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Transitivity of graphs associated with highly symmetric polytopes

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Transitivity of Graphs Associated with Highly Symmetric Polytopes

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ABSTRACT OF DISSERTATION

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Abstract

This dissertation deals with highly symmetric abstract polytopes. Abstract polytopes are combinatorial structures which generalize the classical notion of convex polytopes. There are many different natural graphs associated with an abstract polytope. Particular attention is given to two of these graphs, namely the comparability graph of a polytope and the Hasse diagram of a polytope. We study various types of transitivity in these graphs.

Both the comparability graph and the Hasse diagram for a polytope inherit nicely the rank function associated with a polytope. Using this rank function, we can study specific subgraphs of each graph, by restricting to vertices in a certain range of ranks. If a polytope is of even rank, there is a natural notion of a "medial layer graph," which is the restriction of either the comparability graph, or the Hasse diagram to the middle two ranks. However, if a polytope is of odd rank, and we restrict to the middle three ranks, then we obtain two different graphs, one from the comparability graph, and one from the Hasse diagram.

We study various transitivity properties of these classes of graphs associated with polytopes. We are able to establish polytope properties that are necessary for a graph to have certain transitivities. And conversely, if we require a medial layer graph to have some transitivity, we can understand combinatorial properties of the associated polytope.

In particular, Monson and Weiss proved a theorem in 2005 about arc transitivity of medial layer graphs of 4-polytopes; we prove an analog of that theorem for polytopes of arbitrary even rank. In odd rank, transitivity of the two possible medial layer graphs had not been studied extensively in the past. We provide some results for polytopes of arbitrary rank, as well as a classification for polytopes of rank 3.

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Chapter 1

Introduction

Our purpose in this introductory chapter is to set the scene for the rest of the dissertation. The chapter is broken into four parts. We first give a brief historical sketch of the development of the theory of polytopes. Next, we give the basic definitions and properties related to the study of Abstract Polytopes, and some examples with small rank. Then we demonstrate the relationship between some highly symmetric polytopes and their groups. Finally, we give the basic definitions and properties needed relating to the study of Algebraic Graph Theory.

1.1   Historical Background

The study of polytopes and their symmetries has a rich history dating as far back as the time of the ancient Greeks. For the Greeks, a regular solid was an object with convex regular polygons (all of the same type) as faces, arranged with the same number of them at every vertex. The discovery of the five regular solids was attributed to Pythagoras, and in Euclid's *Elements* there is a proof that demonstrates that these five well known Platonic solids are indeed the only ones (see Figure 1.1).
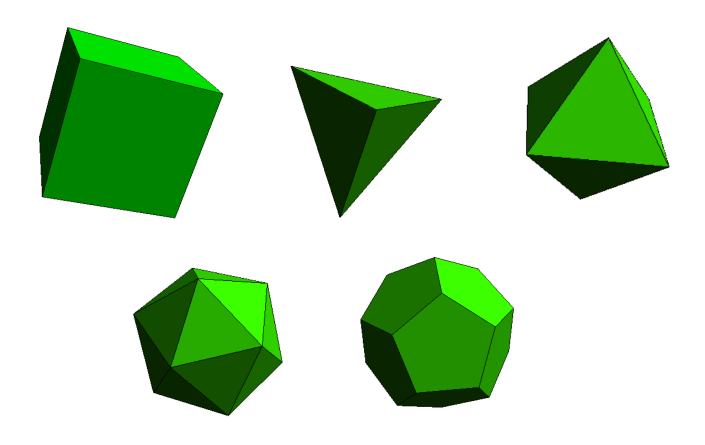
Figure 1.1: The Platonic Solids

In the late sixteenth century, Johannes Kepler began the modern investigation of regular polytopes with the discovery of two new regular polyhedra. The faces of these polyhedra are no longer convex, but are star-shaped, and as a result the points where more than two faces intersect need not be vertices of the polyhedron. In the early nineteenth century Louis Poinsot rediscovered these two regular star-polyhedra, and found two more regular polyhedra (which are dual to Kepler's). These polyhedra had convex faces, but the arrangement of the faces around each vertex (such an arrangement is called a vertex-figure) were star-shaped (see Figure 1.2). Shortly

after Poinsot's polyhedra were discovered, Cauchy proved that the list of such regular star-polyhedra in Euclidean 3-space was complete.
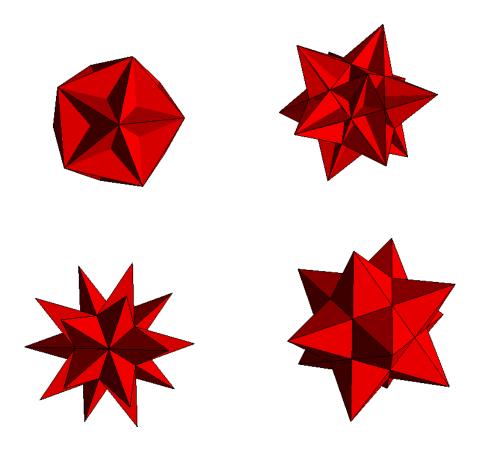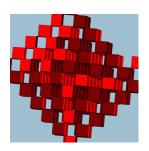
Figure 1.2: The Kepler-Poinsot star-polyhedra

Until the nineteenth century, the study of regular polytopes is limited, almost entirely, to objects in two and three dimensions. However, around 1850 Ludwig Schläfli was able to discover regular polytopes and honeycombs in four and more dimensions. By the beginning of the twentieth century S.L. van Oss proved that the enumeration of such regular polytopes in higher dimensions was complete.

In the early 1920's by allowing the vertex-figures to be non-planar polygons and

the polyhedra to be infinite, Petrie discovered two new regular polyhedra (called skew apeirohedra). When Petrie shared his discovery with H.S.M (Donald) Coxeter, Coxeter immediately found the third skew apeirohedra and proved the enumeration was complete (see Figure 1.3).
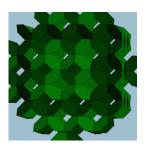
Figure 1.3: The Petrie-Coxeter skew apeirohedra

In 1975 Grünbaum proposed the study of slightly more general polyhedra, by allowing skew polygons as faces as well as vertex-figures. He found eight more examples and twelve infinite families, and Dress completed the classification by discovering one final case and proving the enumeration was complete.

In the classical theory of higher dimensional polytopes, the proper faces of maximal dimension and the vertex-figures are spherical. Danzer and Schulte relax this condition and set out the basic theory for the combinatorial objects which are known as abstract polytopes.

1.2 Abstract Polytopes

We begin this chapter by outlining some of the basic ideas concerning abstract polytopes and their symmetries. While our definition will be combinatorial, we keep in mind that these abstract polytopes are motivated by geometric objects, and thus we will point out the connection between the combinatorial theory and the geometry.

An *abstract polytope* $\mathcal{P}$ of (finite) rank d (≥ -1), or for brevity, an *abstract d-polytope*, is a partially ordered set of faces with the following four properties.

(P1) $\mathcal{P}$ contains a least face and a greatest face (which we will call F_{-1} and F_d respectively).

(P2) Each maximal chain (called a flag) of $\mathcal{P}$ contains $d+2$ faces (including F_{-1} and F_d).

(P3) $\mathcal{P}$ is strongly connected.

(P4) For each $i = 0, 1, ..., d - 1$, if F and G are incident faces of $\mathcal{P}$, of ranks $i - 1$ and $i + 1$ respectively, then there are precisely two $i-$faces H of $\mathcal{P}$ such that $F < H < G$.

The meaning of properties (P1) and (P2) is clear. In particular, if $\mathcal{P}$ is the face lattice of a convex polytope ordered by inclusion, then (P1) says that the empty set and the polytope itself are faces, and (P2) says that each face of dimension $r < d$ is contained in some face of dimension $r + 1$. To clarify (P3) and (P4) more discussion is needed.

From the properties (P1) and (P2), one can show that $\mathcal{P}$ has a natural rank function, with $\mathrm{rank}(F_{-1}) = -1$ and $\mathrm{rank}(F_d) = d$.

Definition 1. For any two faces F and G of $\mathcal{P}$ with $F \leq G$, we call G/F

$$G/F := \{H \mid H \in \mathcal{P}, \ F \leq H \leq G\}$$

a *section* of $\mathcal{P}$.

Let $\mathcal{P}$ be a partially ordered set satisfying properties (P1) and (P2), we say $\mathcal{P}$ is *connected* if either d $\leq$ 1, or d $\geq$ 2 and for any two *proper* faces F and G of $\mathcal{P}$ (meaning any faces other than F_{-1} and F_d) there is a sequence of proper faces $F = H_0, H_1, ..., H_{k-1}, H_k = G$ such that H_i and H_{i-1} are incident for $i = 1, ..., k$. We say that $\mathcal{P}$ is *strongly connected* if each section of $\mathcal{P}$ is connected.

We denote the set of all flags of $\mathcal{P}$ by $\mathcal{F}(\mathcal{P})$. Two flags are said to be *adjacent* if they differ by exactly one face.

Definition 2. Let $\mathcal{P}$ be an abstract polytope. The *flag graph of $\mathcal{P}$* is the simple graph with vertices that are the flags of $\mathcal{P}$, and an edge between two vertices in the graph if and only if the flags are adjacent.

It will be useful to notice the equivalence between $\mathcal{P}$ being strongly connected and $\mathcal{P}$ being strongly flag connected (as long as (P1) and (P2) hold). Here strongly flag connected means that the flag graph of each section is a connected graph.

In the classical theory of polyhedra outlined in the previous section, every edge of a polyhedron belongs to exactly two faces and two vertices. The property (P4) states exactly the same thing when $\mathcal{P}$ is a 3-polytope, and extends this homogeneity to higher ranks. Also notice that if Φ is a flag of $\mathcal{P}$, the condition (P4), which will sometimes be called the diamond condition, tells us that for $i = 0, 1, ..., n - 1$ there is exactly one flag that differs from Φ in the $i-$face. This flag will be denoted Φ^i and called the *i-adjacent flag* to Φ. We quickly notice that $(\Phi^i)^i = \Phi$, and if $|i - j| > 1$ that $(\Phi^i)^j = (\Phi^j)^i$ for $i, j \in \{0, 1, ..., n - 1\}$.

We later require the following lemma. Let $\mathcal{P}$ be an abstract $d-$polytope $(d \geq 2)$, and let $1 \leq k \leq d$. The *k-skeleton $skel_k(\mathcal{P})$* of $\mathcal{P}$ is the ranked partially ordered set

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consisting of the faces of $\mathcal{P}$ of rank at most k.

Lemma 3. *$skel_k(\mathcal{P})$ is connected (with "connectedness" defined as above).*

Proof. We proceed by induction on d. The case $d = 2$ is trivial.

Let $d \geq 3$, and let F and G be two proper faces of $\mathcal{P}$ contained in $skel_k(\mathcal{P})$. By the connectedness of $\mathcal{P}$ there exists a sequence of proper faces $F = H_0, H_1, ..., H_{l-1}, H_l = G$ such that H_{i-1} and H_i are incident for each i. We must show that we can find a sequence inside of $skel_k(\mathcal{P})$.

We alter the above sequence as follows: Let r_i denote the rank of H_i for each i. We first remove unnecessary faces so that either $r_0 \leq r_1 \geq r_2 \leq r_3 \geq ...$ or $r_0 \geq r_1 \leq r_2 \geq r_3 \leq ...$ We do this by removing any face H_i in the sequence of adjacent faces with the property that $\mathrm{rank}(H_{i-1}) < \mathrm{rank}(H_i) < \mathrm{rank}\ (H_{i+1})$ or $\mathrm{rank}(H_{i-1}) > \mathrm{rank}(H_i) > \mathrm{rank}(H_{i+1})$.

Consider the first possibility. If $r_{2j} > k$ for some j, we may replace H_{2j} by a $k-$face incident with it while still maintaining the basic features of the sequence $H_0, ..., H_l$. Thus, we may assume that $r_{2j} \leq k$ for each j. Now if $r_{2j+1} > k$ for some j, then H_{2j} and H_{2j+2} are faces of the $r_{2j+1}-$polytope H_{2j+1}/F_{-1}, and hence, by our inductive hypothesis, can be joined by an appropriate sequence of faces inside this $k-$skeleton; in this case we can replace H_{2j+1} by this sequence (with their initial face H_{2j} and last face H_{2j+2} removed). Proceeding in this way for all such j, we finally arrive at a sequence inside $skel_k(\mathcal{P})$ connecting F and G.

For the second possibility we can argue similarly. This ends the proof of the lemma. $\qquad\square$

Note that it is not really necessary that $\mathcal{P}$ be a polytope. The statement of the lemma remains true for any strongly connected ranked poset.

We will be examining polytopes with high degrees of symmetry. We denote the group of all order preserving bijections (called automorphisms) from $\mathcal{P}$ to $\mathcal{P}$ as $\Gamma(\mathcal{P})$ or $\text{Aut}(\mathcal{P})$. Also when we consider order reversing bijections (called dualities), we denote the group of all automorphisms and dualities of $\mathcal{P}$ as $\text{D}(\mathcal{P})$. If $\mathcal{P}$ admits a duality then we say that $\mathcal{P}$ is self-dual, and then clearly it is the case that $\Gamma(\mathcal{P})$ has index 2 in the group $\text{D}(\mathcal{P})$. Otherwise, $\text{D}(\mathcal{P}) = \Gamma(\mathcal{P})$. The group $\Gamma(\mathcal{P})$ also acts naturally on the set of chains of $\mathcal{P}$, where a chain is any totally ordered subset of $\mathcal{P}$. We can classify chains by looking at the ranks of the faces contained in them. We call a chain $\{F_{i_1}, F_{i_2}, ..., F_{i_m}\}$ of *type* $\{i_1, i_2, ..., i_m\}$ if $rank(F_{i_k}) = i_k$ for each k. Similarly, we call a chain of *cotype* $\{j_1, j_2, ..., j_r\}$ if it has faces of every rank except for $\{j_1, j_2, ..., j_r\}$.

A $d-$polytope ($d \geq 2$) is called *equivelar* if for each $i \in 1, 2, ..., d-1$ there is an integer p_i so that any section F/G defined by an $(i+1)-$face F and an $(i-2)-$face G is a p_i-gon. If $\mathcal{P}$ is equivelar we say it has *Schläfli type* $\{p_1, p_2, ..., p_{d-1}\}$.

For notation, we denote the number of faces of $\mathcal{P}$ as $|\mathcal{P}|$, and the number of i-faces of $\mathcal{P}$ as $f_i(\mathcal{P})$. (Often we simply write f_i where is it clear which polytope is being considered.)

Definition 4. Given any partially ordered set $\mathcal{P}$, in particular an abstract polytope, $\mathcal{H}(\mathcal{P})$ is the Hasse Diagram for $\mathcal{P}$. Here $\mathcal{H}(\mathcal{P})$ is the graph with vertex set equal to the set of faces of $\mathcal{P}$ and with an edge between two vertices F and G of $\mathcal{H}(\mathcal{P})$ if and only if F and G are incident in $\mathcal{P}$ and $|\text{rank}(F)\text{-rank}(G)| = 1$.

1.2.1 Examples with Small Rank

We end this section with some examples of $d-$polytopes for small values of d, and we will describe these polytopes using their Hasse diagrams. A $0-$polytope contains only the two elements required by property (P1), and hence up to isomorphism there is only one $0-$polytope. A $1-$polytope must have a diamond shaped Hasse diagram due to property (P4), and again up to isomorphism there is only one $1-$polytope (see Figure 1.4)
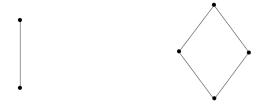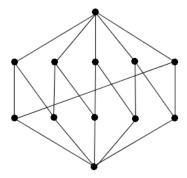
Figure 1.4: Hasse diagrams for rank 0 and 1 polytopes respectively.

If $\mathcal{P}$ is a $2-$polytope, it follows easily from the properties (P1),...,(P4) that the Hasse diagram of $\mathcal{P}$ is necessarily of one of the two kinds illustrated in Figure 1.5. If $\mathcal{P}$ is finite, it is identical to the diagram of a convex $p-$gon, and otherwise to the apeirogon. Notice that for a fixed p, every two $p-$gons are combinatorially equivalent. For example, the convex pentagon and a pentagram, have the same Hasse diagram (see Figure 1.5), and are thus equivalent in this combinatorial setting. Also notice that all $2-$polytopes are equivelar of type $\{p\}$ for $2 \leq p \leq \infty$.
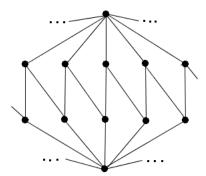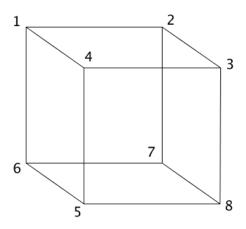
Figure 1.5: Hasse diagrams for the pentagon and the apeirogon.

In rank 3, we can determine if a polyhedron is equivelar simply by checking if its $2-$faces are all of the same type $\{p_1\}$ and vertex-figures are all of the same type $\{p_2\}$. If this is the case, then the polyhedron is equivelar of type $\{p_1, p_2\}$. As a result, we can conclude that when considered as abstract polytopes, the Platonic solids are equivelar; for example the cube is of type $\{4, 3\}$.
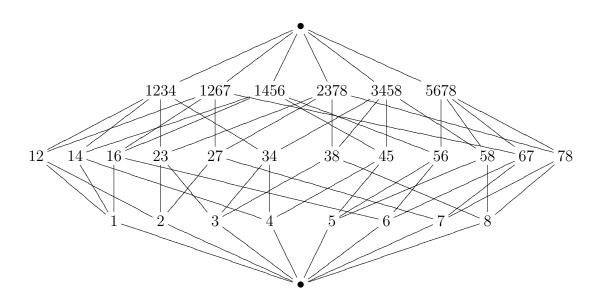
Figure 1.6: 3-cube and its Hasse diagram

While the regular convex polyhedra are uniquely determined by their Schläfli type, this is not true for abstract polytopes of rank at least 3. To see this consider the two following abstract polytopes. First, the regular Platonic solid that is the octahedron, it is of Schläfli type $\{3, 4\}$ as it has triangular faces with four of them

arranged at each vertex. Second, consider what is called the *hemi-octahedron*. This can be constructed by taking the octahedron and identifying opposite faces in each dimension (see Figure 1.7). The hemi-octahedron has only four triangular 2−faces compared to the octahedron's eight, and thus is a different polytope. This polytope can be realized as a projective polyhedron. That is to say, a tessellation of the real projective plane by 4 triangles.
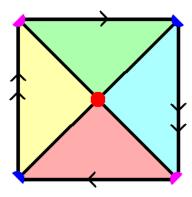
Figure 1.7: Realization of the hemi-octahedron.

Finally, there are examples of abstract polytopes with small rank that are not "geometric," meaning they do not fit well into normal geometric spaces. The examples seen so far can be thought of as tessellations of manifolds. However there exist examples, even with full symmetry, that cannot be thought of in this way. For example the 11-cell (see Chapter 7) which is an abstract 4-polytope consisting of 11 vertices, 55 edges, 55 2-faces, and 11 hemi-icosahedral facets. This polytope was discovered by Grünbaum in 1977, and rediscovered (and more rigorously studied) by Coxeter in 1984. It is, however, locally projective, that is its facets and vertex figures can be thought of as tessellations of real projective planes.

1.3 Regular and Chiral Polytopes from Groups

Throughout this dissertation we will be considering polytopes with high degrees of symmetry. The notions of regularity and chirality both have been studied greatly in the past, and encode this concept of maximum symmetry.

Definition 5. A d-polytope $\mathcal{P}$ is called *regular* if its automorphism group $\Gamma(\mathcal{P})$ has exactly one orbit on the flags of $\mathcal{P}$, or equivalently, if we fix a base flag $\Phi = \{F_{-1}, F_0, F_1, ..., F_d\}$, then for each $j \in \{0, 1, ..., d-1\}$ there exists a (unique) involutory automorphism ρ_j of $\mathcal{P}$ such that $\Phi\rho_j = \Phi^j$ (where Φ^j is the unique j-adjacent flag to Φ guaranteed by (P4)).

We quickly note some consequences of this definition. If $\mathcal{P}$ is a regular d-polytope then all sections of $\mathcal{P}$ are also regular polytopes; any two sections which are defined by faces of the same rank are isomorphic; and $\mathcal{P}$ is equivelar.

For a regular d-polytope $\mathcal{P}$ its automorphism group $\Gamma(\mathcal{P})$ is generated by the involutions $\rho_0, ..., \rho_{d-1}$ given in the above definition for a fixed base flag of $\mathcal{P}$. We call these ρ_j *distinguished generators* of $\Gamma(\mathcal{P})$, and they satisfy the following relations:

$$(\rho_i \rho_j)^{p_{ij}} = 1 \text{ for } i, j \in \{0, ..., d-1\},$$

where $p_{ii} = 1$ for all i, $p_{ji} = p_{ij} =: p_i$ if $j = i - 1$, and $p_{ij} = 2$ otherwise; here the p_i's are given by the type $\{p_1, p_2, ..., p_{d-1}\}$ of the polytope $\mathcal{P}$. The group $\Gamma(\mathcal{P})$ and its distinguished generators also satisify the following *intersection property*:

$$\langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle \text{ for } I, J \subseteq \{0, ..., d-1\}.$$

Any group generated by involutions which has this intersection property is called

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a C-group, and thus $\Gamma(\mathcal{P})$ is always a C-group from what was mentioned above. We note briefly that the "C" in C-group stands for "Coxeter." However, not all C-groups are Coxeter groups as there can be many other relations amongst the generators. In particular the underlying Coxeter diagram for $\Gamma(\mathcal{P})$ will always be a string diagram (since $(\rho_i \rho_j)^2 = \epsilon$ when $|i - j| \geq 2$). Thus $\Gamma(\mathcal{P})$ will always be what is called a *string C-group*.

In [21], it is proved that the combinatorial structure of a regular d-polytope can be completely described in terms of these distinguished generators. We often take this approach when creating a computational example. Instead of trying to describe the polytope combinatorially, we will instead focus on the string C-group and use Todd-Coxeter coset enumeration to understand the partial order on the faces.

To see how this is accomplished we need further notation. Let $\Gamma_i := \langle \rho_j | j \neq i \rangle$, that is for each i, Γ_i is the group generated by all but the ith distinguished generator. Then we can define the faces of $\mathcal{P}$ and the partial order in terms of cosets of these groups. First set $\Gamma_{-1} := \Gamma_d := \Gamma := \Gamma(\mathcal{P})$. Then for all ranks j, we take the set of j-faces of $\mathcal{P}$ to be the set of all right cosets $\Gamma_j \varphi$ in Γ, with $\varphi \in \Gamma$. Finally, we define the partial order by $\Gamma_j \varphi \leq \Gamma_k \psi$ if and only if $-1 \leq j \leq k \leq d$ and $\Gamma_j \varphi \cap \Gamma_k \psi \neq \emptyset$.

Then using the Todd-Coxeter algorithm, we can study the combinatorial structure of a regular polytope $\mathcal{P}$ and any of its associated graphs. Similarly, combinatorics of a chiral polytope can be understood using cosets of its automorphism group.

Definition 6. A d-polytope $\mathcal{P}$ ($d \geq 3$) is called *chiral* if it is not regular, but for any (base) flag Φ and for each $i \in \{1, ..., d - 1\}$ there exists an automorphism σ_i of $\mathcal{P}$ such that σ_i fixes all faces in Φ, except for F_{i-1} and F_i, and cyclically permutes consecutive i-faces of the section F_{i+1}/F_{i-2} of $\mathcal{P}$.

We now summarize some results known about the groups of chiral polytopes

(See [33] for details). Let $\mathcal{P}$ be a chiral d-polytope, then the following can be shown. The automorphism group of $\mathcal{P}$ has only two orbits on the set of flags; the even flags all in one orbit, and the odd flags in the other. Here a flag Ψ of $\mathcal{P}$ is called *even* (with respect to the base flag Φ) if there exists a finite sequence of flags $\Phi = \Phi_0, \Phi_1, ..., \Phi_{2k} = \Psi$ where any two consecutive flags are adjacent (a flag is called *odd* if it is not even). Also, the automorphism group of $\mathcal{P}$ is generated by the σ_i for $i \in \{1, .., d-1\}$. The σ_i's satisfy the relations $\sigma_i^{p_i} = 1$ for $1 \le i \le d-1$ and $(\sigma_i \sigma_{i+1}...\sigma_j)^2 = 1$ for $1 \le i < j \le d-1$, where the p_i are given by the type $\{p_1, ..., p_{d-1}\}$ of the polytope. Also, for any i-face of $\mathcal{P}$, the sections F_d/F_i and F_i/F_{-1} are regular or chiral polytopes. Finally a result which will be useful in later theorems in this dissertation is that for each $i \in \{0, ..., d-1\}$ the group of automorphisms of $\mathcal{P}$ acts transitively on the chains of cotype $\{i\}$. Also the generators of $\Gamma(\mathcal{P})$ satisfy an intersection property which we will not state here.

Similar to the regular case, we would like to utilize the structure of the group $\Gamma(\mathcal{P})$ to reconstruct the combinatorial properties of a chiral polytope $\mathcal{P}$. This is achievable by the following construction. Let $A^{-1} := A^d := \Gamma(\mathcal{P})$. Let $A^0 = \langle \sigma_2, ..., \sigma_{n-1} \rangle$, and let $A^{d-1} = \langle \sigma_1, ..., \sigma_{d-2} \rangle$. Finally let $A^i = \langle \{\sigma_j | j \neq i, i+1\} \cup \{\sigma_i \sigma_{i+1}\} \rangle$.

Then analogous to the regular case, let the set of proper i-faces of $\mathcal{P}$ be the set of left cosets φA^i of A^i for $\varphi \in \Gamma(\mathcal{P})$, and let the greatest and least face be represented by copies of the entire group $\Gamma(\mathcal{P})$. We then give the partial order, by $\varphi A^i \le \psi A^j$ if and only if $-1 \le i \le j \le d$ and $\varphi A^i \cap \psi A^j \neq \emptyset$.

Again using coset enumeration, we can understand the combinatorics of a particular example completely from the structure of the group.

1.4   Symmetries of Graphs

An *automorphism* of a simple graph $\mathcal{G}$ is a permutation π of the vertex set of $\mathcal{G}$ which has the property that there is an edge between two vertices v and w of $\mathcal{G}$ if and only if there is an edge between $\pi(v)$ and $\pi(w)$. The group of all such permutations is the *automorphism group* of $\mathcal{G}$, denoted by $\mathrm{Aut}(\mathcal{G})$. We say that $\mathcal{G}$ is vertex transitive if $\mathrm{Aut}(\mathcal{G})$ acts transitively on the vertex set of $\mathcal{G}$, that is there is just one orbit.

Some basic properties of automorphisms of graphs come as direct consequences of the definition. We first notice that if two vertices v and w belong to the same orbit under the action of $\mathrm{Aut}(\mathcal{G})$, that is there exists and automorphism α in $\mathrm{Aut}(\mathcal{G})$ such that $\alpha(v) = w$, then v and w have the same degree. In a connected graph $\mathcal{G}$, one can define the distance $\delta(v, w)$ between two vertices v and w. Then for any automorphism α of $\mathcal{G}$ we have that $\delta(v, w) = \delta(\alpha(v), \alpha(w))$. In other words, there can be no automorphism of a connected graph which sends a pair of vertices of distance k to a pair of vertices at distance $j \neq k$.

Definition 7. Let $\mathcal{G}$ be a simple graph with vertex set $\mathrm{V}(\mathcal{G})$ and edge set $\mathrm{E}(\mathcal{G})$; a t-arc in $\mathcal{G}$ is a sequence of vertices $(v_0, v_1, ..., v_t)$ such that $\{v_{i-1}, v_i\} \in \mathrm{E}(\mathcal{G})$ for $1 \leq i \leq t$ and $v_i \neq v_{i-2}$ for all i.

Definition 8. Let $\mathcal{G}$ be a simple graph with vertex set $\mathrm{V}(\mathcal{G})$ and edge set $\mathrm{E}(\mathcal{G})$. Then a sequence $S = (v_0, v_1, v_2, ..., v_t)$ is a *t-star* if $v_i \in V(\mathcal{G})$ for all i, $v_i \neq v_j$ for $i \neq j$, and $\{v_0, v_i\} \in \mathrm{E}(\mathcal{G})$ for i$\neq$ 0.

It is clear that the action of $\mathrm{Aut}(\mathcal{G})$ on the vertices of $\mathcal{G}$ can be extended naturally to an action on the t-arc and the t-stars in $\mathcal{G}$.

Definition 9. $\mathcal{G}$ is *t-arc transitive* if $\mathrm{Aut}(\mathcal{G})$ acts transitively on the t-arcs of $\mathcal{G}$ but not on the (t+1)-arcs of $\mathcal{G}$.

For vertex transitive graphs, Tutte has shown that there exists a maximal value of t so that $\mathrm{Aut}(\mathcal{G})$ is transitive on $t-$arcs (allowing t=0 here). Using the classification theorems of finite simple groups, Weiss (1983) showed that there are no finite graphs (other than the cycles) that have an automorphism group which acts transitively on the t-arcs for $t \geq 8$. However, 7-arc transitive graphs do exist so this bound is sharp. Tutte also proved that for a k-regular graph $(k \geq 3)$, $\mathrm{Aut}(\mathcal{G})$ is transitive on $t-$arcs $(t \geq 1)$ if and only if for some t-arc $[v]$ there exist (k-1) special automorphisms of the graph $g_1, g_2, ..., g_{k-1}$, which are called shunts, such that each g_i takes $[v]$ to a different successor of the arc. In other words if $[v] = (v_0, v_1, v_2, ...v_t)$ and the k neighbors of v_t are $\{v_{t-1}, w_1, w_2, ..., w_{k-1}\}$ then $g_i([v]) = (v_1, v_2, ...v_t, w_i)$ for each i.

For a fixed $t-$arc [v], the *stabilizer sequence* for [v] is:

$$Aut(\mathcal{G}) \supset B_t \supset B_{t-1} \supset ... \supset B_1 \supset B_0$$

where the subgroup B_i is the pointwise stabilizer of $\{v_0, ..., v_{t-i}\}$. We can use this stabilizer sequence to understand the arc transitivity of a graph. In particular, in the 3-regular case if a graph $\mathcal{G}$ is t-arc transitive, for $t \geq 1$, then we can determine the arc transitivity of $\mathcal{G}$ by examining the subgroup B_t, which is the vertex stabilizer of any vertex. This result is summarized in the following lemma (see [1], [31]).

Lemma 10. *If $\mathcal{G}$ is a finite, connected, t-transitive, 3-regular graph, with $t \geq 1$, then $t \leq 5$, and the stabilizer B_t is determined up to isomorphism by t. Moreover, if $t = 1$, then $B_t \cong \mathbb{Z}_3$; if $t = 2$, then $B_t \cong \mathbb{S}_3$; if $t = 3$, then $B_t \cong \mathbb{D}_{12}$; if $t = 4$, then $B_t \cong \mathbb{S}_4$; and if $t = 5$, then $B_t \cong \mathbb{S}_4 \times \mathbb{Z}_2$.*

Unfortunately for graphs with higher valencies no such elegant description is known for the stabilizers. For calculations later in this dissertation we require the following lemma.

Lemma 11. *Let $\mathcal{G}$ be a simple k-regular graph and $[v] = (v_0, ..., v_t)$ be a t-arc ($t \geq 1$) in $\mathcal{G}$; then Aut($\mathcal{G}$) acts transitively on the t-arcs if and only if Aut($\mathcal{G}$) acts transitively on the (t-1)-arcs, and B_1 acts transitively on the k neighbors of v_0 if $t = 1$, or the $k - 1$ neighbors of v_{t-1} distinct from v_{t-2} if $t > 1$.*

Proof. Let us begin with the case where $t = 1$. To show that Aut($\mathcal{G}$) acts transitively on the 1-arcs, we require that $\mathcal{G}$ is vertex transitive and that the vertex stabilizer of v_0 acts transitively on the k neighbors of v_0. Denote the neighbors of v_0 by $v_1, w_1, ..., w_{k-1}$, and let $\beta_1, \beta_2, .., \beta_{k-1}$ be (k-1) automorphisms in the vertex stabilizer of v_0 so that $\beta_i(v_1) = w_i$. Then to show that Aut($\mathcal{G}$) acts transitively on the 1-arcs, let $[x] = (x_0, x_1)$ be any 1-arc in $\mathcal{G}$. We will find an automorphism that sends $[v]$ to $[x]$. Since $\mathcal{G}$ is vertex transitive, there exists an automorphism α such that $\alpha(v_0) = x_0$. Then $\alpha(v_1)$ will be a neighbor of x_0, If it is equal to x_1 then we are done. Otherwise $\beta_i(v_1) = \alpha^{-1}(x_1)$ for some i, and thus $\alpha\beta_i(v_1) = x_1$. Since $\beta_i(v_0) = v_0$, we have $\alpha\beta_i(v_0) = x_0$. So $\alpha\beta_i([v]) = [x]$ as we needed.

If $t > 1$ the argument is similar. Denote the neighbors of v_{t-1} by $v_{t-2}, v_t, w_1, ..., w_{k-2}$, and let $\beta_1, \beta_2, .., \beta_{k-2}$ be (k-2) automorphisms in the pointwise stabilizer of $(v_0, v_1, ..., v_{t-1})$ so that $\beta_i(v_t) = w_i$. To show that Aut($\mathcal{G}$) acts transitively on the t-arcs, let $[x] = (x_0, x_1, ..., x_t)$ be any t-arc in $\mathcal{G}$. Again we will find an automorphism that sends $[v]$ to $[x]$. Since Aut($\mathcal{G}$) acts transitively on the (t-1)-arcs there exists an α such that $\alpha((v_0, v_1, ..., v_{t-1})) = (x_0, ..., x_{t-1})$. Then $\alpha(v_t)$ will be a neighbor of x_{t-1}, If it is equal to x_t then we are done. Otherwise $\beta_i(v_t) = \alpha^{-1}(x_t)$ for some i, and thus $\alpha\beta_i(v_t) = x_t$. Since β_i is in the appropriate stabilizer, $\beta_i((v_0, ..., v_{t-1})) = (v_0, ..., v_{t-1})$ and thus $\alpha\beta_i([v]) = [x]$ as we needed. The other implication of the claim is trivial, so this completes the proof. $\square$

Definition 12. $\mathcal{G}$ is *t-star transitive* if Aut($\mathcal{G}$) acts transitively on the t-stars of $\mathcal{G}$

but not on the (t+1)-stars of $\mathcal{G}$.

Definition 13. $\mathcal{G}$ is *distance transitive* if for all vertices u, v, x, y of $\mathcal{G}$ such that $\delta(u, v) = \delta(x, y)$, there exists an automorphism α in $\text{Aut}(\mathcal{G})$ such that $\alpha(u) = x$ and $\alpha(v) = y$.

We also will refer to the notion of distance regularity of a graph. If a graph is distance transitive it will automatically be distance regular (however the converse is not true). We say that a graph $\mathcal{G}$ is distance regular if the number of vertices at fixed distances from a pair of vertices at any given distance is independent of the pair chosen at that distance. That is to say, for all pairs of vertices $\{x, y\}$ of $\mathcal{G}$, let $s_{i,j}(x, y)$ be the number of vertices of $\mathcal{G}$ that are of distance i from vertex x and, at the same time, of distance j from vertex y. Then $\mathcal{G}$ is *distance regular* if the numbers $s_{i,j}(x, y)$ do not depend on $\{x, y\}$ but only the distance between them; formally if $\delta(x, y) = \delta(u, v)$, then $s_{i,j}(x, y) = s_{i,j}(u, v)$ for all i, j (where δ is the distance function).

There is a natural hierarchy between some of the previously mentioned graph theoretical properties, which will be useful in later chapters. For example, for a graph to be either t-arc or t-star transitive, it must be vertex transitive. Also if a graph is distance transitive, then all the 1-arcs (and similarly the 1-stars) are in one orbit (see [1], Ch 20). Also notice, for a graph to be vertex transitive, it must be k-regular (where k-regular means that all of the vertices have k neighbors).

Chapter 2

Graphs of More than Three Levels

2.1 Hasse Diagrams

In this section we study the first of several possible graphs associated with an abstract polytope. Recall that $\mathcal{H}(\mathcal{P})$ is the graph with vertex set equal to the set of faces of $\mathcal{P}$ and with an edge between two vertices F and G of $\mathcal{H}(\mathcal{P})$ if and only if F and G are incident in $\mathcal{P}$ and $|\text{rank}(F) - \text{rank}(G)| = 1$. Given a highly symmetric polytope, it is natural to ask how much graph symmetry can there be in $\mathcal{H}(\mathcal{P})$.

Rank 2

In the rank 2 case, all polytopes are self-dual and regular. Given a rank 2 polytope $\mathcal{P}$, the property (P4) tells us that any 0-face F_0 of $\mathcal{P}$ has two incident 1-faces to it. Also property (P1) tells us that F_0 is incident to F_{-1}. Thus the degree of F_0 in $\mathcal{H}(\mathcal{P})$ is 3. So for $\mathcal{H}(\mathcal{P})$ to have any of the transitivity being studied, $\mathcal{H}(\mathcal{P})$ must be a 3-regular graph. Then the degree of F_2 and F_{-1} is 3, which implies that $f_0 = f_1 = 3$. Thus the only case in rank 2, where $\mathcal{H}(\mathcal{P})$ is k-regular is the abstract polytope that is the partially ordered set of faces for a triangle.
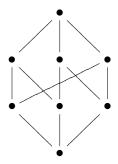
Figure 2.1: Unique vertex transitive Hasse diagram of a 2-polytope

In this case $\mathcal{H}(\mathcal{P})$ is not only vertex transitive, but is 2-arc transitive, and 3-star transitive. (Here and many places to follow, supporting calculations for specific graphs are done in GAP, using the packages *Grape* and *nauty*.) This is the complete classification for transitivity of Hasse Diagrams in the rank 2 case.

Rank 3

Let $\mathcal{P}$ be a rank 3 polytope. If we want $\mathcal{H}(\mathcal{P})$ to be k-regular, this will imply that $f_2 = k$. Also every 2-face of $\mathcal{P}$ must be incident to (k-1) 1-faces. Similarly, we know that $f_0 = k$, and every 0-face of $\mathcal{P}$ must be incident to (k-1) 1-faces. This implies that $\mathcal{P}$ is equivelar and thus has a Schläfli symbol of {k-1,k-1}.

Now consider a 1-face F_1 of $\mathcal{P}$. The property (P4) tells us that there are two 0-faces of $\mathcal{P}$ and two 2-faces of $\mathcal{P}$ incident to F_1. Thus the degree of F_1 in $\mathcal{H}(\mathcal{P})$ is 4. Therefore, k=4 and $\mathcal{P}$ is of type {3,3}. The only polytope of type {3,3} is the regular polytope which is isomorphic to the face lattice of the three dimensional simplex.

In this case $\mathcal{H}(\mathcal{P})$ is 2-arc transitive, and 4-star transitive.
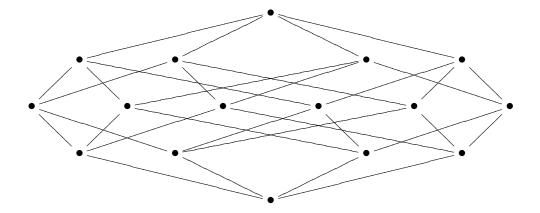
Figure 2.2: Unique vertex transitive Hasse diagram of a 3-polytope

Rank 4

Let $\mathcal{P}$ be a rank 4 polytope. Again, if $\mathcal{H}(\mathcal{P})$ is k-regular, then we know the number of facets of $\mathcal{P}$ is equal to k. Then looking at any face of rank 3, we see that it must have (k-1) faces of rank 2 incident to it. Considering the property (P4) we know that any 2-face will have two 3-faces incident to it. Thus the k-regularity of $\mathcal{H}(\mathcal{P})$ implies that every 2-face has (k-2) 1-faces incident to it. On the other hand, looking at faces of lower ranks, we see dually the same arguments give us information about the vertex figures of $\mathcal{P}$. Putting this all together, we can conclude that $\mathcal{P}$ is equivelar and has Schläfli type {k-2,q,k-2} with $f_3 = f_0 = k$ and $f_2 = f_1$.

We now want to get an idea of the number of flags of $\mathcal{P}$. First recall that if a polytope is of rank 2 and of type {p}, then it has 2p flags. Now in the rank 4 case, we have a polytope $\mathcal{P}$ with a k-regular Hasse Diagram. Counting the number of flags by starting at F_4 and going toward the lower ranks, we get $|\mathcal{F}(\mathcal{P})| = k \times (k-1) \times 2(k-2)$.

Counting the flags differently, instead starting at any specific 2-face of $\mathcal{P}$, we know that any 2-face F_2 is incident with two 3-faces, and the number of flags in the section F_2/F_{-1} is 2(k-2) where F_2/F_{-1} is of type {k-2}. This gives us $|\mathcal{F}(\mathcal{P})| = f_2 \times 2 \times 2(k-2)$,

and thus $f_2 = \dfrac{k(k-1)}{2}$. Thus for $\mathcal{H}(\mathcal{P})$ to be k-regular, $\mathcal{P}$ must have an f-vector of $(k, \dfrac{k(k-1)}{2}, \dfrac{k(k-1)}{2}, k)$.

To understand this fully, we consider the examples of polytopes of type $\{p, q, p\}$ that arise while varying the value of k. The first possible example is p=2 or k=4. In this case for each value of q, the polytope $\mathcal{P}$ would have to be the unique regular degenerate polytope of type $\{2,q,2\}$. However, the f-vector for these polytopes is (2,q,q,2). This does not match up with our previous calculations. So there are no 4-regular Hasse Diagrams for polytopes of rank 4.

Next when p=3 or k=5, we get an f-vector of (5,10,10,5). Similar to the rank 2 and rank 3 cases, this is the f vector for the regular simplex. In this case $\mathcal{H}(\mathcal{P})$ is 2-arc transitive and 5-star transitive.

When p=4 or k=6, we get an f-vector of (6,15,15,6). As we will show in the final chapter, there exists a chiral polytope with this f-vector. However, while its Hasse diagram is a 6-regular graph, there are two orbits of vertices under the action of the graph's automorphism group, and thus it is not vertex-transitive.

This classification is not complete in rank 4. There is still room to study the connection between possible f-vectors for a polytope of fixed Schläfli type in order to completely understand the rank 4-case. However, it seems likely, due to the restrictive nature of this f-vector condition, that the only possible polytope with a vertex-transitive Hasse diagram is the simplex.

Rank d (≥ 5)

When the rank of $\mathcal{P}$ is arbitrary, a few things can be said. First, if $\mathcal{H}(\mathcal{P})$ is k-regular then we know that $f_{d-1} = f_0 = k$, and that the number of (d-2)-faces incident to any (d-1)-face is equal to the number of 1-faces incident to a 0-face of $\mathcal{P}$, which are both

equal to (k-1). Similarly, the number of (d-3)-faces incident to any (d-2)-face is equal to the number of 2-faces incident to a 1-face of $\mathcal{P}$, which are both equal to k-2.

Counting the flags of $\mathcal{P}$, we get

$$|\mathcal{F}(\mathcal{P})| = f_{d-2} \times 2|\mathcal{F}(F_{d-2}/F_{-1})| = f_1 \times 2|\mathcal{F}(F_d/F_1)|.$$

Now if $\mathcal{H}(\mathcal{P})$ is vertex transitive, then we know that there will be an automorphism of $\mathcal{H}(\mathcal{P})$ that sends F_d to F_{-1}. This will imply that $f_{d-2} = f_1$. To understand this claim, let us build up the possible images of an automorphism α of the graph $\mathcal{H}(\mathcal{P})$ with this property that it sends the least face of $\mathcal{P}$ to the greatest face of $\mathcal{P}$. Since $\alpha(F_{-1}) = F_d$ and α must preserve edges of the graph, we know that $\alpha(\mathcal{P}_0) = \mathcal{P}_{d-1}$. There is no other possible place to send the 0-faces of $\mathcal{P}$, since in $\mathcal{H}(\mathcal{P})$ the only vertices of the graph that are adjacent to F_d are the facets. Then once you know the images of both $\mathcal{P}_{-1}$ and $\mathcal{P}_0$ under α, the only possibility for the image of $\mathcal{P}_1$ under α is $\mathcal{P}_{d-2}$. This is true since a graph automorphism must send a 1-face to a face adjacent to the image of a 0-face. With the image of the 0-faces being the facets, the only options are sending a 0-face to a (d-2)-face or a d-face. However, the unique d-face already has something mapped to it, namely F_{-1}. Therefore, we can see that the number of (d-2)-faces is the same as the number of 1-faces.

2.1.1 The d-simplex and the (d+1)-hypercube

Also we can notice that the Hasse diagram of the d-simplex will always be 2-arc transitive, and (d+1)-star transitive. This can be seen by noticing that there is a graph automorphism between $\mathcal{H}(\mathcal{P})$ for $\mathcal{P}$ the d-simplex, and the 1-skeleton of the (d+1)-hypercube. Then studying symmetries of the Hasse diagram of the simplex, can just be thought of as studying the symmetries of the graph of 0-faces and 1-faces

of the (d+1)-hypercube. To see this explicitly, let us give a description of the faces of the d-simplex. We recall an d-simplex in Euclidean d-space is simply the convex hull of (d+1) points in general position, where every collection of (m+1) points defines a face of dimension m. Let us label these points by (1,2,3,...,d+1), then every face of the d-simplex can be thought of as a binary string of length d+1, where the value in the kth position is 1 if and only if k is the label of one of the points that determines the face (for example the whole simplex itself is the binary string 111...1). Then two vertices F and G of the Hasse diagram are adjacent if their corresponding faces in the simplex are incident and $|\text{rank}(F)-\text{rank}(G)| = 1$. This corresponds to the binary strings only differing in one position. This description of the vertices and edges of the Hasse diagram for the d-simplex is a well known realization of the (d+1)-dimensional hypercube. For the example of the 3-simplex and 4-cube see Figure 2.3.
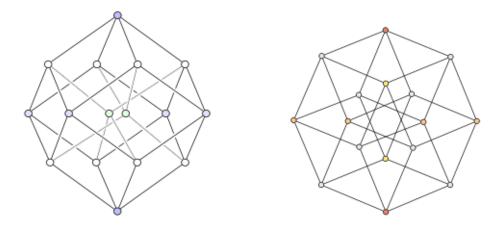
Figure 2.3: Hasse diagram of 3-simplex and 1-skeleton of 4-cube

To understand that the 1-skeleton of the (d+1)-cube $\mathcal{P}$ is (d+1)-star transitive, we switch back to thinking of the cube as an abstract polytope. The cube is regular and of type {4,3,3,..,3}. This implies two useful things. First the regularity tells us that there is an automorphism of the cube (and thus an automorphism of its 1-skeleton)

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that sends any vertex to any other vertex. Thus we see that our graph is vertex transitive. Also, the vertex figure at any vertex in the cube is of type {3,3,...,3}, which is that of a simplex. The symmetry group of the simplex can be thought of as the full group of permutations S_{d+1} on the (d+1) vertices of the simplex. Thus once you have fixed a vertex of our graph $\mathcal{H}(\mathcal{P})$, there is still full freedom for where the adjacent vertices can be mapped. And thus $\mathcal{H}(\mathcal{P})$ is (d+1)-star transitive.

The argument for 2-transitivity is simpler. Being (d+1)-star transitive implies that $\mathrm{Aut}(\mathcal{H}(\mathcal{P}))$ is transitive on the 2-arcs. To show there is more than one orbit on the 3-arcs of $\mathcal{H}(\mathcal{P})$, consider one 3-arc that is defined by three edges which are in a path in a 2-dimensional square in the hypercube, and another 3-arc defined by three edges which do not lie in a single square. There cannot be a graph automorphism which sends the first arc to the second, because a graph automorphism must preserve distance between vertices. The first and last vertex in the original 3-arc will be at distance 1, whereas the first vertex and the last vertex of the second 3-arc will be at a distance which is strictly larger. Thus $\mathcal{H}(\mathcal{P})$ is not 3-arc transitive.

Understanding the structure of the hypercube gives us many other results for graphs related to the Hasse diagram of the simplex. In particular many of the results from Chapter 3 and Chapter 5 are known in this specific case (see Donovan [10]).

2.2 Restrictions of the Hasse Diagram to Levels

Definition 14. Let $\mathcal{P}$ be an abstract d-polytope and let I be a subset of the set of ranks {-1,0,1,...,d}. Then we can define a new graph, which is the restriction of $\mathcal{H}(\mathcal{P})$ to the faces with ranks in I. We denote this graph $\mathcal{H}_I(\mathcal{P})$.

Note that $\mathcal{H}_I(\mathcal{P})$ is connected if and only if I is of the form $\{i, i+1, ..., j\}$ for some i and j with $-1 \leq i < j \leq d$. If $I = \{i, i+1, ..., j\}$ with $-1 \leq i < j \leq d$, then $\mathcal{H}_I(\mathcal{P})$ is the Hasse diagram of $\mathrm{skel}_j((\mathrm{skel}_{d-i-1}(P^*))^*)$, where the $*$ represents the duality (in the class of ranked partially ordered sets) and is connected by Lemma 3.

Next we want to study what possible transitivity there can be in $\mathcal{H}_I(\mathcal{P})$ for various sets I. In particular, if $\mathcal{P}$ is an even rank polytope, say of rank 2k, then we pay special attention to when I={k-1,k}. Also if $\mathcal{P}$ is an odd rank polytope, say of rank 2k+1, then we pay special attention to the case when I={k-1,k,k+1}. Both of these cases will be studied in their own sections to follow. We can then generalize this to when $|I| = 2$ or 3 for some of the results.

Another interesting possible value for I occurs when $\mathcal{P}$ is of any rank n, and I={0,1,2,...n-1}. Simply removing the faces of $\mathcal{P}$ that exist in every polytope due to property (P1), gives an interesting class of graphs to study in the future.

2.3 Restrictions of the Comparability Graph to Levels

Definition 15. Given any partially ordered set $\mathcal{P}$, in particular an abstract polytope, $\mathcal{C}(\mathcal{P})$ denotes the Comparability Graph for $\mathcal{P}$. Recall that $\mathcal{C}(\mathcal{P})$ is the graph with vertex set equal to the set of faces of $\mathcal{P}$ and with an edge between two vertices F and G of $\mathcal{C}(\mathcal{P})$ if and only if F and G are incident in $\mathcal{P}$.

This definition only differs slightly from that in the last section. Here we allow edges between faces of $\mathcal{P}$ that differ greatly in rank. Quickly we can notice that $\mathcal{H}(\mathcal{P})$ is a subgraph of $\mathcal{C}(\mathcal{P})$, since they have the same vertex set and every edge of $\mathcal{H}(\mathcal{P})$ is also an edge of $\mathcal{C}(\mathcal{P})$. Also similar to the previous section, we can define $\mathcal{C}_I(\mathcal{P})$ for any set I of ranks of $\mathcal{P}$.

To study transitivity in $\mathcal{C}(\mathcal{P})$ for an n-polytope $\mathcal{P}$, first we consider $\mathcal{C}(\mathcal{P})$ to be k-regular. If $\mathcal{C}(\mathcal{P})$ is k-regular, then in particular the degree of F_n is k in $\mathcal{C}(\mathcal{P})$. However, F_n is comparable to every face (other than itself) in $\mathcal{P}$. Thus $k = |\mathcal{P}| - 1$. Now let F_{n-1} be any (n-1)-face of $\mathcal{P}$. Then in $\mathcal{C}(\mathcal{P})$, F_{n-1} is adjacent to F_n and all the faces (other than itself) contained in the section F_{n-1}/F_{-1}. Thus $k = 1 + |F_{n-1}/F_{-1}| - 1$. Combining the two, we get $|\mathcal{P}| = |F_{n-1}/F_{-1}| + 1$. Therefore there is only one more face of $\mathcal{P}$ other than what is in F_{n-1}/F_{-1}, and F_n itself is that one face. Therefore other than the trivial case of rank $\mathcal{P}$ being 0, $\mathcal{C}(\mathcal{P})$ is never k-regular, and thus never has any of the desired transitivity.

Similar to the last section, a few interesting classes of graphs to be studied come from letting I be very small or very big. In the previous section we were limited to I containing adjacent ranks, in order for the graph $\mathcal{H}_I(\mathcal{P})$ to be connected. However, with $\mathcal{C}_I(\mathcal{P})$ that is no longer an issue. Also we notice that if I consists of two ranks differing by 1, then $\mathcal{C}_I(\mathcal{P})$ and $\mathcal{H}_I(\mathcal{P})$ are the same graph.

Chapter 3

Arc Transitivity of M($\mathcal{P}$)

Our concern in this chapter will be of abstract polytopes of even rank $d = 2n$. The main theorem in this chapter is an analog for polytopes of arbitrary even rank to a theorem of Monson and Weiss [31] about 4-polytopes. When $\mathcal{P}$ is of rank 2n we have the following definition.

3.1   Introduction

Definition 16. Let $\mathcal{P}$ be a 2n-polytope. The associated *medial layer graph* M($\mathcal{P}$) is the simple graph with vertex set consisting of the faces F $\in \mathcal{P}$ such that $rank(F) = n$ or $rank(F) = n - 1$. The edge set of M($\mathcal{P}$) is defined naturally such that F and G are adjacent in M($\mathcal{P}$) when F and G are incident in $\mathcal{P}$.

We quickly notice that M($\mathcal{P}$) is a simple bipartite graph, and that

$$M(\mathcal{P}) = skel_n((skel_n(P^*))^*);$$

then the connectedness of M($\mathcal{P}$) follows from Lemma 3. We now will examine the

relationships between various types of transitivity in M($\mathcal{P}$) and the symmetries of $\mathcal{P}$. Also notice that M($\mathcal{P}$) = $\mathcal{C}_I(\mathcal{P})$ = $\mathcal{H}_I(\mathcal{P})$, with $I = \{n - 1, n\}$.

For the medial layer graphs of rank 2n abstract polytopes we can partition the arcs of a graph into two classes, according to the initial vertex of an arc. We call a t-arc $(v_0, v_1, ..., v_t)$ in M($\mathcal{P}$) of type n if $rank(v_0) = n$ and of type (n-1) if $rank(v_0) = n - 1$.

Definition 17. Let $\mathcal{P}$ be an abstract 2n-polytope with medial layer graph M($\mathcal{P}$), we say M($\mathcal{P}$) is t-arc transitive if D($\mathcal{P}$) acts transitively on the t-arcs but not on the (t+1)-arcs. (Recall that D($\mathcal{P}$) is the group of all automorphisms and dualities of $\mathcal{P}$).

Notice immediately that this is a different definition than is classically given in graph theory. All the graphs studied in this thesis will be associated to a polytope. It is thus natural to study the action of the polytope automorphisms on these graphs. D($\mathcal{P}$) will be isomorphic to a subgroup of Aut(M($\mathcal{P}$)), and in many cases we will have D($\mathcal{P}$)= Aut(M($\mathcal{P}$)). However, while studying arc transitivity in an arbitrary graph, there is no polytope in the background, and thus the only option is to consider the action of the entire automorphism group of the graph on the arcs. The result of this altered definition is that, if M($\mathcal{P}$) is t-arc transitive, in our definition of the term, then classically it is k-arc transitive for some $k \geq t$.

3.2 Dual Neighborliness

Definition 18. Let $\mathcal{P}$ be an abstract polytope. Then $\mathcal{P}$ is called *neighborly* if any two 0-faces are incident with at least one common 1-face.

Our first result of this chapter relates the arc transitivity of a medial layer graph of a 2n-polytope to the neighborliness of the dual of some of its faces.

Proposition 19. *Let $\mathcal{P}$ be a rank 2n abstract polytope. If $M(\mathcal{P})$ is k-arc transitive for some $k \geq 1$, then $\mathcal{P}$ is self-dual; moreover, if $k \geq 2$ then for every n-face $F_n \in \mathcal{P}$ the dual of F_n/F_{-1} is a neighborly polytope.*

Proof. First, it is clear that $\mathcal{P}$ is self-dual as there must exist a duality of $\mathcal{P}$ that maps an arc of type n to an arc of type (n-1). Now, let G_n be an n-face of $\mathcal{P}$, and let G_{n-1} and G_{n-2} be (n-1)-faces and (n-2)-faces incident with G_n such that G_{n-2} is incident with G_{n-1}. Let G'_{n-1} be the other (n-1)-face incident with G_n and G_{n-2}; we know G'_{n-1} exists and is unique by (P4). Let $[w] := (F_{n-1}, F_n, F'_{n-1})$ be any 2-arc of type (n-1) in $M(\mathcal{P})$, and let $[v] := (G_{n-1}, G_n, G'_{n-1})$. Then the conditions of the proposition, imply that there exists an α such that $\alpha([v])=[w]$. Then $\alpha(G_{n-2})$ will be incident with F_{n-1} and F'_{n-1}. Since α is in $D(\mathcal{P})$, it will either preserve rank or reverse it, and since rank$(\alpha(G_{n-1}))=$(n-1), then rank$(\alpha(G_{n-2}))$ will be (n-2). Now consider the dual of F_n/F_{-1}. Then F_{n-1} and F'_{n-1} are vertices in this polytope, and $\alpha(G_{n-2})$ is an edge incident with both of them. Therefore since $[w]$ was arbitrary, we are done. $\qquad\square$

Now we have seen there is a relationship between the transitivity of a medial layer graph and the dual neighborliness of a middle rank face. We next impose this neighborly condition on a polytope with a high degree of symmetry, and see that we can get back the transitivity of the medial layer graph.

Theorem 20. *Let $\mathcal{P}$ be a rank 6 abstract polytope, which is self-dual, has symmetry group acting transitively on the chains of type {1,2,3,4}, and any two 2-faces in a 3-face meet in exactly one 1-face. Then $M(\mathcal{P})$ is 3-arc transitive.*

Before we begin the proof, let us look more closely at two of the conditions imposed on the polytope $\mathcal{P}$. First, we require that the symmetry group of $\mathcal{P}$ act transitively on the chains of a certain type. This condition is a relaxing of a regularity type condition. For example if $\mathcal{P}$ is regular or chiral, the symmetry group will act appropriately. Next you see something like the dual neighborliness condition that came out of the last proposition. We know that this condition is necessary for medial layer 3-arc transitivity, we can now see that with the proper symmetry of the polytope, it is also sufficient. The only difference is that we also require something of a "lattice like property." That is, since an abstract polytope need not be a lattice the neighborliness will allow for multiple edges incident to a pair of vertices, we require, however, that the edge be unique.

Proof. The vertices of $M(\mathcal{P})$ are the faces of $\mathcal{P}$ of ranks 2 and 3. Let $[v] = (v_0, v_1, v_2, v_3)$ and $[w] = (w_0, w_1, w_2, w_3)$ be two 3-arcs in $M(\mathcal{P})$. We want to show there exists an $\alpha \in D(\mathcal{P})$ such that $\alpha(v_0, v_1, v_2, v_3) = (w_0, w_1, w_2, w_3)$. We will show that any 3-arc of type 2 can be mapped via α to any other 3-arc of type 2. To show that this is sufficient, consider the other following three cases. In all cases let δ be a duality of $\mathcal{P}$ and let α be the automorphism between the appropriate 3-arcs of type 2.

1. If $[v]$ is of type 2, and $[w]$ is of type 3, then $\delta([w])$ is a 3-arc of type 2. So there exists α such that $\alpha([v]) = \delta([w])$. Then, $\delta^{-1}\alpha([v]) = [w]$.

$$[v] \xrightarrow{\delta^{-1}\alpha} [w]$$
$$\downarrow_{\varepsilon} \qquad \uparrow_{\delta^{-1}}$$
$$[v] \xrightarrow{\alpha} \delta([w])$$

2. If $[v]$ is of type 3, and $[w]$ is of type 2, then $\delta([v])$ is a 3-arc of type 2. So there exists α such that $\alpha\delta([v]) = [w]$.

$$[v] \xrightarrow{\alpha\delta} [w]$$
$$\downarrow_{\delta} \qquad \uparrow_{\varepsilon}$$
$$\delta([v]) \xrightarrow{\alpha} [w]$$

3. If $[v]$ is of type 3, and $[w]$ is of type 3, then $\delta([v])$ is a 3-arc of type 2 and $\delta([w])$ is a 3-arc of type 2. So there exists α such that $\alpha\delta([v]) = \delta([w])$. Then, $\delta^{-1}\alpha\delta([v]) = [w]$.
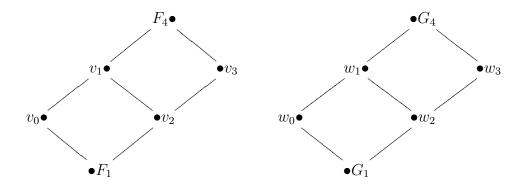
$$[v] \xrightarrow{\delta^{-1}\alpha\delta} [w]$$
$$\downarrow_{\delta} \qquad \uparrow_{\delta^{-1}}$$
$$\delta([v]) \xrightarrow{\alpha} \delta([w])$$

Thus we can restrict to the case where $[v]$ and $[w]$ are 3-arcs of type 2 in $M(\mathcal{P})$. In other words, $rank(v_0) = rank(v_2) = rank(w_0) = rank(w_2) = 2$ and $rank(v_1) = rank(v_3) = rank(w_1) = rank(w_3) = 3$. From the conditions of the theorem every two 2-faces in a 3-face of $\mathcal{P}$ intersect in a 1-face. Therefore, since the 2-faces v_0 and v_2 lie in the 3-face v_1, there exists a 1-face, F_1, such that F_1 is incident to both v_0 and v_2. Similarly there exists another 1-face G_1, such that G_1 is incident to both w_0 and w_2. Also, since $\mathcal{P}$ is self-dual, any two 3-faces which share a 2-face are incident to a common 4-face. So, since the 3-faces v_1 and v_3 share the 2-face v_2, there exists a

4-face, F_4, such that F_4 is incident to both v_1 and v_3. Similarly there exists another 4-face G_4, such that G_4 is incident to both w_1 and w_3. Thus to summarize, the Hasse diagram of $\mathcal{P}$ contains the following subgraphs:

Now consider the chains (F_1, v_0, v_1, F_4) and (G_1, w_0, w_1, G_4) in $\mathcal{P}$. From the conditions of the theorem, we know that $\Gamma(\mathcal{P})$ acts transitively on the chains of type $\{1,2,3,4\}$. So there exists an α in $\Gamma(\mathcal{P})$ such that $\alpha(F_1, v_0, v_1, F_4) = (G_1, w_0, w_1, G_4)$.

By property (P4) the only proper faces of w_1/G_1 are w_0 and w_2. So, since $\alpha(v_0) = w_0$, it must be that $\alpha(v_2) = w_2$. Again by property (P4) the only proper faces of the section G_4/w_2 are w_1 and w_3. Since $\alpha(v_2, v_1, F_4) = (w_2, w_1, G_4)$, it must be that $\alpha(v_3) = w_3$. So $\alpha(v_0, v_1, v_2, v_3) = (w_0, w_1, w_2, w_3)$ as we wanted.
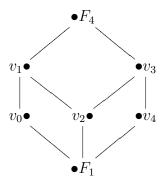
The action of $\mathrm{D}(\mathcal{P})$ is transitive on the set of 3-arcs of type 2, and therefore transitive on the set of all 3-arcs. To show that $\mathrm{M}(\mathcal{P})$ is 3-arc transitive, all that remains is to show that $\mathrm{D}(\mathcal{P})$ is not transitive on the set of 4-arcs in $\mathrm{M}(\mathcal{P})$.

To show this we will show that there exist two 4-arcs $[v] = (v_0, v_1, v_2, v_3, v_4)$ and $[w] = (w_0, w_1, w_2, w_3, w_4)$ in $\mathrm{M}(\mathcal{P})$, both of type 2, such that there cannot be an automorphism α of $\mathcal{P}$ with $\alpha([v]) = [w]$.

We construct the 4-arcs by extending a 3-arc of type 2 in two non-equivalent ways. To begin with, suppose (v_0, v_1, v_2, v_3) is any 3-arc of $\mathrm{M}(\mathcal{P})$ of type 2. Let F_1 be the

rank 1 face incident to v_0 and v_2 like before, and let F_6 be the greatest face of the polytope $\mathcal{P}$. Each section of $\mathcal{P}$ is a polytope itself, with F_6/F_1 a rank 4 polytope. As to not confuse the rank function of this restricted polytope, with that of $\mathcal{P}$, let us call this new polytope $\mathcal{Q}$ and its rank function $rank_Q$. We then have that F_1 is the least face of $\mathcal{Q}$ so $rank_Q(F_1) = -1$, similarly $rank_Q(v_0) = 0 = rank_Q(v_2)$, and $rank_Q(v_1) = 1 = rank_Q(v_3)$.
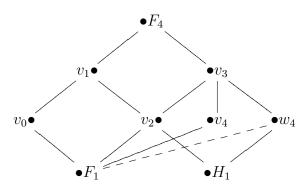
The polytope Q also contains the face F_4 associated with v_1 and v_3 as before, and $rank_Q(F_4) = 2$. Relative to Q, the face F_4 (strictly speaking F_4/F_1) is a p-gon for some $p \geq 3$, containing the faces v_0, v_1, v_2, and v_3. This is true since if $p = 2$ the polytope will not satisfy the conditions of the theorem. We now choose v_4 to be a face of $rank_Q$ equal to 0, which is incident with v_3 but distinct from v_2. Now let $[v] = (v_0, v_1, v_2, v_3, v_4)$.

In other words, we have the following arrangement in the Hasse Diagram (in terms of convex geometry, we have two adjacent edges of a polygon defining three vertices of that polygon):

Now, let us consider the pointwise stabilizer of the 3-arc (v_0, v_1, v_2, v_3) in $\Gamma(\mathcal{P})$. Let β be any automorphism in this stabilizer group. Then β fixes v_0, v_1, and v_2, so β fixes F_1 as well. Note here that, by our assumptions on $\mathcal{P}$, the 2-faces v_0 and v_2 have only one 1-face in common, namely F_1. Similarly $\beta(F_4) = F_4$. Therefore, the image

under β of the section F_4/F_1 is F_4/F_1 itself.

Let w_4 be a 2-face incident with v_3 which is not incident with F_1. Thus $w_4 \neq v_0, v_1, ..., v_4$. The neighborliness condition of the 3-face v_3 of $\mathcal{P}$ implies that v_2 and w_4 are incident to a common 1-face distinct from F_1, which we will call H_1. We now set $[w] := (v_0, v_1, v_2, v_3, v_4)$. Note that any automorphism of $\mathcal{P}$ that maps $[v]$ to $[w]$ must necessarily fix the 3-arc (v_0, v_1, v_2, v_3) pointwise.
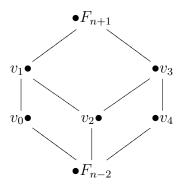
Now we saw that any automorphism β that is in the pointwise stabilizer of the 3-arc (v_0, v_1, v_2, v_3) has the property $\beta(F_4/F_1) = F_4/F_1$, and we know $w_4 \notin F_4/F_1$. So there can be no automorphism of $\mathcal{P}$ that sends the 4-arc $(v_0, v_1, v_2, v_3, v_4)$ to $(v_0, v_1, v_2, v_3, w_4)$, and thus $M(\mathcal{P})$ is not 4-arc transitive.

$\square$

Theorem 21. *Let $\mathcal{P}$ be a rank 2n abstract polytope $(n \geq 2)$, which is self-dual, has automorphism group acting transitively on the chains of type $\{n\text{-}2, n\text{-}1, n, n+1\}$, and for every n-face F_n of $\mathcal{P}$ the dual of F_n/F_{-1} is a neighborly polytope in which any two vertices are incident with exactly one edge. Then $M(\mathcal{P})$ is 3-arc transitive.*

Proof. Here the statement that for every n-face F_n of $\mathcal{P}$ the dual of F_n/F_{-1} is a neighborly polytope of the type mentioned means that in any n-face of $\mathcal{P}$ any two (n-1)-faces share a common (n-2)-face. The proof of the rank 2n case is an easy

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generalization of the prior theorem, with the neighborliness condition providing the needed faces of rank (n-2) and (n+1) for the chains. The only change in the proof is in establishing the existence of the following subgraph of the Hasse diagram:

$$
\begin{array}{c}
\bullet F_{n+1} \\
v_1 \bullet \qquad \bullet v_3 \\
v_0 \bullet \qquad v_2 \bullet \qquad \bullet v_4 \\
\bullet F_{n-2}
\end{array}
$$

In the general case, we replace F_4 and F_1 with F_{n+1} and F_{n-2} respectively. To establish the existence of this graph instead of looking at the geometry of F_4/F_1 we can instead consider F_{n+1}/F_{n-2}. This section is also a p-gon for some p≥3, because again if p=2 the polytope will not satisfy the conditions of the theorem, so the argument follows completely. □

We saw in the previous two results that we exclude polytopes with particular 2-gon sections. We now consider a proposition to take care of this case.

Proposition 22. *Let $\mathcal{P}$ be a self-dual polytope of type $\{p_1, .., p_{2n-1}\}$ with $p_n = 2$. Then M($\mathcal{P}$) is a 3-arc transitive complete bipartite graph.*

Proof. The Schläfli type of the polytope implies that for any $(n+1)$−face F_{n+1} of $\mathcal{P}$ and any incident $(n-3)$−face F_{n-3} that the section F_{n+1}/F_{n-3} is the unique 3-polytope of type $\{p_{n-1}, 2\}$. This polytope has the property that each of the two 2-faces is incident to every vertex and edge of the polytope. The 2-faces of these sections are the same as the n-faces of the larger polytope. This along with the strong connectedness is enough to guarantee that every n-face of $\mathcal{P}$ is incident to

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every (n-1)-face of $\mathcal{P}$. Since $\mathcal{P}$ is assumed to be self-dual, $M(\mathcal{P})$ is the complete bipartite graph with independent sets of the same size, and is thus vertex transitive, and in fact 3-arc transitive as well. $\square$

3.3 Existence Results

The last theorem creates a class of polytopes with 3-arc transitive medial layer graphs. Polytopes in this class have a high degree of symmetry, and their faces of middle rank satisfy a "lattice like property," and a "dual neighborliness property." To strengthen this result, we also would like to guarantee that there are some polytopes that have these required properties, and thus medial layer 3-arc transitivity is actually achievable. The first example of this has been studied extensively by Monson, Weiss, and others.

Corollary 23. *If $\mathcal{P}$ is a finite, regular, self-dual 4-polytope of type $\{3,q,3\}$, then $M(\mathcal{P})$ is bipartite, trivalent, and 3-arc transitive.*

Proof. It is clear that $M(\mathcal{P})$ is bipartite and trivalent. Now the group $\Gamma(\mathcal{P})$ acts transitively on the chains of type $\{0,1,2,3\}$ (that is the flags) since $\mathcal{P}$ is regular. Moreover, since every 2-face of $\mathcal{P}$ is a triangle, for every 2-face F_2 in $M(\mathcal{P})$ the dual of F_2/F_{-1} is a neighborly polytope with exactly one 1-face between adjacent 0-faces. So $\mathcal{P}$ meets the conditions of the prior theorem, and thus $M(\mathcal{P})$ is 3-arc transitive. $\square$

In 2005 Barry Monson and Asia Ivic Weiss proved that, in the $\{3,q,3\}$ case, $M(\mathcal{P})$ is actually 3-arc transitive in the stronger classical definition as a graph not just in the polytope-based definition as a medial layer graph. That is to say, $\text{Aut}(M(\mathcal{P}))=D(\mathcal{P})$.

Corollary 24. *If $\mathcal{P}$ is the regular 2n-simplex, then $M(\mathcal{P})$ is 3-arc transitive.*

Proof. Let $\mathcal{P}$ be the regular 2n-simplex. Then $\mathcal{P}$ is self-dual, and $\Gamma(\mathcal{P})$ acts transitively on the chains of type $\{n-2,n-1,n,n+1\}$. Moreover, every n-face of $\mathcal{P}$ is an n-simplex, and any two vertices of an n-simplex are incident with exactly one edge. Since the n-simplex is self-dual, the conditions of Theorem 21 are met, and $M(\mathcal{P})$ is 3-arc transitive. $\square$

Using the relationship between the 2n-simplex and the (2n+1)-hypercube, we can see that this transitivity in the medial layer graph of a simplex gives transitivity in $\mathcal{Q}_{2n+1}(L_n, L_{n+1})$ (see [10]). Here $\mathcal{Q}_{2n+1}(L_n, L_{n+1})$ is the graph obtained by restricting the graph of the 1-skeleton of the $(2n + 1)-$cube to vertices of weight n or $(n + 1)$; recalling that the vertices of a cube can be described using binary strings, we can define the *weight* of a given vertex as the number of 1's in its associated binary string. In particular, $\mathcal{Q}_{2n+1}(L_n, L_{n+1})$ is 3-arc transitive. In Chapter 5 of the same work, it is proved that $\mathcal{Q}_{2n+1}(L_n, L_{n+1})$ is distance transitive. This alone, implies that M($\mathcal{P}$) is at least 1-arc transitive.

Because of the ability to think of M($\mathcal{P}$) as either the medial layer graph of a simplex or as $\mathcal{Q}_{2n+1}(L_n, L_{n+1})$, many more results about the medial layer graph of a simplex of even rank are also known.

Corollary 25. *If $\mathcal{P}$ is self-dual regular 6-polytope of type {3,3,4,3,3}, then M(P) is 3-arc transitive.*

Examples of regular 6-polytopes of this kind are described in Chapter 12D of [21]. Since they are regular, they satisfy the requirement on transitivity of chains in Theorem 21. Their 3-faces are tetrahedral. Therefore the 3-faces are lattices and are dual-neighborly. The graph M($\mathcal{P}$) will be studied more in Chapter 7 for a specific example from this family.

Since the simplex is a self-dual neighborly lattice, if we can find classes of self-dual regular 2n-polytopes with simplices as n-faces, then we can apply the main theorem of this chapter to obtain classes of 3-arc transitive graphs. Recently such polytopes have been discovered using modular reduction techniques (see [23], [24], [25], [26]).

Corollary 26. *For each prime $p > 3$ there is a finite, self-dual 4-polytope of type $\{3, 6, 3\}$, obtained using reduction techniques modulo p, with a 3-regular, bipartite,*

3-arc transitive medial layer graph.

Corollary 27. *For each prime $p \geq 3$ and each $n \geq 2$ there is a finite, self-dual $2n$-polytope of type $\{3, ..., 3, p, 3, ..., 3\}$, obtained using reduction techniques modulo p, with a $(n+1)$-regular, bipartite, 3-arc transitive medial layer graph.*

Chapter 4

Stars in M($\mathcal{P}$)

4.1 Introduction

After examining the transitivity defined in terms of one particular subgraph, a t-arc, of the medial layer graph of a 2n-polytope, we wish to take these concepts further and apply them to other subgraphs.

Recall the definition of a k-star in a simple graph $\mathcal{G}$. A sequence $S = (v_0, v_1, v_2, ..., v_k)$ is a *k-star* if $v_i \in V(G)$ for all i, $v_i \neq v_j$ for $i \neq j$, and $\{v_0, v_i\} \in \mathrm{E}(G)$ for all i$\neq$ 0. We will be considering k-stars in M($\mathcal{P}$) for $\mathcal{P}$ an abstract polytope of rank 2n.

Definition 28. Let $\mathcal{P}$ be an abstract 2n-polytope with medial layer graph M($\mathcal{P}$), we say M($\mathcal{P}$) is *k-star transitive* if D($\mathcal{P}$) acts transitively on the k-stars but not on the (k+1)-stars.

Note again that we are only considering automorphisms of M($\mathcal{P}$) which are induced by automorphisms or dualities of $\mathcal{P}$. Moreover, observe that $\mathcal{P}$ must necessarily be self-dual if M($\mathcal{P}$) is k-star transitive for any $k \geq 0$.

We notice a couple things to start. First k-star transitivity and k-arc transitivity will turn out to be the same for $k \leq 2$. However, this is clearly not the case for larger

values of k. Second, we notice that obviously if a graph has any star transitivity, then it clearly is vertex transitive as well, and thus is r-regular for some $r \geq k$.

Also similar to the chapter on arc transitivity, if a graph is distance transitive, it will then necessarily be 1-star transitive as well. However, distance transitivity need not imply higher k-star transitivity for larger values of k.

4.2   3-star Transitivity

In this section we try to understand the combinatorial properties of polytopes $\mathcal{P}$ in which the extended automorphism group acts transitively on the 3-stars in M($\mathcal{P}$). For the remaining lower values of k, k-star transitivity can be expressed in other well known notions. In particular, 2-star transitivity is the same as 2-arc transitivity studied in the last section. If a graph is 1-star transitive it is equivalently 1-arc transitivie, or a *symmetric* graph. And finally 0-star transitivity is simply vertex transitivity of a graph.

We start this section with a class of polytopes that have high star transitivity.

Proposition 29. *If $\mathcal{P}$ is a self dual regular 2n-polytope of type $\{p_1, p_2, ..., p_{2n-1}\}$ with $p_n = 2$, then M($\mathcal{P}$) is k-star transitive where $k = f_n$.*
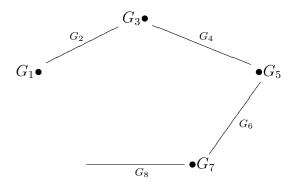
Proof. To prove this we use a result of [21], that says if $\mathcal{P}$ is of type $\{p_1, p_2, ..., p_{2n-1}\}$ with $p_n = 2$ then every n-face of $\mathcal{P}$ is incident to every (n-1)-face of $\mathcal{P}$. Since $\mathcal{P}$ is also self dual, M($\mathcal{P}$) is the bipartite complete graph on $2f_n$ vertices. This graph is easily shown to be f_n-star transitive. $\qquad\square$

We next get a result relating to the Schläfli type of the polytope. Similar to the last section, we need a "lattice like condition" to avoid degenerate cases.

Theorem 30. *If $\mathcal{P}$ is an equivelar 2n-polytope of type $\{p_1, p_2, ..., p_{2n-1}\}$ such that any three (n-1)-faces are incident with at most one (n-3)-face, and if M($\mathcal{P}$) is k-star transitive for $k \geq 3$, then $p_{n-1} = p_{n+1} = 3$.*

Proof. Fix a base flag $(F_{-1}, F_0, ..., F_{2n})$ in $\mathcal{P}$. The section F_n/F_{n-3} is isomorphic to a $p_{(n-1)}$-gon. Thus $\Gamma(F_n/F_{n-3}) \cong D_{2(p_{n-1})}$, the dihedral group of order $2p_{n-1}$. Assume to the contrary that $p_{n-1} \geq 4$, then there exists an alternating sequence of rank (n-2) and rank (n-1) faces in F_n/F_{n-3} of length 8. Denote this sequence

[G]=$(G_1, G_2, ..., G_8)$. If this is unclear, imagine in the convex case, 4 adjacent edges in a cycle and the sequence of vertices obtained from their intersections. For example in the case $p_{n-1} = 5$ the convex realization of [G] in F_n/F_{n-3} is:
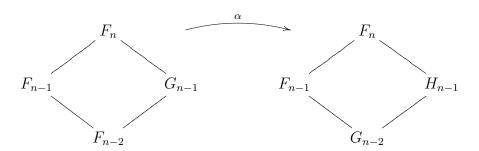
$$G_3 \bullet$$
$$G_2 \qquad\qquad G_4$$
$$G_1 \bullet \qquad\qquad\qquad\qquad \bullet G_5$$
$$G_6$$
$$\underline{\qquad\qquad} \bullet G_7$$
$$G_8$$

Consider the two 3-stars $S_1 = (F_n, G_2, G_4, G_6)$ and $S_2 = (F_n, G_2, G_4, G_8)$. Since M($\mathcal{P}$) is k-star transitive for k$\geq$ 3, there exists an automorphism α of $\mathcal{P}$ that takes S_1 to S_2. That is, α fixes F_n, G_2, and G_4, and maps G_6 to G_8. Now recall our assumptions on $\mathcal{P}$. Since F_{n-3} is incident with G_2, G_4, G_6, its image $\alpha(F_{n-3})$ is incident with G_2, G_4, G_8, and hence must coincide with F_{n-3} the only (n-3)-face that is incident with G_2, G_4, and G_8. Thus α fixes F_{n-3} and induces an automorphism of the section F_n/F_{n-3}. Thus α acts like an element of $D_{2(p_{n-1})}$ on the faces of F_n/F_{n-3}. However, no such automorphism can exist that fixes two adjacent 1-faces of this section and permutes two others.

Therefore, there is no automorphism $\mathcal{P}$ that would send S_1 to S_2. So M($\mathcal{P}$) is at most 2-star transitive, which is a contradiction to the conditions of the theorem. Therefore, our assumption that $p_{n-1} \geq 4$ must be false. If $p_{n-1} = 2$ then $\mathcal{P}$ would fail to satisfy the lattice type condition on the intersection of three (n-1)-faces. Thus we have $p_{n-1} = 3$ as desired. Using the self-duality of $\mathcal{P}$ we get $p_{n+1} = 3$ by similar arguments. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We next are able to prove a strong relationship between polytopes that are 3-arc transitive and those that are 3-star transitive.

Theorem 31. *Let $\mathcal{P}$ be a rank 2n abstract polytope. If M($\mathcal{P}$) is 3-star transitive, then for any n-face F_n of $\mathcal{P}$ the dual of F_n/F_{-1} is a neighborly polytope.*

Proof. Fix a base flag $(F_{-1}, F_0, ..., F_{2n})$. By property (P4), the section F_n/F_{n-2} is diamond shaped. Let F_{n-1} and G_{n-1} be the proper faces in this section. Also, let H_{n-1} be another (n-1)-face of F_n and consider the 3-stars $(F_n, F_{n-1}, G_{n-1}, H_{n-1})$ and $(F_n, F_{n-1}, H_{n-1}, G_{n-1})$. Since $M(\mathcal{P})$ is 3-star transitive, there is an $\alpha \in \Gamma(\mathcal{P})$ such that $\alpha((F_n, F_{n-1}, G_{n-1}, H_{n-1})) = (F_n, F_{n-1}, H_{n-1}, G_{n-1})$. Since α is an automorphism of the polytope it must map a section of $\mathcal{P}$ to another. And since $\alpha(F_n) = F_n$ it must send the section F_n/F_{n-2} to the section F_n/G_{n-2}, where G_{n-2} is incident to F_{n-1} and H_{n-1}.

Therefore, since H_{n-1} was arbitrary, any two (n-1)-faces of P must be incident to a common (n-2)-face. Thus, when we take the dual of F_n, any two 0-faces must be incident to a common 1-face. So the dual of F_n is neighborly. $\square$

Corollary 32. *Let $\mathcal{P}$ be a self dual regular rank 2n abstract polytope such that any two (n-1)-faces of an n-face are incident with at most one (n-2)-face. If M($\mathcal{P}$) is 3-star transitive, then M($\mathcal{P}$) is 3-arc transitive.*

Proof. Let $\mathcal{P}$ be a self-dual regular rank 2n abstract polytope with M($\mathcal{P}$) 3-star

transitive. By Theorem 31 for any n-face $F_n \in \mathcal{P}$ the dual of F_n/F_{-1} is a neighborly polytope, and by our assumptions on $\mathcal{P}$, any two vertices in this dual are incident with exactly one edge. Further $\mathcal{P}$ being regular implies that $\Gamma(\mathcal{P})$ acts transitively on the chains of type {n-2,n-1,n ,n+1}. So by Theorem 21, $M(\mathcal{P})$ is 3-arc transitive. $\square$

Theorem 33. *Let $\mathcal{P}$ be a regular self-dual rank 6 abstract polytope which is also a lattice. If M($\mathcal{P}$) is 3-star transitive, then $\mathcal{P}$ is either of type {3,3,r,3,3} or {5,3,r,3,5} for some $r \geq 3$. In fact, the 3-faces of $\mathcal{P}$ must be tetrahedra {3,3} or hemi-dodecahedra {5,3}$_5$.*

Proof. Let $\mathcal{P}$ be a regular self-dual rank 6 abstract polytope, which is also a lattice. Then it is clear that all sections of $\mathcal{P}$ are also lattices. If $M(\mathcal{P})$ is 3-star transitive, then by Theorem 30, $\mathcal{P}$ is of type $\{p_1, 3, p_3, 3, p_4\}$. Note that the condition of Theorem 30 on the (n-1)-faces and the (n-3)-faces is satisfied since $\mathcal{P}$ is a lattice. In particular for any 3-face F_3 the section F_3/F_{-1} is of type $\{p_1, 3\}$. So the dual of any 3-face is of type $\{3, p_1\}$. Also by Theorem 31, we know that the dual of F_3/F_{-1} is a neighborly polytope. The only neighborly regular 3-polytopes of type $\{3, p_1\}$ which are lattices are the tetrahedron and the hemi-icosahedron (Lemma 11C6 in [21]). So we can conclude p_1 must be 3 or 5. $\square$

Chapter 5

Medial Comparability Graphs for (2n+1)-Polytopes

5.1 Introduction

Unlike in the even rank case, where there is a natural way to define the medial layer graph of a rank 2n abstract polytope, when the rank is (2n+1) there are two possible natural definitions for a similar object. We first study the 1-skeleton of the order complex restricted to faces of ranks (n-1), n, and (n+1). This is known as the comparability graph on the faces of those ranks. In the next chapter, we will study the restriction of the Hasse diagram to the ranks (n-1), n, and (n+1). In the rank 2n case these approaches, for faces of rank (n-1) and n, lead to studying the same object, which we defined as the medial layer graph. However, in the (2n+1) case, the graphs are different. The Hasse diagram hides the transitivity of the partial order, so that there is not an edge between comparable faces that differ in rank by 2 or more. Whereas in the comparability graph there is an edge between any comparable faces regardless of rank.

In the even rank case we were able to use the group $D(\mathcal{P})$ to study the graph automorphisms of $M(\mathcal{P})$. The group $D(\mathcal{P})$ gave us the flexibility to map faces of different ranks to each other. However, when the rank is (2n+1) the extended group of $\mathcal{P}$ will not give us enough flexibility. For example, $D(\mathcal{P})$ will not contain any bijections that send an (n-1)-face to an n-face. For this reason, we will need to use the entire group of graph automorphisms of the graph to study its symmetry, not just the ones that come from $\mathcal{P}$ itself.

Definition 34. Let $\mathcal{P}$ be a (2n+1)-polytope. The (comparability graph based) *medial layer graph* $M_C(\mathcal{P})$ is the simple graph with vertex set consisting of the faces F of $\mathcal{P}$ such that $rank(F) = (n-1), n,$ or $(n+1)$. The edge set of $M_C(\mathcal{P})$ is defined such that F and G are adjacent in $M_C(\mathcal{P})$ if and only if F and G are incident in $\mathcal{P}$.

This is a particular example of the graphs $\mathcal{C}_I(\mathcal{P})$ for I={n-1,n,n+1} (see section 2.3). We will be examining the transitivity of this medial layer graph $M_C(\mathcal{P})$. First let us look at the least restrictive transitivity, and try to understand when $M_C(\mathcal{P})$ is vertex transitive. It is clear that if $M_C(\mathcal{P})$ is going to be vertex transitive, it must be a k-regular graph.

5.2 Connectivity

To understand the transitivity properties on $M_C(\mathcal{P})$ we first need to understand the connectivity of some related graphs.

Definition 35. For any graph $\mathcal{G}$ with edge set $E(\mathcal{G})$ and vertex set $V(\mathcal{G})$, and for any set $J \subset V(\mathcal{G})$, define a new graph $\mathcal{G} - J$ with vertex set $V(\mathcal{G} - J) = \{v \in V(\mathcal{G}) | v \notin J\}$ and edge set $E(\mathcal{G} - J) = \{\{v, w\} \in E(\mathcal{G}) | v \notin J \text{ and } w \notin J\}$.

Lemma 36. *Let $\mathcal{P}$ be a rank (2n+1) polytope with medial layer graph $M_C(\mathcal{P})$. Then, for $i \in \{$ n-1,n,n+1$\}$, $M_C(\mathcal{P}) - \mathcal{P}_i$ is a connected graph, where $\mathcal{P}_i$ is the set of $i-$faces of $\mathcal{P}$.*

Proof. First note that $M_C(\mathcal{P})$ - $\mathcal{P}_{n-1} = \text{skel}_{n+1}((\text{skel}_n(P^*))^*)$ and $M_C(\mathcal{P})$ - $\mathcal{P}_{n+1} = \text{skel}_n((\text{skel}_{n+1}(P^*))^*)$, so both $M_C(\mathcal{P})$ - $\mathcal{P}_{n-1}$ and $M_C(\mathcal{P})$ - $\mathcal{P}_{n+1}$ are connected by Lemma 3.

We can treat $M_C(\mathcal{P})$ - $\mathcal{P}_n$ directly. Suppose we have two vertices of $M_C(\mathcal{P})$ - $\mathcal{P}_n$ representing faces F and G of $\mathcal{P}$, each of rank (n-1) or (n+1). First consider the case that F and G both have rank (n-1), and exploit the connectedness of $M_C(\mathcal{P})$ - $\mathcal{P}_{n+1}$. Then there exists a sequence of (vertices representing) faces $F = H_0, H_1, ..., H_{l-1}, H_l = G$ such that H_{i-1} and H_i are incident for each i, and H_i has rank (n-1) or n for each i. We may assume that the faces H_i are mutually distinct, so exactly the faces H_i with i odd, have rank n. If we now replace each H_i, with i odd, by an (n+1)-face incident with it, we arrive at a sequence in $M_C(\mathcal{P})$ - $\mathcal{P}_n$ connecting F and G in $M_C(\mathcal{P})$ - $\mathcal{P}_n$. This settles the case where F and G have rank (n-1). A very similar argument applies dually to the case where F and G have rank (n+1).

Finally, then, if the ranks of F and G are distinct, with $\text{rank}(F) = n - 1$ and $\text{rank}(G) = n + 1$, we first replace G by a (n-1)-face G' incident with it. Next join
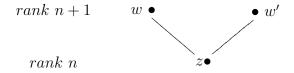
F and G' by a sequence in $\mathrm{M}_C(\mathcal{P})$ - $\mathcal{P}_n$, and then append G to this sequence. This gives a sequence joining F and G in $\mathrm{M}_C(\mathcal{P})$ - $\mathcal{P}_n$, as desired.

$\square$

5.3 Vertex Transitivity

Proposition 37. *Let $\mathcal{P}$ be a rank (2n+1) polytope. If $M_C(\mathcal{P})$ is vertex transitive, for $n \geq 1$, then every graph automorphism of $M_C(\mathcal{P})$ maps vertices of equal ranks to vertices of equal ranks. That is, for all $\alpha \in Aut(M_C(\mathcal{P}))$, all $i \in \{n-1, n, n+1\}$, and all $w, w' \in \mathcal{P}_i$, we have rank($\alpha(w)$)=rank($\alpha(w')$).*
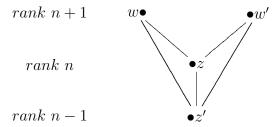
Proof. Let $\mathcal{P}$ be a rank (2n+1) polytope and $M_C(\mathcal{P})$ be its associated vertex transitive medial layer graph. Assume to the contrary that there exists i $\in$ {n-1,n,n+1}, $\alpha \in$ Aut($M_C(\mathcal{P})$), and $w, w' \in \mathcal{P}_i$ such that rank($\alpha(w)$) $\neq$ rank($\alpha(w')$). By the prior lemma, we know that $M_C(\mathcal{P}) - \mathcal{P}_i$ is a connected graph. We will break the proof into three cases, one for each possible value of i, and prove each case by inducting on the length of a shortest possible path between w and w' in these restricted graphs.

First let i=n+1. Let m denote the length of a shortest path between w and w' in $M_C(\mathcal{P}) - \mathcal{P}_{n-1}$. Then $m \geq 2$, by our assumption on w and w'. If $m = 2$ consider the diagram.

In this case wzw' is a path from w to w' in $M_C(\mathcal{P}) - \mathcal{P}_{n-1}$. So we want to show that rank($\alpha(w)$) = rank($\alpha(w')$).

Both w and w' are incident with z in $\mathcal{P}$. Let z' be any rank (n-1) face of $\mathcal{P}$ incident with z. Then w and w' are both comparable to z'. So in our graph $M_C(\mathcal{P})$ we have the following subgraph:

rank n + 1 $w\bullet$ $\bullet w'$

rank n $\bullet z$

rank n − 1 $\bullet z'$

Now α preserves adjacency, so it sends that subgraph to another one of the same shape. By our assumption we have rank($\alpha(w)$)$\neq$rank($\alpha(w')$). Also since in $M_C(\mathcal{P})$ there are only edges between vertices of differing ranks, and z and z' each are joined to all other vertices, the ranks of $\alpha(w)$, $\alpha(w')$, $\alpha(z)$, and $\alpha(z')$ must necessarily be mutually distinct. However there are only three possible ranks in $M_C(\mathcal{P})$, and we have four different ranks that are not equal. This is a contradiction. Therefore our assumption is not true, and rank($\alpha(w)$)=rank($\alpha(w')$).

Now assume by induction that for any two vertices $v, v' \in \mathcal{P}_{n+1}$ such that the length of a shortest path between them in $M_C(\mathcal{P}) - \mathcal{P}_{n-1}$ is smaller than m, that rank($\alpha(v)$)=rank($\alpha\ (v')$).
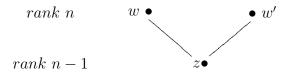
Let w and w' be as before. Let $T = v_0, v_1, ..., v_m$, with $v_0 = w$ and $v_m = w'$ be a shortest path of length m between w and w' in $M_C(\mathcal{P}) - \mathcal{P}_{n-1}$. Since T is a path in a bipartite graph, its vertices alternate rank. So $rank(v_{m-2}) = rank(v_m) = n + 1$. Now $T' = v_0, v_1, ..., v_{m-2}$ is a path of length $< m$ in $M_C(\mathcal{P}) - \mathcal{P}_{n-1}$ so by induction we have $rank(\alpha(v_{m-2})) = rank(\alpha(v_0))$. From the base case m=2 we know that $rank(\alpha(v_{m-2})) = rank(\alpha(v_m))$, thus $rank(\alpha(v_m)) = rank(\alpha(v_0))$ where $v_0 = w$ and $v_m = w'$. This completes the proof for $i = n + 1$.

For i=n-1, thinking dually, the proof is almost identical to the proof of the case $i = n + 1$. We omit the details.

Finally let i=n. Again let m denote the length of a shortest path between w and

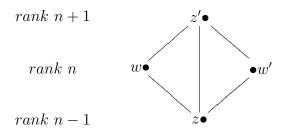w', now in $M_C(\mathcal{P}) - \mathcal{P}_{n+1}$. Then $m \geq 2$, and like in the prior cases we will induct on m.
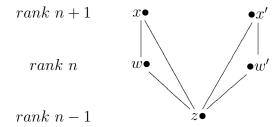
For m=2 consider the diagram

$$rank\ n \qquad w \bullet \qquad\qquad\qquad \bullet\ w'$$
$$rank\ n-1 \qquad\qquad\qquad\qquad z \bullet$$

Let S be a shortest path from w to w' in $M_C(\mathcal{P}) - \mathcal{P}_{n-1}$. If S has length 2, then there exists a vertex z', with rank(z')=n+1, such that z' is incident with w and w'. In $M_C(\mathcal{P})$ we then have the following subgraph:

$$rank\ n+1 \qquad\qquad\qquad z' \bullet$$
$$rank\ n \qquad w\bullet \qquad\qquad \bullet w'$$
$$rank\ n-1 \qquad\qquad\qquad z \bullet$$

Again in $M_C(\mathcal{P})$ edges only exist between vertices of differing ranks (and z, z' are joined to all other vertices), and any automorphism of $M_C(\mathcal{P})$ preserves adjacency, so we know that the ranks of $\alpha(w)$, $\alpha(z)$, and $\alpha(z')$ are all mutual(ly distinct, as are the ranks of $\alpha(w')$, $\alpha(z)$, and $\alpha(z')$. Since there are only three ranks in $M_C(\mathcal{P})$ we know that rank($\alpha(w)$)=rank($\alpha(w')$).

If S has length greater than 2, then there exist a vertex x and a vertex x', both of rank n+1, incident with w and w' respectively. In $M_C(\mathcal{P})$ we then have the following subgraph:

Now by our considerations for $i = n + 1$ we know that $\text{rank}(\alpha(x))=\text{rank}(\alpha(x'))$. However, this forces the ranks of $\alpha(w)$, $\alpha(z)$, and $\alpha(x)$ to be mutually distinct, and similarly for $\alpha(x')$, $\alpha(w')$, and $\alpha(z)$. So $\text{rank}(\alpha(w)) = \text{rank}(\alpha(w'))$ as we wanted.

To finish, the inductive step for the case $i = n$ is proved identically to that of for the case $i = n+1$. So all automorphisms of the medial layer graph have the property that if two vertices have equal ranks, then they are mapped to two vertices of equal ranks.

$\square$

Corollary 38. *If $\mathcal{P}$ is a rank $(2n + 1)$ polytope and $M_C(\mathcal{P})$ is vertex transitive then $f_{n-1} = f_n = f_{n+1}$*

Proof. Let $x \in \mathcal{P}_{n+1}$, $y \in \mathcal{P}_n$, and $z \in \mathcal{P}_{n-1}$. Since $M_C(\mathcal{P})$ is vertex transitive, there exists an $\alpha \in \text{Aut}(M_C(\mathcal{P}))$ such that $\alpha(x)=y$. Then by the proposition, for every $v \in \mathcal{P}_{n+1}$ we have $\text{rank}(\alpha(v))=n$. Thus $f_{n+1} \leq f_n$. Similarly, there exists an $\beta \in \text{Aut}(M_C(\mathcal{P}))$ such that $\beta(y)=z$. Then by the proposition, for every $v \in \mathcal{P}_n$ we have $\text{rank}(\beta(v))=$n-1. Thus $f_n \leq f_{n-1}$. And also there exists an $\gamma \in \text{Aut}(M_C(\mathcal{P}))$ such that $\gamma(z)=x$. Then by the proposition, for every $v \in \mathcal{P}_{n-1}$ we have $\text{rank}(\gamma(v))=$n+1. Thus $f_{n-1} \leq f_{n+1}$. Putting all the inequalities together we get $f_{n-1} = f_n = f_{n+1}$ as we claimed. $\square$

5.4   Rank 3

We continue our investigation of this class of graphs by completely classifying the 3-polytopes with transitivity in $M_C(\mathcal{P})$.

Theorem 39. *If $\mathcal{P}$ is a 3-polytope and $M_C(\mathcal{P})$ is vertex transitive, then $\mathcal{P} = \{2, 2\}$, the universal regular polytope of type $\{2, 2\}$ (with $f_0 = f_1 = f_2 = 2$) and $M_C(\mathcal{P})$ is isomorphic to the (1-arc transitive) 1-skeleton of the octahedron.*

Proof. Let G_0 be a vertex, G_1 an edge, and G_2 a 2-face of $\mathcal{P}$. Their valencies in $M_C(\mathcal{P})$ are $2p$, 4, and $2q$ respectively, so $p = q = 2$. Hence $\mathcal{P}$ is equivelar of type $\{2, 2\}$. Moreover, there is only one polytope of type $\{2, 2\}$, namely the universal regular polytope of this type, denoted $\{2, 2\}$. To see this consider the figure below:
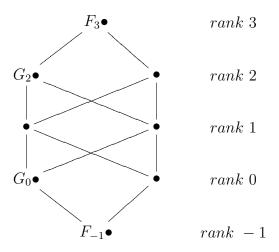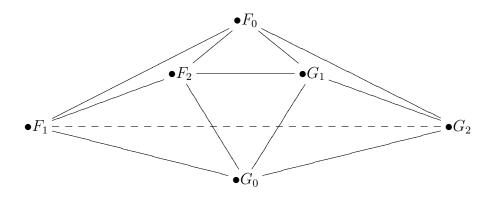
Figure 5.1: Unique 3-Polytope with vertex transitive $M_C(\mathcal{P})$

If $\mathcal{P}$ is of type $\{2, 2\}$, then G_2/F_{-1} and F_3/G_0 are 2-gons, forcing the diagram for $\mathcal{P}$ to close up as indicated. In this diagram, every flag has a unique $i-$adjacent flag for each i, so no further faces can be added without violating the diamond condition. Thus the diagram shown is the Hasse diagram for $\{2, 2\}$, and $\mathcal{P} = \{2, 2\}$.

Now inspection shows that $\mathrm{M}_C(\mathcal{P})$ is just the 1-skeleton of the octahedron. The pairs of faces of $\mathcal{P}$ of the same rank correspond to pairs of antipodal vertices of the octahedron. Notice that the latter is 1-arc transitive.

Chapter 6

Medial Hasse Diagrams for

(2n+1)-Polytopes

6.1   k-Regularity

We saw in the last chapter, that the comparability graph gives one possible way to define a medial layer graph for a polytope of odd rank. In this chapter we provide another option for a medial layer graph, based on the Hasse diagram, and get rather different results.

Definition 40. Let $\mathcal{P}$ be a (2n+1)-polytope. The (Hasse diagram based) *medial layer graph $M_H(\mathcal{P})$* is the simple graph with vertex set consisting of the faces F of $\mathcal{P}$ such that rank(F) = (n-1), n, or (n+1). The edge set of $M_H(\mathcal{P})$ is defined such that F and G are adjacent in $M_H(\mathcal{P})$ if and only if F and G are incident in $\mathcal{P}$ and $|rank(F) - rank(G)| = 1$.

This is the special case of $\mathcal{H}_I(\mathcal{P})$ for I={(n-1),n,(n+1)} (see Section 2.2). We would like to understand what types of polytopes have transitive medial layer graphs

using this definition. If a graph is to have any of the transitivities as defined in previous chapters, a minimum requirement is that it be k-regular for some k. We will see that this weaker condition yields interesting results.

Lemma 41. *If $\mathcal{P}$ is a self-dual (2n+1)-polytope and $M_H(\mathcal{P})$ is k-regular for some k, then $f_n = 2f_{n-1} = 2f_{n+1}$.*

Proof. Let $\mathcal{P}$ be a (2n+1)-polytope with $M_H(\mathcal{P})$ k-regular. Then $M_H(\mathcal{P})$ is a bipartite graph, with one partition being $\mathcal{P}_n$ and the other partition the union of $\mathcal{P}_{n-1}$ and $\mathcal{P}_{n+1}$. Now we can count all the edges of $M_H(\mathcal{P})$ by summing up the degrees of vertices in one partition. Doing this we get $|E(M_H(\mathcal{P}))|=(k)(f_n)$ one way and $|E(M_H(\mathcal{P}))|=(k)(f_{n-1} + f_{n+1})$ the other. $\mathcal{P}$ being self-dual ensures that $f_{n-1} = f_{n+1}$. $\square$

6.2 Maps of Type {4,4}

As in the previous chapter, we will attempt to classify the rank 3 case completely.

Lemma 42. *If $\mathcal{P}$ is a self-dual 3-polytope and $M_H(\mathcal{P})$ is k-regular for some k, then $\mathcal{P}$ is equivelar of type $\{4, 4\}$ and $k = 4$.*

Proof. Let G_0, G_1, and G_2 be a vertex, an edge, and a 2-face of $\mathcal{P}$. Their valencies in $M_H(\mathcal{P})$ are p, 4, and q respectively for some $p, q \geq 2$. Since $M_H(\mathcal{P})$ is k-regular, we must have $k = p = 4 = q$. In particular, $\mathcal{P}$ is of type $\{4, 4\}$ and $k = 4$. $\qquad\square$

Example 43. *Let $\mathcal{P}$ be the regular self-dual 3-polytope of type $\{4, 4\}_{(2,0)}$. Then $M_H(\mathcal{P})$ is drawn below:*
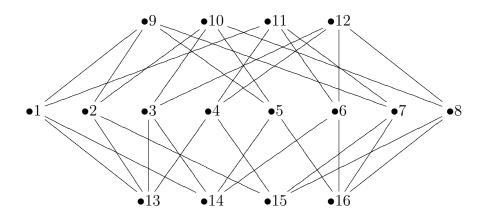
Figure 6.1: Unique 2-arc and distance transitive $M_H(\mathcal{P})$

We notice immediately that the girth of $M_H(\mathcal{P})$ is 4, so we have a bound on the t-arc transitivity of $t \leq 3$. Consider the pointwise stabilizer of the 2-arc (1,9,2) in $\mathrm{Aut}(M_H(\mathcal{P}))$. In this group, it is clear that no graph automorphism could send the vertex labeled (10) to the vertex labeled (13), since they are both neighbors of (2), and (13) is adjacent to (1) while (10) is not. So we know that the group of automorphisms

of $M_H(\mathcal{P})$ does not act transitively on the 3-arcs by Lemma 11. In fact, the 3-arc (1,9,2,10) cannot be mapped onto the 3-arc (1,9,2,13).

Using the GAP package GRAPE, we can fully understand all automorphisms of this graph. In particular $|\mathrm{Aut}(M_H(\mathcal{P}))|$=384. From looking at the pointwise arc stabilizer sequence, we know that for any k-regular graph on n vertices, the group of automorphisms has size $(n)(k)(k-1)^{(t-1)}(|A|)$, where $t(\geq 1)$ is the arc transitivity of the graph and A is the pointwise stabilizer of any t-arc. Now 384=(16)(4)(3)(2), so if $M_H(\mathcal{P})$ is 2-arc transitive the size of the pointwise stabilizer group of any 2-arc should be 2. Using GAP, we see that indeed the pointwise stabilizer of the 2-arc [1,9,2] is the subgroup of S_{16} consisting of identity and the element (3,4)(5,7)(10,15)(11,14). Moreover, $\mathrm{M}_H(\mathcal{P})$ is 2-arc transitive.
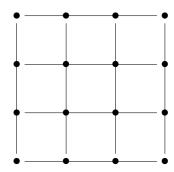
One last thing to notice about this example is that $M_H(\mathcal{P})$ is also distance transitive. Meaning for all vertices u,v,x,y of $M_H(\mathcal{P})$ such that $\delta(u,v) = \delta(x,y)$ there is a graph automorphism α such that α(u)=x and α(v)=y where δ is the distance between two vertices.

Proposition 44. *If $\mathcal{P}$ is a toroidal edge-transitive 3-polytope of type {4,4} but $\mathcal{P} \neq$ {4,4}$_{(2,0)}$, then $M_H(\mathcal{P})$ is 1-arc transitive.*
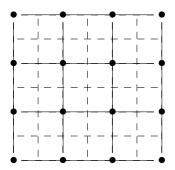
Proof. Let $\mathcal{P}$ be a 3-polytope of type {4,4}. When $\mathcal{P}$ is of type {4,4} it can be considered as a map on the torus or on the Klein bottle. We have ruled out the Klein bottle. We saw in the previous example, if $\mathcal{P}$ is of type {4,4}$_{(b,c)}$ for (b,c)=(2,0) or (0,2), then $M_H(\mathcal{P})$ is 2-arc transitive. Siran, Tucker, and Watkins classified all maps on the torus which are finite and edge transitive into three possible cases. Let $\mathcal{T}$ be the regular tessellation of the Euclidean plane by squares with vertices at points in the integer lattice $\mathbb{Z} \times \mathbb{Z}$. We can then think of $\mathcal{P}$ as the quotient of $\mathcal{T}$ by a sub-lattice. In two of the cases the sub-lattice is generated by two orthogonal integer vectors (b,c)

and (-c,b), and $\mathcal{P}$ is denoted by $\{4,4\}_{(b,c)}$. If bc(b-c)=0 then this map is regular, if not then it is chiral. In the third case, the sub-lattice is generated by two non-zero integer vectors (b,c) and (d,d), where if b+c≠0 then d=b+c, otherwise d can be any integer greater than 1. If d=b+c then this construction gives a polytope if and only if $|b-c| > 1$. If b+c=0, then this construction gives a polytope if and only if $|b|, |d| > 1$.

When we consider $M_H(\mathcal{P})$ we notice that $M_H(\mathcal{P})$ can be thought of as the 1-skeleton of a related self dual 3-polytope of the same type. To see this, consider the picture that represents the 1-skeleton of $\mathcal{P}$.
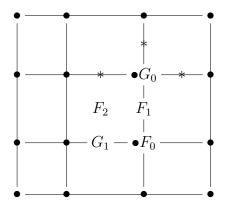

Then leaving the realization of $\mathcal{P}$, and adding edges of $M_H(\mathcal{P})$ in as dotted lines we get

Thus we can quickly see that $M_H(\mathcal{P})$ is the edge graph of another map of type $\{4,4\}$ with the same quotient operation as before, but now instead the tessellation of the Euclidean plane is by squares with vertices at points in the lattice $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. In the first two cases of the quotient of the tessellation, this is equivalent to letting $b' = 2b$ and $c' = 2c$, and giving us the 1-skeleton of the polytope $\{4,4\}_{(b',c')}$. In the last case, this is equivalent to letting $b' = 2b$, $c' = 2c$, and $d' = 2d$, and giving us the 1-skeleton of the polytope $\mathcal{P}' = \{4,4\}_{(b',c')(d',d')}$. In all of these cases the new polytope is again edge transitive and self-dual giving us that $M_H(\mathcal{P})$ is at least 1-arc transitive. To finish we need to show that it is not transitive on the 2-arcs. To see this, fix a base flag of $\mathcal{P}$, (F_0, F_1, F_2), and let G_0 be the unique 0-face of $\mathcal{P}$ distinct from F_0 and incident with F_1, and G_1 the unique 1-face of $\mathcal{P}$ distinct from F_1 and incident with both F_0 and F_2.

We claim that there is no graph automorphism of $M_H(\mathcal{P})$ that sends the 2-arc (F_0, F_1, F_2) to the 2-arc (F_0, F_1, G_0). To see this, notice that (F_0, F_1, F_2) is part of a 4-cycle in (F_0, F_1, F_2, G_1) in $M_H(\mathcal{P})$. If there exists a graph automorphism that sends the 2-arc (F_0, F_1, F_2) to the 2-arc (F_0, F_1, G_0), the latter will need to be part of a 4-cycle in $M_H(\mathcal{P})$. Assume to the contrary that this 4-cycle exists, and let H_1 be the remaining vertex of $M_H(\mathcal{P})$ in this cycle. The existence of H_1 implies that $\mathcal{P}$ is not a lattice, as F_0, and G_0 do not have a unique supremum. Now consider what this implies about the geometric realization of $\mathcal{P}$. We know that $\mathcal{P}$ can be thought of as arising from the regular tessellation of the Euclidean plane by squares with vertices at points in the integer lattice $\mathbb{Z} \times \mathbb{Z}$.

We can see that H_1 has to be one of the edges of $\mathcal{P}$ labeled with an $*$, and F_0 must be the other vertex adjacent to that edge. Therefore, one of the vectors (1,-1) or (1,1) or (0,2) must be in the translation group whose quotient with the integer lattice gave $\mathcal{P}$. For a map from the first two infinite families to be polytopal we must have $b^2 + c^2 \geq 2$. Thus since we have ruled out the polytope $\{4,4\}_{(2,0)}$ by assumption, $\mathcal{P}$ cannot be chiral or regular. Finally, in the third infinite family, if (b,c)=(2,0) then the translation group for the quotient is generated by the two vectors (2,0) and (2,2). It is easy to see that this quotient is the same as the quotient by (2,0) and (0,2) which gives the regular polytope $\{4,4\}_{(2,0)}$ ruled out by assumption. If (b,c)=(1,1) or (1,-1) then the map is not polytopal. It follows that no edge of $\mathcal{P}$ labeled with a $*$ can be part of a 4-cycle that contains (F_0, F_1, G_0). Thus no graph automorphism can take (F_0, F_1, F_2) to (F_0, F_1, G_0). This completes the proof.

$\square$

Proposition 45. *If $\mathcal{P}$ is a 3-polytope of type $\{4,4\}_{(b,c)}$ for $(b,c)\neq(2,0)$ or $(0,2)$, then $M_H(\mathcal{P})$ is not distance transitive.*

Proof. Let $\mathcal{P} = \{4,4\}_{(b,c)}$ with $(b,c)\neq(2,0)$ or $(0,2)$. Fix a base flag (F_0, F_1, F_2) and let G_0 and G_1 be defined in the same way as the previous proof. We will show that

there is no automorphism α that sends (F_0, G_0) to (F_0, F_2) where both these are pairs of vertices at distance two from each other. From the last proof we saw that if F_0 and G_0 do not have a unique 1-face that is incident to them both, then $\mathcal{P}$ is not a lattice and $\mathcal{P} = \{4,4\}_{(2,0)}$. Therefore F_1 is the unique 1-face of $\mathcal{P}$ that is incident to them both, and thus F_0 and G_0 are not part of any 4-cycles in $M_H(\mathcal{P})$. On the other hand, F_0 and F_2 are part of the 4-cycle (F_0, F_1, F_2, G_1) and therefore there can be no automorphism that sends G_0 to F_2 while fixing F_0. Thus $M_H(\mathcal{P})$ is not distance transitive. $\qquad \square$

After showing that this class of 3-polytopes has vertex transitive medial layer graphs, it is natural to ask if these are the only 3-polytopes with this property. We can show, however, that this is not the case. Since there can be graph automorphisms of $M_H(\mathcal{P})$ which are not obtained from polytope automorphisms, understanding vertex transitivity of a graph using the entire group of graph automorphisms doesn't narrow down the class of polytopes as far.

Example 46. *A vertex transitive medial layer graph, not obtained from an edge transitive polytope.*

Let $\mathcal{P}$ be the polytope of type $\{4,4\}$ obtained by taking the quotient of the regular tessellation of the Euclidean plane by squares (with vertices at points in the integer lattice $\mathbb{Z} \times \mathbb{Z}$) by a sub-lattice generated by the two orthogonal vectors $(3,0)$ and $(0,2)$. This will not be an edge transitive polytope (see [35]). However, when you consider all the graph automorphisms of $M_H(\mathcal{P})$, you will see that $M_H(\mathcal{P})$ is vertex transitive.

However, we can get around this in a similar manner as in the sections on arc and star transitivity. We restrict the group of graph automorphisms to those which are obtained from the polytope. Unfortunately under this restriction, no (2n+1)-polytope will have a vertex transitive medial layer graph. This is because there will

be no applicable automorphism of the polytope that maps an n-face to an (n+1)-face, for example.

So we will need a slightly relaxed version of vertex transitivity as well. We want the vertices to be essentially of the same type. So we require that the graph is still k-regular for some k. We then require that the polytope is i-face transitive for i=n-1,n,n+1; that is to say if $F, G \in \mathcal{P}_i$ for $i \in \{n-1, n, n+1\}$, then there exists an automorphism α of $\mathcal{P}$ such that $\alpha(F) = G$.

Under this more relaxed definition, and the restriction to graph automorphisms derived from polytope automorphisms, we can show that our classification of 3-polytopes with vertex transitive medial layer graphs is complete.

Corollary 47. *Let $\mathcal{P}$ be a 3-polytope, then $M_H(\mathcal{P})$ is "vertex transitive", in the sense that for rank=0,1, or 2, there is one orbit of the vertices of $M_H(\mathcal{P})$ of that rank under the action of $\Gamma(\mathcal{P})$, if and only if $\mathcal{P}$ is a (regular or chiral) polytope $\{4,4\}_{(b,c)}$ or a polytope with exactly 2 flag orbits of type $\{4,4\}_{(b,c)(d,d)}$.*

Proof. Let $\mathcal{P}$ be a 3-polytope. If $M_H(\mathcal{P})$ is "vertex transitive", then $M_H(\mathcal{P})$ is k-regular. So by Lemma 42, $\mathcal{P}$ is of type $\{4,4\}$ and thus self-dual. Since $M_H(\mathcal{P})$ is "vertex transitive" then in particular it is transitive on the set of 1-faces of $\mathcal{P}$. Using the classification of edge transitive maps, $\mathcal{P}$ must be one of the three types mentioned in Proposition 44. On the other hand, if $\mathcal{P}$ is one of those three types, then Example 43 and Proposition 44 show that $\mathcal{P}$ is at least 1-arc transitive, and therefore "vertex transitive". $\square$

Chapter 7

Notable Polytopes

In various past sections we have seen examples of highly symmetric polytopes and their associated graphs which illustrate specific theorems that we have proved. We end this thesis with a collection of graph theoretical information about graphs associated with polytopes which are interesting in their own right, independent of results proved in this dissertation.

For reasons of completeness, we will include some important polytopes whose associated graphs have already been studied in past sections. In rank 3 and 5, we consider the graphs, $\mathcal{H}(\mathcal{P})$, $\mathrm{M}_H(\mathcal{P})$, and $\mathrm{M}_C(\mathcal{P})$. In rank 4 and 6, $\mathcal{H}(\mathcal{P})$ and $\mathrm{M}(\mathcal{P})$ are considered.

7.1   3-simplex

We start our discussion with the 3-simplex, that is to say the abstract polytope that is isomorphic to the face lattice of a convex 3 dimensional simplex (or tetrahedron). This universal polytope of type {3,3} has four triangular 2-faces, six edges, and four vertices. Its automorphism group is isomorphic to S_4, and can be presented as

$$\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 | \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^3 = \epsilon \rangle.$$

$\mathcal{H}(\mathcal{P})$ is a bipartite 4-regular graph on 16 vertices. We have seen that it is isomorphic to the 1-skeleton of the 4-cube. $\mathrm{Aut}(\mathcal{H}(\mathcal{P}))$ has 384 elements, where $384 = 16 \times 4 \times 3 \times 2$. This leads us to believe that $\mathcal{H}(\mathcal{P})$ is 2-arc transitive, which we saw to be the case in section 2.1.1; computation, or verification on the 4-cube, confirms this result.

As we also saw in section 2.1.1, $\mathcal{H}(\mathcal{P})$ will be 4-star transitive. To check this computationally, or again on the 4-cube, we look at the structure of the vertex stabilizer of any vertex of the graph. We will see that a vertex stabilizer is isomorphic to S_4, and thus $\mathcal{H}(\mathcal{P})$ will be 4-star transitive. $\mathcal{H}(\mathcal{P})$ is a distance transitive graph of diameter 4 and girth 4.

$\mathrm{M}_H(\mathcal{P})$ is a bipartite graph on 14 vertices with vertices of degree 3 or 4. It too has diameter and girth of 4, since it is not k-regular it cannot be vertex transitive or distance regular. $\mathrm{M}_C(\mathcal{P})$ is a graph on the same 14 vertices, now with degrees 4 or 6. $\mathrm{M}_C(\mathcal{P})$ has diameter 3 and girth 3. Similar to the last case, $\mathrm{M}_C(\mathcal{P})$ is not vertex transitive or distance regular.

7.2 3-cube

As an abstract polytope the 3-cube is isomorphic to the face lattice of the platonic solid that is the convex 3 dimensional cube. This universal polytope of type {4,3} has six square 2-faces, twelve edges, and eight vertices. Its automorphism group has 48 elements, is isomorphic to the octahedral group, and can be presented as

$$\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 | \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^3 = \epsilon \rangle.$$

$\mathcal{H}(\mathcal{P})$ is a bipartite graph on 28 vertices with vertices of degrees 4,5,6, or 8. It has diameter 4 and girth 4. $M_H(\mathcal{P})$ is a bipartite graph restricted to 26 of the 28 vertices of $\mathcal{H}(\mathcal{P})$ with vertices of degrees 3 or 4. It has diameter 6 and girth 4. $M_C(\mathcal{P})$ is a graph on the same 26 vertices now with degrees 4,6, or 8. It has diameter 4 and girth 3.

7.3   Hemi-icosahedron

The hemi-icosahedron is an abstract polytope that is built from half the faces of the regular convex icosahedron. It is a projective polyhedron and can be realized as a tessellation of the real projective plane by 10 triangles. Notably the facets of the 11-cell are all hemi-icosahedral. The hemi-icosahedron has ten triangular faces, fifteen edges, and six vertices. It is of type $\{3, 5\}_5$ and its group can be presented as

$$\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 | \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^5 = (\rho_0\rho_1\rho_2)^5 = \epsilon \rangle.$$

$\mathcal{H}(\mathcal{P})$ is a bipartite graph on 33 vertices with vertices of degrees 4,6, or 10. It has diameter 4 and girth 4. $\mathrm{M}_H(\mathcal{P})$ is a bipartite graph restricted to 31 of the 33 vertices of $\mathcal{H}(\mathcal{P})$ with vertices of degree 3,4, or 5. It has diameter 5 and girth 4. $\mathrm{M}_C(\mathcal{P})$ is a graph on the same 31 vertices now with degrees 4,6, or 10. It has diameter 3 and girth 3.

7.4 Digonal Dihedron

This degenerate polytope of type {2,2} was seen in chapter 5. It has many interesting properties such as being: universal, flat, locally spherical, self-Petrie, and self-dual. It has two digonal 2-faces, two edges, and two vertices. Its automorphism group is isomorphic to Z_2^3 and can be presented as

$$\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 | \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_1)^2 = (\rho_1\rho_2)^2 = \epsilon \rangle.$$

$\mathcal{H}(\mathcal{P})$ is a bipartite graph on 8 vertices with vertices of degrees 2,3, or 4. It has diameter 4 and girth 4. $M_H(\mathcal{P})$ is a bipartite graph on 6 of the 8 vertices of $\mathcal{H}(\mathcal{P})$ with vertices of degrees 2 or 4. It has diameter 2 and girth 4. Finally, $M_C(\mathcal{P})$ is a 4-regular graph on the same 6 vertices. It has diameter 2 and girth 3. Since it is regular, we can also check for other transitivities. $\mathrm{Aut}(M_C(\mathcal{P}))$ has $48 = 6 \times 4 \times 2$ elements leading us to think that $M_C(\mathcal{P})$ might be 1-arc transitive, and calculation shows this is the case. Also $M_C(\mathcal{P})$ is distance regular, and by examining the orbits of a vertex under its stabilizer in $\mathrm{Aut}(M_C(\mathcal{P}))$ we can see that $M_C(\mathcal{P})$ is also distance transitive.

7.5 Smallest Regular Polytope of Type {4,4}

Regular maps on the torus have been studied extensively, and we demonstrate the smallest such map (in terms of its automorphism group) here. The polytope $\{4,4\}_{(2,0)}$ was seen in chapter 6. It has four square 2-faces, eight edges, and four vertices. Its automorphism group is isomorphic to SmallGroup(32,27) and can be presented as

$$\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 | \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_1\rho_2\rho_1\rho_0)^2 = \epsilon \rangle.$$

$\mathcal{H}(\mathcal{P})$ is a bipartite graph on 18 vertices with vertices of degrees 4 or 5. It has diameter 4 and girth 4. $M_H(\mathcal{P})$ is a bipartite 4-regular graph on 16 of the 18 vertices of $\mathcal{H}(\mathcal{P})$. In fact, computation reveals that there is a graph isomorphism between $M_H(\mathcal{P})$ for this toroidal polytope and $\mathcal{H}(\mathcal{P})$ for the 3-simplex. Thus $M_H(\mathcal{P})$ is 2-arc transitive and distance transitive as well. $M_C(\mathcal{P})$ has vertices of degrees 4 or 8, and has diameter 3 and girth 3.

7.6 Smallest Self-dual Chiral 3-polytope

The problem of finding the smallest 3-polytope with certain properties comes down to a computational algebra problem of finding normal subgroups of a small index from a larger group. For example, if we want a chiral polytope of type $\{4, 4\}$, then we start with the automorphism group of the infinite universal regular polytope of type $\{4, 4\}$. This is the Coxeter group with presentation

$$\Gamma = \langle \rho_0, \rho_1, \rho_2 | \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = \epsilon \rangle.$$

We then take its rotation subgroup generated by the generators $\sigma_1 = \rho_0\rho_1$ and $\sigma_1 = \rho_1\rho_2$ which can be presented as

$$\Gamma^+ = \langle \sigma_1, \sigma_2 | \sigma_1^4 = \sigma_2^4 = (\sigma_1\sigma_2)^2 = \epsilon \rangle.$$

Now to find a group that could be a potential automorphism group for a chiral polytope, we need to find a group that is normal in Γ^+ but not normal in Γ (see [34] for details).

Γ^+ has 2242 proper subgroups of index less than or equal to 100. However only 27 of them are normal in Γ^+. There are two normal subgroups of Γ^+ or smallest index, K_1 and K_2, that are not normal in Γ and they both have index 20 in Γ^+. When you take the quotients of Γ^+ by either of them you see that the two quotients are in fact isomorphic as groups.

Then, once we have found the smallest possible group, we can reconstruct the combinatorics of the polytope by looking at right cosets of the images of the subgroups mentioned in section 1.3 for constructing a polytope from its group.

The polytope obtained in this manner is not only the smallest chiral polytope of

type $\{4, 4\}$ but is in fact the smallest of all chiral 3-polytopes. It is of type $\{4, 4\}_{(2,1)}$, having five square faces, ten edges, and five vertices. Its automorphism group has order 20 and is isomorphic to a subgroup of the symmetric group S_{20} acting on the twenty faces of $\mathcal{P}$ generated by the two permutations

$$\sigma_1 = (1, 2, 4, 8)(3, 6, 11, 16)(5, 10, 15, 19)(7, 12, 13, 17)(9, 14, 18, 20) \text{ and}$$

$$\sigma_2 = (1, 3, 7, 10)(2, 5, 11, 14)(4, 9, 15, 17)(6, 8, 13, 18)(12, 16, 19, 20)$$

.

$\mathcal{H}(\mathcal{P})$ is a bipartite graph with 22 vertices each having degrees of 4 or 5. It has diameter 4 and girth 4. $\mathrm{M}_C(\mathcal{P})$ is a graph with 20 vertices each having degree 4 or 8. It has diameter 3 and girth 3. $\mathrm{M}_H(\mathcal{P})$ is a 4-regular bipartite graph with 20 vertices. $\mathrm{Aut}(\mathrm{M}_H(\mathcal{P}))$ has $80 = 20 \times 4$ elements, suggesting that $\mathrm{M}_H(\mathcal{P})$ could be 1-arc transitive. We saw in Proposition 44 that this is the case, and computation confirms it as well. Also we know from Proposition 45 that $\mathrm{M}_H(\mathcal{P})$ cannot be distance transitive. Computation shows that it is not distance regular either.

7.7 4-simplex

The first 4-polytope that we consider is the 4-simplex. This abstract polytope is isomorphic to the face lattice of a convex 4-simplex; it is of type $\{3, 3, 3\}$ has five tetrahedral facets, ten triangular 2-faces, ten edges, and five vertices. Its automorphism group is isomorphic to S_5 and can be presented as

$$\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2, \rho_3 | \rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_3)^2$$

$$= (\rho_0\rho_1)^3 = (\rho_1\rho_2)^3 = (\rho_2\rho_3)^3 = \epsilon \rangle.$$

$\mathcal{H}(\mathcal{P})$ is a bipartite 5-regular graph on 32 vertices. $\mathrm{Aut}(\mathcal{H}(\mathcal{P}))$ has 3840 elements, where $3840 = 32 \times 5 \times 4 \times 6$. This leads us to believe that $\mathcal{H}(\mathcal{P})$ is 2-arc transitive, which we saw to be the case in section 2.1.1; computation reveals that this is the case, as the orbit of a point adjacent to a vertex at the end of a 2-arc, under the pointwise stabilizer of the 2-arc, has only three of its neighbors. For $\mathrm{Aut}(\mathcal{H}(\mathcal{P}))$ to act transitively on the 3-arc, all four neighbors of this vertex (not on the arc) would have to be in the mentioned orbit.

As we also saw in section 2.1.1, $\mathcal{H}(\mathcal{P})$ will be 5-star transitive. To check this computationally we look at the structure of the vertex stabilizer of any vertex of the graph. We see that a vertex stabilizer is isomorphic to S_5, and thus $\mathcal{H}(\mathcal{P})$ will be 5-star transitive. $\mathcal{H}(\mathcal{P})$ is a distance transitive graph of diameter 5 and girth 4.

$\mathrm{M}(\mathcal{P})$ is a bipartite 3-regular graph on 20 vertices. $\mathrm{Aut}(\mathrm{M}(\mathcal{P}))$ has 240 elements, where $240 = 20 \times 3 \times 2^2$. This leads us to believe that $\mathrm{M}(\mathcal{P})$ will be 3-arc transitive, and we saw this to be the case in Corollary 23. Also computation shows that $\mathrm{M}(\mathcal{P})$ is 3-star transitive as well. $\mathrm{M}(\mathcal{P})$ is a distance transitive graph of diameter 5 and girth 6, which is isomorphic to $\mathcal{Q}_5(L_2, L_3)$.

7.8   11-cell

The 11-cell is an abstract regular 4-polytope with many interesting properties. It was discovered by Grünbaum in 1977 (and then independently by Coxeter in 1984). It is the universal polytope of type $\{\{3,5\}_5, \{5,3\}_5\}$; has 11 hemi-icosahedral 3-faces, 55 triangular 2-faces, 55 edges, 11 vertices; and is self-dual. Its automorphism group has 660 elements, is isomorphic to the projective special linear group $L_2(11)$, and can be presented as follows.

Let $\rho_0, \rho_1, \rho_2, \rho_3$ be involutions that are generators of the Coxeter Group for the 3-dimensional regular hyperbolic honeycomb $\{3,5,3\}$. That is to say $\rho_i^2 = \epsilon$ for i=0,1,2,3; $(\rho_i \rho_j)^2 = \epsilon$ if $|i - j| \geq 2$; $(\rho_i \rho_{i+1})^3 = \epsilon$ for $i = 0, 2$; and $(\rho_1 \rho_2)^5 = \epsilon$.

$\Gamma(\mathcal{P})$ is a quotient of that Coxeter group by two more relations:

$$(\rho_0 \rho_1 \rho_2)^5 = (\rho_1 \rho_2 \rho_3)^5 = \epsilon.$$

M($\mathcal{P}$) is a 3-regular bipartite graph on 110 vertices. Aut(M($\mathcal{P}$)) has 1320 elements, where $1320 = 110 \times 3 \times 2^2$. We know from Theorem 21 that M($\mathcal{P}$) is k-arc transitive for $k \geq 3$. Computation confirms that it is indeed 3-arc transitive with trivial stabilizer of a 3-arc (symmetric trivalent graphs always have this trivial arc stabilizer; see [1]). Since M($\mathcal{P}$) is 3-arc transitive, we can also ask about its star transitivity. M($\mathcal{P}$) is 3-star transitive, which is the maximum possible giving the degree of the vertices. M($\mathcal{P}$) is not distance regular, and thus not distance transitive. It has diameter 7 and girth 10.

7.9   57-cell

In [6] Coxeter discusses what happens when the roles of the hemi-icosahedron and the hemi-dodecahedron of the 11-cell are reversed, as to create a 4-polytope with hemi-dodecahedral facets and hemi-icosahedral vertex figures. The resulting abstract polytope is called the 57-cell. It is the universal polytope of type $\{\{5,3\}_5, \{3,5\}_5\}$, and has 57 hemi-dodecahedral facets, 171 pentagonal 2-faces, 171 edges, and 57 vertices. It is self dual and its symmetry group has 3240 elements, is isomorphic to the projective special linear group $L_2(19)$, and can be presented using the standard Coxeter group relations along with the same two extra relations that determined the 11-cell.

$\rho_i^2 = \epsilon$ if i=0,1,2,3; $(\rho_i \rho_j)^2 = \epsilon$ if $|i - j| \geq 2$; $(\rho_i \rho_{i+1})^5 = \epsilon$ for $i = 0, 2$; and $(\rho_1 \rho_2)^3 = \epsilon$.

$\Gamma(\mathcal{P})$ is a quotient of that Coxeter group by two more relations:

$$(\rho_0 \rho_1 \rho_2)^5 = (\rho_1 \rho_2 \rho_3)^5 = \epsilon.$$

M$(\mathcal{P})$ is a 5-regular bipartite graph on 342 vertices. Aut(M$(\mathcal{P})$) has 6840 elements, where $6840 = 342 \times 5 \times 4$. This would lead us to believe that either M$(\mathcal{P})$ is 2-arc transitive with trivial stabilizer of a 1-arc, or 1-arc transitive, and having a group of order 4 as the pointwise stabilizer of any 2-arc. Computation reveals that the latter is the case, and M$(\mathcal{P})$ is 1-arc transitive. M$(\mathcal{P})$ is not distance regular, and has diameter 6 and girth 6.

7.10   Smallest Self-dual Chiral 4-polytope

As in the case for the smallest chiral 3-polytope, finding small chiral 4-polytopes becomes a computational algebra problem of finding normal subgroups of a small index from a larger group. To construct the smallest chiral 4-polytope of type $\{4,4,4\}$ (which is the smallest self-dual chiral 4-polytope of any type), we start with involutions that generate the Coxeter group presented as:

$$\Gamma = \langle \rho_0, \rho_1, \rho_2, \rho_3 | \rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_3)^2$$

$$= (\rho_0\rho_1)^4 = (\rho_1\rho_2)^4 = (\rho_2\rho_3)^4 = \epsilon \rangle.$$

We then take its rotation subgroup generated by the generators $\sigma_1 = \rho_0\rho_1$, $\sigma_2 = \rho_1\rho_2$, and $\sigma_3 = \rho_2\rho_3$, which can be presented as

$$\Gamma^+ = \langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1^4 = \sigma_2^4 = \sigma_3^4 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = \epsilon \rangle.$$

Then to find a group that could be a potential automorphism group for a chiral polytope, we need to find a group that is normal in Γ^+ but not normal in Γ (again see [34] for details). The smallest such polytope obtained in this way is the universal polytope of type $\{\{4,4\}_{(2,1)}, \{4,4\}_{(1,2)}\}$ (see [4] for more details about this polytope and a description of its medial layer graph). $\mathcal{P}$ is a self-dual polytope which has six facets, fifteen square 2-faces, fifteen edges, and six vertices.

$\mathrm{M}(\mathcal{P})$ is the 4-regular bipartite graph with 30 vertices drawn in Figure 7.1. $\mathrm{Aut}(\mathrm{M}(\mathcal{P}))$ has 720 elements and is isomorphic to $\mathbb{S}_6$. Since $720 = 30 \times 4 \times 3 \times 2$, we are led to believe that either $\mathrm{M}(\mathcal{P})$ is 2-arc transitive with a group of size two acting as the stabilizer of a 2-arc, or is 1-arc transitive with a group of size six acting as the stabilizer

of a 1-arc. Computation shows us that M($\mathcal{P}$) is a 2-arc transitive graph.

Also since Aut(M($\mathcal{P}$)) acts transitively on the 2-arcs of M($\mathcal{P}$), we can ask if M($\mathcal{P}$) is k-star transitive. It turns out that Aut(M($\mathcal{P}$)) acts transitively on the 4-stars, which is maximum given the degree of the graph. M($\mathcal{P}$) is not distance regular, and has diameter 4 and girth 6.
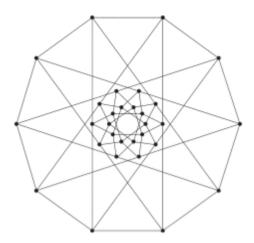
Figure 7.1: Medial layer graph of the smallest self-dual chiral 4-polytope

7.11   4-polytope with a Semisymmetric Graph

In [30] polytopes which have associated with them semisymmetric graphs are studied. A semisymmetric graph is a k-regular graph $\mathcal{G}$ such that $\text{Aut}(\mathcal{G})$ acts transitively on the edges of $\mathcal{G}$ but not transitively on the vertices of $\mathcal{G}$. Proposition 15.1 in [1], shows us that $\mathcal{G}$ must be bipartite, and thus has vertices of two types. If we consider medial layer graphs, we can find an example of such a graph coming from a 4-polytope.

Let $\mathcal{P}$ be the universal 4-polytope of type $\{\{3,6\}_{(1,1)}, \{6,3\}_{(3,0)}\}$. Then $\mathcal{P}$ has facets of type $\{3,6\}_{(1,1)}$ and vertex figures of type $\{6,3\}_{(3,0)}$, and thus cannot be self-dual. $\mathcal{P}$ has nine facets, twenty seven triangular 2-faces, 27 edges, and only 3 vertices. Its automorphism group has 324 elements and has presentation as follows:

Let $\rho_0, \rho_1, \rho_2, \rho_3$ be involutions that are generators of the Coxeter Group for the infinite universal polytope $\{3,6,3\}$. That is to say $\rho_i^2 = \epsilon$ for i=0,1,2,3; $(\rho_i\rho_j)^2 = \epsilon$ if $|i-j| \geq 2$; $(\rho_i\rho_{i+1})^3 = \epsilon$ for $i = 0,2$; and $(\rho_1\rho_2)^6 = \epsilon$.

$$\bullet \overset{}{\underset{3}{\rule{2cm}{0.4pt}}} \bullet \overset{}{\underset{6}{\rule{2cm}{0.4pt}}} \bullet \overset{}{\underset{3}{\rule{2cm}{0.4pt}}} \bullet$$

$\Gamma(\mathcal{P})$ is a quotient of that Coxeter group by two more relations (see [30]):

$$(\rho_2\rho_1\rho_2\rho_1\rho_0)^2 = (\rho_1\rho_2\rho_3)^6 = \epsilon.$$

M($\mathcal{P}$) is a 3-regular bipartite graph on 54 vertices, and is known as the Gray graph. Aut($\mathcal{P}$) has 1296 elements. However, the orbit of any vertex of M($\mathcal{P}$) under Aut($\mathcal{P}$) has size of only 27, and thus M($\mathcal{P}$) is not vertex transitive. However, the orbit of any pair of adjacent vertices (considered without order) is 81 under Aut($\mathcal{P}$). Thus M($\mathcal{P}$) is edge-transitive. Since it is 3-regular and edge transitive but not vertex transitive, we have that M($\mathcal{P}$) is semisymmetric.

Even though we do not have vertex transitivity, we can still consider another type of arc transitivity in $M(\mathcal{P})$. For each of the two types of vertex in our bipartite graph, we can consider only arcs that begin with vertices of that type, and see if $\text{Aut}(\mathcal{P})$ acts transitively on these two sets. If you consider an arc that starts with a 2-face, then there is only one orbit of 3-arcs of this kind under the action of $\text{Aut}(\mathcal{P})$, and the pointwise stabilizer of any 3-arc is trivial. Thus $M(\mathcal{P})$ is 3-arc transitive on arcs of this kind. On the other hand, $M(\mathcal{P})$ is 4-arc transitive on arcs with initial vertex corresponding to a 1-face.

7.12 Smallest Known Chiral 5-polytope

As in the case in ranks three and four, finding small chiral 5-polytopes becomes a computational algebra problem of finding normal subgroups of a small index from a larger group. To construct the smallest known chiral 5-polytope, which is of type {3,4,4,3}, we start with involutions that generate the Coxeter group presented as:

$$\Gamma = \langle \rho_0, \rho_1, \rho_2, \rho_3, \rho_4 | \rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = \rho_4^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_0\rho_4)^2$$

$$= (\rho_1\rho_3)^2 = (\rho_1\rho_4)^2 = (\rho_2\rho_4)^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^4 = (\rho_2\rho_3)^4 = (\rho_3\rho_4)^3 = \epsilon \rangle.$$

We then take its rotation subgroup generated by the generators $\sigma_1 = \rho_0\rho_1$, $\sigma_2 = \rho_1\rho_2$, $\sigma_3 = \rho_2\rho_3$, and $\sigma_4 = \rho_3\rho_4$, which can be presented as:

$$\Gamma^+ = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 | \sigma_1^3 = \sigma_2^4 = \sigma_3^4 = \sigma_4^3 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_3\sigma_4)^2$$

$$= (\sigma_1\sigma_2\sigma_3)^2 = (\sigma_2\sigma_3\sigma_4)^2 = (\sigma_1\sigma_2\sigma_3\sigma_4)^2 = \epsilon \rangle.$$

Then to find a group that could be a potential automorphism group for a chiral polytope, we need to find a group that is normal in Γ^+ but not normal in Γ (again see [34] for details). The smallest such polytope obtained in this way is the universal polytope of type $\{\{\{3, 4\}, \{4, 4\}_{(2,1)}\}, \{\{4, 4, \}_{(2,1)}, \{4, 3\}\}\}$. It has six facets, fifteen octahedral 3-faces, forty triangular 2-faces, fifteen edges, and six vertices (see [4] for a detailed description).

$M_H(\mathcal{P})$ is a bipartite graph on 70 vertices, each with degree 6 or 8. It has diameter 4 and girth 4. $M_C(\mathcal{P})$ is a graph on the same 70 vertices with, each with degree 6 or 20. It has diameter 3 and girth 3.

7.13   6-polytope with Tetrahedral 3-faces

We end our discussion of notable polytopes with one of rank 6, in particular the universal polytope $\{\{3,3,4,3\}_{(2,0,0,0)}, \{3,4,3,3\}_{(2,0,0,0)}\}$. We saw as a corollary to Theorem 21 that this polytope will have a 3-arc transitive medial layer graph.

$\mathcal{P}$ has 32 facets, 768 4-faces, 4096 tetrahedral 3-faces, 4096 triangular 2-faces, 768 edges, and 32 vertices. Its automorphism group has 589824 elements and can be presented as follows:

Let $\rho_0, \rho_1, ..., \rho_5$ be involutions that are generators of the Coxeter Group with the string diagram below. That is to say $\rho_i^2 = \epsilon$ for i=0,...,5; $(\rho_i\rho_j)^2 = \epsilon$ if $|i - j| \geq 2$; $(\rho_i\rho_{i+1})^3 = \epsilon$ for $i = 0, 1, 3, 4$; and $(\rho_2\rho_3)^4 = \epsilon$.

$$\bullet \overset{}{\underset{3}{\rule{1cm}{0.4pt}}} \bullet \overset{}{\underset{3}{\rule{1cm}{0.4pt}}} \bullet \overset{}{\underset{4}{\rule{1cm}{0.4pt}}} \bullet \overset{}{\underset{3}{\rule{1cm}{0.4pt}}} \bullet \overset{}{\underset{3}{\rule{1cm}{0.4pt}}} \bullet$$

$\Gamma(\mathcal{P})$ is a quotient of that Coxeter group by two more relations:

$$(\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1\rho_4\rho_3\rho_2\rho_3\rho_4\rho_1\rho_2\rho_3\rho_2\rho_1)^2 = (\rho_5\rho_4\rho_3\rho_2\rho_3\rho_4\rho_1\rho_2\rho_3\rho_2\rho_1\rho_4\rho_3\rho_2\rho_3\rho_4)^2 = \epsilon.$$

For a full explanation of these relations see [21]. Now since this polytope is self-dual, when considering symmetries of $M(\mathcal{P})$, we need to consider the group of all rank preserving and rank reversing automorphisms of $\mathcal{P}$. $D(\mathcal{P})$ has 1179648 elements and is isomorphic to the semi-direct product of $\Gamma(\mathcal{P})$ and $\mathcal{C}_2$; here $\mathcal{C}_2$ is the group generated by the involution that reverses a base flag of $\mathcal{P}$.

$M(\mathcal{P})$ is a bipartite 4-regular graph on 8192 vertices. $\text{Aut}(M(\mathcal{P}))$ has 1179648 elements, and $1179648 = 8192 \times 4 \times 3^2 \times 4$. This would lead us to believe that $M(\mathcal{P})$ is 3-arc transitive with group of size 4 acting as a non-trivial stabilizer of any given 3-arc. Computation using GAP shows that this is the case.

Similar to the 4-simplex, we knew from Theorem 21 that D($\mathcal{P}$) would act transitively on the 3-arcs of M($\mathcal{P}$) but not on the 4-arcs. In this case, as well as in the case of the simplex, the full group of graph automorphisms still did not act transitively on the 4-arcs. In fact every graph automorphism of M($\mathcal{P}$) comes from a polytope automorphism of $\mathcal{P}$. In other words, Aut(M($\mathcal{P}$))=D($\mathcal{P}$). M($\mathcal{P}$) has diameter 16 and girth 8. Unlike in the case of the 4-simplex, M($\mathcal{P}$) is not distance regular, and thus not distance transitive.

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