hat{\lambda} \\ (k^d) \\ \lambda}}$ does not vanish on all of $T^*\mathrm{Grass}(k,n)$.

The associated graded object is

$$gr(\mathcal{F}) = \bigoplus_{d,\lambda} L_\lambda \mathcal{Q} \otimes L_\lambda \mathcal{R}^* \otimes L_{(k^d)} \mathcal{R}^*. \tag{2.13}$$

We have shown there is a surjection

$$L_{\substack{\hat{\lambda} \\ (k^d) \\ \lambda}} E \to F_{\leq_{part}((k^d),\lambda)} / F_{<_{part}((k^d),\lambda)} \to 0. \tag{2.14}$$

Let $N(d,\lambda)$ be the kernel of the map

$$N(d,\lambda) \to L_{\substack{\hat{\lambda} \\ (k^d) \\ \lambda}} \to F_{\leq_{part}((k^d),\lambda)} / F_{<_{part}((k^d),\lambda)} \to 0.$$

We want to show that $N(d,\lambda) = 0$. If $N(d,\lambda)$ is non-zero then there is a relation among the standard tableaux of shape $\substack{\hat{\lambda} \\ (k^d) \\ \lambda}$.

Let us proceed by induction. The base case is similar to those in the previous proofs. Let us then assume that the standard tableaux of shapes $\substack{\hat{\mu} \\ (k^{d'}) \\ \mu} <_{rev} \substack{\hat{\lambda} \\ (k^d) \\ \lambda}$ are linearly independent. We know $F_{\leq_{part}((k^d),\lambda)} / F_{<((k^d)_{part},\lambda)}$ has a U invariant of weight $\substack{\hat{\lambda} \\ (k^d) \\ \lambda}'$. We know that $L_{\substack{\hat{\lambda} \\ (k^d) \\ \lambda}} E$ has a U invariant of the same weight. By Lemma 3.9 we know that all weights of $N(d,\lambda)$ are less than $\substack{\hat{\lambda} \\ (k^d) \\ \lambda}'$. Thus the standard tableaux are linearly independent.

□

Corollary 2.16. *The filtration* $\{F_{\leq_{part}((k^d),\lambda)}\}$, *defined in equation 2.11, is a good filtration.*

2.4. Arithmetic Cohen-Macaulayness of $T^*\mathrm{Grass}(k,n)$ in arbitrary characteristic.

DEFINITION 2.17. Let E be an n-dimensional K vector space. By the *partial flag variety* $\mathrm{Flag}(E; a_1, a_2, \ldots, a_r)$ we mean the space of all flags $\{0\} \subset E_{a_1} \subset \cdots \subset E_{a_r} \subset E$ where $\dim E_{a_i} = a_i$.

THEOREM 2.18 ([**6**]). *The coordinate ring of the cone over the incomplete flag variety* $\mathrm{Flag}(E; a_1, \ldots, a_r)$ *is Cohen-Macaulay.*

Theorem 2.15 we have a SL_n filtration of $B(k,n)$ with associated graded algebra a direct sum of Schur modules.

THEOREM 2.19. *For $k \neq n/2$ the ring $B(k,n)$ is arithmetically Cohen-Macaulay.*

PROOF. By Theorem 4.3 there is a flat family with generic fiber isomorphic to $B_{n,k}$ and special fiber $gr(\mathcal{F})$. The ring $gr(\mathcal{F})$ is isomorphic to the coordinate ring on the cone of the incomplete flag variety $\mathrm{Flag}(E; 1, 2, \ldots, k, n-k, n-k+1, \ldots, n)$. By Theorem 2.18 we know the special fiber is Cohen-Macaulay. Hence by Theorem 4.9 we have that $B(k,n)$ is Cohen-Macaulay. $\square$

When $k = n/2$, i.e. we are considering the coordinate ring to $T^*\widehat{\mathrm{Grass}(k,2k)}$, there are repetitions in the terms of the filtration.

3. The Isotropic Grassmannian

DEFINITION 3.1. Let E be a k vector-space of dimension $2n$. Let $\langle \cdot, \cdot \rangle \colon E \times E \to k$ be a symmetric bilinear form. Choose a basis of E, $\{e_1, e_2, \ldots, e_{2n-1}, e_{2n}\}$ such that $\langle e_i, e_j \rangle = \delta_{i,2n-j+1}$. Such a basis is called a *hyperbolic basis*.

DEFINITION 3.2. Let $V \subset E$ be a subspace. We say that V is an *isotropic subspace* if the form $\langle \cdot, \cdot \rangle$ vanishes identically on V.

The following is a well known consequence of the Witt Theorem.

PROPOSITION 3.3. *The maximum dimension of an isotropic subspace is n. Further every isotropic subspace is contained in one of maximal dimension.*

Let F be the subspace of E spanned by $\{e_1, \ldots, e_n\}$.

DEFINITION 3.4. Let $\mathrm{IGrass}(n) = \{V \in \mathrm{Grass}(E, n) \mid \langle V, V \rangle \equiv 0\}$. The set $\mathrm{IGrass}(n)$ is a subvariety of $\mathrm{Grass}(E, n)$ given by the vanishing of the bilinear form. The space $\mathrm{IGrass}(n)$ has two connected components. Let $\mathrm{IGrass}(n)_0$ be the component containing the subspace F.

3.1. The Good filtration. Let $\mathcal{B}(Spin_{2n}, P_{\omega_n})$ be the sheaf of rings on $T^*\mathrm{IGrass}(n)_0$

$$\mathcal{B}(Spin_{2n}, P_{\omega_n}) := Sym(\Omega \otimes \mathcal{L}(\omega_n)). \tag{3.1}$$

Let $B(Spin_{2n}, P_{\omega_n})$ be the ring

$$B(Spin_{2n}, P_{\omega_n}) := H^0(\mathrm{IGrass}(n)_0, \mathcal{B}(Spin_{2n}, P_{\omega_n})). \tag{3.2}$$

THEOREM 3.5. *The ring $B(Spin_{2n}, P_{\omega_n})$ has a natural filtration with associated graded object*

$$\bigoplus_{a_1, \ldots, a_s, a_n} L^{Spin_{2n}}_{a_1\omega_{n-2} + a_2\omega_{n-4} \cdots + a_s\omega_{n-2s} + a_n\omega_n} \tag{3.3}$$

where $s = \lfloor (n-1)/2 \rfloor$.

THEOREM 3.6 ([**1**]). *The variety $Spin_{2n}/B$ is arithmetically Cohen-Macaulay.*

THEOREM 3.7. *The ring $B(Spin_{2n}, P_{\omega_n})$ is Cohen-Macaulay and normal in all characteristics.*

PROOF. By Theorem 4.3 of chapter 1 we know that there is a flat deformation from $B(Spin_{2n}, P_{\omega_n})$ to the associated graded object $gr(B(Spin_{2n}, P_{\omega_n})$ in Theorem 3.5. The ring $gr(B(Spin_{2n}, P_{\omega_n})$ is the coordinate ring of $Spin_{2n}/B$. By Theorem 3.6 we know that the special fiber of the deformation is Cohen-Macaulay. Hence by Theorem 4.9 $B(Spin_{2n}, P_{\omega_n})$ is Cohen-Macaulay.

To see normality we only need to note that $gr(B(Spin_{2n}, P_{\omega_n}))$ is a sum of Schur modules, hence is a good filtration of $B(Spin_{2n}, P_{\omega_n})$. Then applying Theorem 4.7 we find that $B(Spin_{2n}, P_{\omega_n})$ is normal. $\square$

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