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FINITE ELEMENT FORMULATION FOR THE NUMERICAL SOLUTION OF THREE-DIMENSIONAL NONLINEAR MAGNETOSTATIC FIELD PROBLEMS AS APPLIED TO THE DESIGN OF ELECTRIC MACHINERY

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Summary

Recent years have witnessed considerable research activity in the application of digital-computer methods for the determination of the electromagnetic fields in electrical machinery through the solution of Maxwell's equations, while taking full account of the magnetic saturation. Two distinct numerical approaches are evident in the literature: Finite-Difference Method and Finite-Element Method. The author has presented in the recent years a finite-difference formulation for 3-dimensional numerical solutions of the nonlinear electromagnetic field problems in terms of potential functions, and has applied for the analysis of the end-zone fields of aerospace homopolar alternators and solid-rotor induction motors. The present work is directed towards the finite-element formulation for the numerical solution of three-dimensional nonlinear magnetostatic field problems. A variational principle is developed here utilizing the vector potential concept. The approach is based on variational methods in which a correspodning energy functional for the nonlinear case is minimized over the entire region. The minimization is performed by means of the finite-element method and the resultant set of nonlinear algebraic equations is solved through iterative schemes.

1. Introduction

An accurate knowledge of the magnetic as well as other field distributions is of great importance in design optimization. The continuing rapid increase in the rating of the generating plant and the consequent increased electric and magnetic loadings of electrical machines are forcing manufacturers to work

with greatly reduced safety factors in designing alternators. In order to meet the specifications of an electrical apparatus, the designer must be able to analyze and modify several fields of interest (such as an electric field, a magnetic field, a current field, a field of mechanical forces, a field of heat flow, a field of elastic forces in the material stressed by mechanical forces, a field of fluid flow of the cooling medium, etc.) with the ultimate aims of safety, reliability, simplicity, efficiency and economy. The accurate knowledge of these fields is becoming more and more essential for the machine designer of the present day to meet the challeneges of his profession.

During the past fifteen years, the numerical methods have been greatly stimulated by the advent of high-speed digital computational robots which have made routine solutions possible to a high degree of accuracy for many types of problems for which the solution would otherwise be extremely or even prohibitively laborious. There has been considerable research activity for the numerical analysis and determination of the electromagnetic fields in electrical machinery through the solution of Maxwell's equations, while taking full account of the magnetic saturation. Since rigorous solution of the field problem by analytical methods is limited to simple geometric shapes of the region of interest and boundaries, and idealized magnetization characteristics of the ferromagnetic material parts, engineers and designers have taken recourse to numerical methods.

Concepts of both scalar and vector magnetic potentials have been used for computing the magnetic fields. Two distinct numerical approaches are evident in the literature: Finite-Difference Method and Finite-Element Method. The first one replaces the relevant nonlinear partial differential equations by a set of difference equations through the use of finite-difference techniques, and the resultant set of nonlinear algebraic equations is solved through iterative procedures consisting of relaxation techniques. The second approach is based on variational methods, in which a corresponding energy functional

for the nonlinear case is minimized over the entire region. The minimization is performed by means of the finite-element method and the resultant set of nonlinear algebraic equations is solved through iterative schemes such as Newton-Raphson method.

Literature survey reveals the development of a number of digital computer programs for the solution of the general nonlinear electromagnetic field problem in two dimensions[1,2]. However, the nonlinear problem for general three-dimensional geometries has not yet been solved efficiently and extensive research work needs to be done in that direction. Three-dimensional field determination is being sought these days for a great number of important engineering problems, including the end zones of conventional and superconducting alternators, solid-rotor induction machines, heat flow in large turboalternators, and design of special-purpose magnets. The author has presented[3] in the recent years a finite-difference formulation for three-dimensional numerical solutions of the nonlinear electromagnetic field problems in terms of potential functions, and has applied for the analysis of the end-zone fields of aerospace homopolar alternators[4] and solid-rotor induction motors.[5] Computer storage limitations as well as the length of computational time required for reasonable convergence seem to be the present limiting factors. It is the purpose of this paper to present a finite-element formulation for the numerical solution of three-dimensional nonlinear magnetostatic field problems. The relative merits of finite-difference and finite-element methods are yet to be explored, after the application of these formulations to a sufficient number of significant realistic engineering field problems.

2. Formulation

The fundamental laws governing all electromagnetic fields can be expressed by Maxwell's equations. Magnetostatic fields excited by impressed current

density sources require the application of only two of these four laws given below:

$$\nabla\cdot\bar{B} = 0 \; ; \; \nabla x\bar{H} = \bar{J} \qquad (1,2)$$

and the constituent relation is given by

$$\bar{B} = \mu\bar{H} \qquad (3)$$

where $\bar{B}$ is the magnetic induction, $\bar{H}$ the magnetic field intensity, $\bar{J}$ the current density, and μ the permeability of the ferromagnetic material. All materials will be taken to be isotropic and homogeneous. Hysteresis effects will be neglected, thereby making the magnetization characteristic single-valued.

The solution of field problems that include discrete current carrying regions demands the use of the vector potential concept. Since the divergence of the curl of any vector function is zero, one can express

$$\bar{B} = \nabla x\bar{A} \qquad (4)$$

where $\bar{A}$ is the magnetic vector potential. From the above equations, one obtains

$$\nabla x\{\nu(\nabla x\bar{A})\} - \bar{J} = 0 \qquad (5)$$

where ν is the reluctivity, defined as the reciprocal of the magnetic permeability μ. Eq. (5) is then the fundamental equation of the vector potential of the magnetostatic field. The general boundary conditions to be satisfied at the interfaces of stationary dissimilar media may be derived from the limiting integral forms of the Maxwell's equations, and are given by

$$B_{n1} - B_{n2} = 0 \; ; \; H_{t1} - H_{t2} = 0 \; ; \; J_{n1} - J_{n2} = 0 \qquad (6,7,8)$$

where subscripts 1 and 2 indicate the media under consideration, and n and t denote the normal and tangential components respectively. It is possible that a surface current density J_s may exist at the interface, when one of the regions has infinite conductivity; in such a case Eq. (7) needs to be modified accordingly.

Choosing V to be a closed simply connected region, the magnetic energy functional

in V may be defined as

$$F = \iiint_V \left(\int_0^B \nu\, b db\right) dv - \iiint_V (\bar{J} \cdot \bar{A})\, dv \qquad (9)$$

The true functional formulation for the nonlinear field problem is that for which the Euler equation yields the nonlinear partial differential equation (5). Introducing a new function W such that

$$\frac{dW}{d\{(\nabla \times \bar{A}) \cdot (\nabla \times \bar{A})\}} = \nu \qquad (10)$$

one can express the energy functional as follows:

$$F = \frac{1}{2} \iiint_V (W - 2\bar{J} \cdot \bar{A})\, dv \qquad (11)$$

The first variation of F must vanish for F to be stationary in V.

$$\delta F(\bar{A}) = 0 \qquad (12)$$

Through the use of vector identities and divergence theorem, one can obtain

$$\delta F(\bar{A}) = \iiint_V [\nabla \times \{\nu(\nabla \times \bar{A})\} - \bar{J}] \cdot \delta\bar{A}\, dv + \oiint_S [\{\nu(\nabla \times \bar{A})\} \times \bar{n}] \cdot \delta\bar{A} ds = 0 \qquad (13)$$

From the fundamental lemma of the calculus of variations, each bracketed term in the above must vanish independently. Thus

$$\nabla \times \{\nu(\nabla \times \bar{A})\} - \bar{J} = 0 \text{ in } V \qquad (14)$$

and

$$\{\nu(\nabla \times \bar{A})\} \times \bar{n} = 0 \text{ on } S \qquad (15)$$

where S is the closed surface bouding the volume V and $\bar{n}$ is the unit outward normal vector to the surface S.

Eq. (5) is the Euler equation of the functional defined by Eq. (9) or Eq. (11), satisfying homogeneous Dirichlet-and Neumann-type boundary conditions if no others are specified. The energy functional may be chosen in a modified form given below

$$F = \frac{1}{2} \iiint_V (W - 2\bar{J} \cdot \bar{A})\, dv + \oiint_S \bar{C} \cdot \bar{A}\, ds \tag{16}$$

so as to satisfy a more general boundary condition of the type

$$\{\nu(\nabla \times \bar{A})\} \times \bar{n} + \bar{C} = 0 \text{ on } S \tag{17}$$

3. Finite-Element Equations

Finite-element equations approximating the variational principle using Eq. (11) are then developed[6]. The region V is divided into a finite number of subregions, called elements, such that no two elements intersect and the union of all the elements constitutes the entire region V and its bounding surface S. To start with, the elements are chosen to be triangular prisms. In the usual manner, it is assumed that the variation of $\bar{A}$ over a given element and its surface is given by

$$\bar{A}^e\,(x,y,z) = \sum_i N_i\, A_i \tag{18}$$

where the index i ranges over the vertices (1 through 6) of the triangular prism, and x,y,z are the rectangular right-handed coordinates of a Cartesian system. By substituting Eq. (18) into Eq. (11), and requiring that the partial derivative of F with respect to each of the nodal values vanish, one can obtain a set of nonlinear algebraic equations of the matrix-form

$$[S]\,[A] = [J] \tag{19}$$

where S is the coefficient symmetric square matrix, A is the column vector of potentials, and J is the column vector of current densities. For a quasi-linear and uniformly elliptic equation (5), one obtains a positive-definite banded matrix S, whose band structure is determined by the form chosen for the localization functions N_i.

4. Solution of Nonlinear Equations

Having represented the nonlinear continuum problem by a set of nonlinear

algebraic equations, their solution needs to be found. The nature of the combined coefficient matrix S for the entire region depends not only on the shape and size of each triangular prism, but also on the reluctivities which are functions of $\bar{A}$. The order of the matrix S equals the number of vertex potentials in the region, and it should be borne in mind that $\bar{A}$ has in general all three components at each of the vertices. The sparse nature as well as the symmetric and band-structured properties of the coefficient matrix S may be taken advantage of in solving the resultant set of equations.

Based on the author's previous experience with iterative solution methods, it is proposed to solve the resultant nonlinear system of equations essentially in two steps. In each iteration, the vector potentials at each of the nodal points will be calculated using the reluctivities calculated during the previous iteration by successive point relaxation method; later, the reluctivities will be recomputed using the newly claculated vector potentials and will be under-relaxed. The newly proposed method [7] of computing an appropriate optimum relaxation factor for the reluctivity at each nodal point depending on its location on the magnetization characteristic may be implemented. Newton-Raphson iterative scheme [8] may also be employed with advantage. There are other methods of solution such as nonlinear successive over-relaxation used by some researchers [9] in solving two-dimensional magnetostatic problems. Whether such a method would be an improvement over the suggested method of the author could only be decided after having experience with different methods as applied to several three-dimensional field problems.

5. Representation of the Field Solutions

In two-dimensional fields, it is easy to show that contours of constant magnetic vector potential are the flux lines. But similar conclusions cannot be drawn in three-dimensional field problems. That is the reason $\bar{B}$ has to be computed

from $\bar{A}$ with the use of Eq. (4). The representation of the three-dimensional results of the flux densities is extremely complicated by itself and one may have to limit the regions of representation, before any meaningful pictures can be drawn. With the advent of the three-dimensional plotting routines available on the modern computing systems and with some ingenuity, it should be possible to plot meaningful pictures such that surface contours of constant $|B|$ could be conceived and a good physical picture of the flux distribution is given to the designer. Much innovative work needs to be done in this direction.

6. Concluding Remarks

The techniques presented here are of a fundamental nature and can be applied for three-dimensional analyses of various kinds of fields satisfying analogous mathematical relations. The particular system of equations that is presented here can be used as a general mathematical and computer aid for developing, modifying, and optimizing the designs of various parts of electrical machinery from an electromagnetic field point of view. The three-dimensional field distribution can be rather accurately determined with the use of modern digital computers, while taking into account the intricate shapes of the windings and boundaries having different values of varying permeability. The finite-element formulation in three space variables needs to be extended to take care of the induced eddy currents in magnetic structures.

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