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Exhaustive Weakly Wandering Sequences for Alpha Type Transformations

A Dissertation Presented
by

John Lindhe

to
The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

in the field of
Mathematics

Northeastern University
Boston, Massachusetts

July 2009

Exhaustive Weakly Wandering Sequences for
Alpha Type Transformations

by

John Lindhe

ABSTRACT OF DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate School of Arts and Sciences of
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Abstract

In this thesis, new conditions are given which allow for the control of the exhaustive weakly wandering sequence of an ergodic, infinite measure preserving transformation. It is also shown how to control the α-type of a transformation. These are then extended to permit the simultaneous control of both the exhaustive weakly wandering sequence and the α-type. Applying these results, new transformations are presented. The first known examples are given of 0-type and $\frac{1}{3}$-type with known exhaustive weakly wandering sequences. In addition, we present an explicit sequence of integers which is exhaustive weakly wandering for α-type transformations of every $\alpha \in [0, 1]$. This is the first known example of an exhaustive weakly wandering sequence which works for more than one α-type.

Acknowledgments

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1 Introduction

Two properties that differentiate infinite measure preserving transformations from finite measure preserving transformations are exhaustive weakly wandering sequences and α-type, $0 \leq \alpha < 1$. That is, a transformation which has an exhaustive weakly wandering sequence cannot preserve a finite measure; and a transformation which is α-type for $0 \leq \alpha < 1$, cannot preserve a finite measure.

These two properties are usually studied separately. In particular, there is only one published example for which the α-type is calculated and an explicit exhaustive weakly wandering sequence is determined [15]. This example has $\alpha = \frac{1}{2}$ and the exhaustive weakly wandering sequence $\oplus_{i=0}^{\infty}\{0, 2^{2i+1}\}$.

The example is easily generalized to α-type transformations with $\alpha = \frac{p-1}{p}$ for any integer $p > 1$ and associated exhaustive weakly wandering sequence $\oplus_{i=0}^{\infty}\{0, 2^i p^{i+1}\}$.

For all other $\alpha \in [0, 1)$, there are no transformations of α-type where explicit exhaustive weakly wandering sequences are known. This includes $\alpha = 0, \frac{1}{3}$ and every irrational α.

Furthermore, it is unknown if a single sequence $\mathbb{B}$ can be exhaustive weakly wandering for transformations of different α-type. (It is known that transformations can have more than one exhaustive weakly wandering sequence.)

In this thesis, we present a general method for controlling the exhaustive weakly wandering sequence in the classical rank-one construction of an infinite ergodic transformation. The technique connects the rank-one construction to factorizations of finite abelian groups and it is this connection which supplies the proof of exhaustiveness for the transformations. The technique is general enough that it allows one to simultaneously control the α-type and

in particular allows one to obtain any α-type for $\alpha \in [0,1]$. This, in turn, further shows that non-isomorphic transformations of different α-type can have the same exhaustive weakly wandering sequence.

The techniques involved include the standard cutting and stacking construction and the lesser known infinite adding machines. In addition, tools from additive combinatorics on complementing pairs of integers, and factorizations of finite cyclic group theory are employed.

1.1 Outline

In Section 3, the general rank one construction is reviewed. All constructions in this thesis are based on this technique.

In Section 3.1, we show how to modify the general rank-one construction and obtain conditions which allow the control of the α-type.

In Section 3.2, we show how to modify the general rank-one construction in order to determine an exhaustive weakly wandering sequence. The sequence is defined as a direct sum of finite sets of integers and we employ the theory of factorizations of finite cyclic groups in proving the exhaustiveness.

The exhaustive weakly wandering sequences derived from the theorems have a direct sum structure. This is examined in Section 3.3. A dual to this structure is studied in Section 3.5.

Some of the proofs of the theorems are quite long and are presented at the end in Chapter 5.

We apply the main construction theorems in a novel way and, in Section 3.4, an explicit exhaustive weakly wandering sequence is presented which works for transformations of all α-type, $\alpha \in [0,1]$.

Section 4 presents the classical Hajian-Kakutani example and generalizes it. The methods in this section are simpler than the earlier techniques as

bounds are placed on the construction.

1.2 Historical Notes

In this section we present a brief history of the field.

Weakly wandering sets and sequences were introduced by A. Hajian and S. Kakutani [14] in 1964. They showed that all ergodic measure preserving transformations on an infinite σ-finite measure space have such sets and sequences while measure preserving transformations on finite measure spaces do not. Furthermore, weakly wandering sequences are an isomorphism invariant and the measure of the associated weakly wandering set is also invariant under isomorphisms.

In [15], Hajian and Kakutani presented the first example of an exhaustive weakly wandering sequence. Subsequently, Jones and Krengel [16] proved a theorem which implies that every ergodic, invertible, infinite measure preserving transformation has exhaustive weakly wandering sets and sequences. In 1987, Eigen and Hajian [3] connected exhaustive weakly wandering sequences to pairs of infinite complementing subsets of the integers. Exhaustive weakly wandering sequences are significant because they are tied more "tightly" to the transformation than just weakly wandering sequences. For example, a given weakly wandering sequence for a fixed transformation has weakly wandering sets of varying measures, but an exhaustive weakly wandering sequence for a fixed transformation will have exhaustive weakly wandering sets all of the same measure [5].

In their 1964 paper [14], Hajian and Kakutani introduced the concepts of zero and positive-type transformations. The concept of α-type is a special case of positive type. In 1971, [17] Osikawa and Hamachi proved the existence of transformations of α-type for all $0 \leq \alpha \leq 1$. Eigen, Hajian and Halverson in 1998 [4] connected the α-type of a transformation to its multiple-recurrence properties. Ellis and Friedman, [8], studied 0-type trans-

formations and connected them to the existence of gap sequences.

However, except for the 1970 paper [15], there are no other papers connecting exhaustive weakly wandering sequences with the α-type of a transformation.

2 Preliminaries

Throughout this thesis, all transformations T, either by construction or definition, will be invertible (one-to-one and onto), measure preserving and ergodic on an infinite measure, σ-finite, non-atomic Lebesgue space $(X, \mathcal{B}, \mu)$. Statements about sets are to be understood in the measure theoretic sense as being true up to sets of measure zero.

2.1 Definitions

We refer the reader to the books of Friedman [9], and Aaronson [1] for general definitions in ergodic theory. The following definitions are included for easy reference.

Definition 1 *An infinite set of integers* $\mathbb{A} = \{a_i\}$ *is* **exhaustive weakly wandering** *for the transformation* T*, if there exists a set of positive measure* W *satisfying the two conditions*

1. $T^{a_i}W \cap T^{a_j}W = \emptyset$, $i \neq j$ *(weakly wandering),*

2. $X = \cup_{a \in \mathbb{A}} T^a W$, *(exhaustive).*

Note that, the set W may be of finite or infinite measure. Each sequence $\mathbb{A}$ may have more than one set W, and each set W may be exhaustive weakly wandering for more than one sequence.

It follows from Jones and Krengel [16] that exhaustive weakly wandering sequences always exist for ergodic, infinite measure preserving, invertible transformations. Despite this however, there are very few transformations for which explicit exhaustive weakly wandering sequences are known.

Definition 2 *The transformation T is said to be of α-type for a fixed $\alpha \in [0,1]$ if*

$$\limsup_{n\to\infty} \mu(T^n A \cap A) = \alpha \cdot \mu(A)$$

for all A satisfying $\mu(A) < \infty$.

The notion of α-type was first introduced by Hajian and Kakutani. In [17], Hamachi and Osikawa showed that for all $\alpha \in [0,1]$ there exist transformations of α-type. The proof is by construction but gives no information for any exhaustive weakly wandering sequences.

3 Rank One Construction

In this section, we set the notation and review the general, rank one, cutting and stacking construction as used in this thesis, see [9]. (In the literature, this is also referred to as a class one construction and a tower construction.)

A rank one construction defines a transformation, T, in stages. At the end of each stage the transformation is defined on a larger set. The set, at each stage, is a single column of height h_n, consisting of intervals all of the same size,

$$C_n = \{L_0^n,\ L_1^n,\ \cdots,\ L_{h_n-1}^n\}.$$

The transformation, T_{C_n} at this stage is defined by mapping each interval linearly to the interval above it, $T_{C_n} : L_i^n \to L_{i+1}^n$, for $0 \leq i < h_n - 1$. In this thesis, we define $T_{C_n} : L_{h_n-1}^n \to L_1^n$ so that it is periodic. (Note, alternately, many authors leave T_{C_n} undefined on the top level.) In either case, we have $T_{C_{n+1}} = T_{C_n}$ on the column C_n, and as the common width of the columns are going to 0, it follows that the transformations T_{C_n} converge pointwise to a transformation T.

The basic construction at each stage is to cut the column into equal length subcolumns, add 'spacers' above some or all of the subcolumns, and then stack each subcolumn onto the subcolumn to its left.

It is to be understood that when we "cut" a column, we always cut it into subcolumns of equal widths. It is to be understood, that spacers are intervals of the same width as the columns they are being placed above. Stacking, means each subcolumn is placed on top of the subcolumn to its immediate left. The transformation being constructed is defined as going up the columns linearly - which makes the transformation measure preserving. All intervals should be assumed left-closed, right open.

The construction starts with the unit interval $W = [0, 1)$. This is column

C_0 of height $h_0 = 1$.

At stage one, the column C_0 is cut into c_1 subcolumns. Spacers (intervals) are placed above each subcolumn. The subcolumns are then stacked - each subcolumn placed above the column to its immediate left.

At the nth stage, column C_{n-1} is cut into c_n subcolumns. Above the ith subcolumn, $s_i^{(n)}$ spacers are placed. The subcolumns are stacked, forming the new column C_n (see figure 1).

The construction is therefore completely defined by the infinite sequence of cuts $c_n \geq 2$, and the infinite sequence of sets of spacers $\mathcal{S}_n = \{s_1^{(n)}, \cdots, s_{c_n}^{(n)}\}$.

Using the above notation, the total number of spacers added at stage n is $|\mathcal{S}_n| = \sum_{i=1}^{c_n} s_i^{(n)}$. The height of column C_n is then $h_n = c_n \cdot h_{n-1} + |\mathcal{S}_n|$.

The spacers are intervals taken from the real line. In what follows, the total measure of all the spacers will be infinite. Thus, the transformation may be viewed as a piecewise linear map on the real line with the σ-algebra given by the Borel sets and the measure given by the usual linear Lebesgue measure.

From Friedman [9], we have the following.

Proposition 1 *The transformation T defined above and given by the cuts c_n and spacers $\mathcal{S}_n$ is ergodic, invertible and measure preserving.*

The following may also be found in Friedman's book. It gives a useful property concerning rank one transformations which will be used in the sequel to show the various transformations are of α-type.

Lemma 1 *The levels of the columns C_k generate the sigma-algebra.*

That is, let $\mathscr{C} = \{L : L \text{ a level in some } C_k\}$. Then this countable collection of sets generates the sigma algebra of measurable sets.

The nth stage of the general construction

0. Start with the previous column C_{n-1}.
1. Cut into $c_n \geq 2$ subcolumns.
2. Add spacers above each subcolumn

$$\mathcal{S}_n = \{s_1^{(n)}, \cdots, s_{c_n}^{(n)}\}.$$

3. Stack right subcolumn on top of left subcolumn

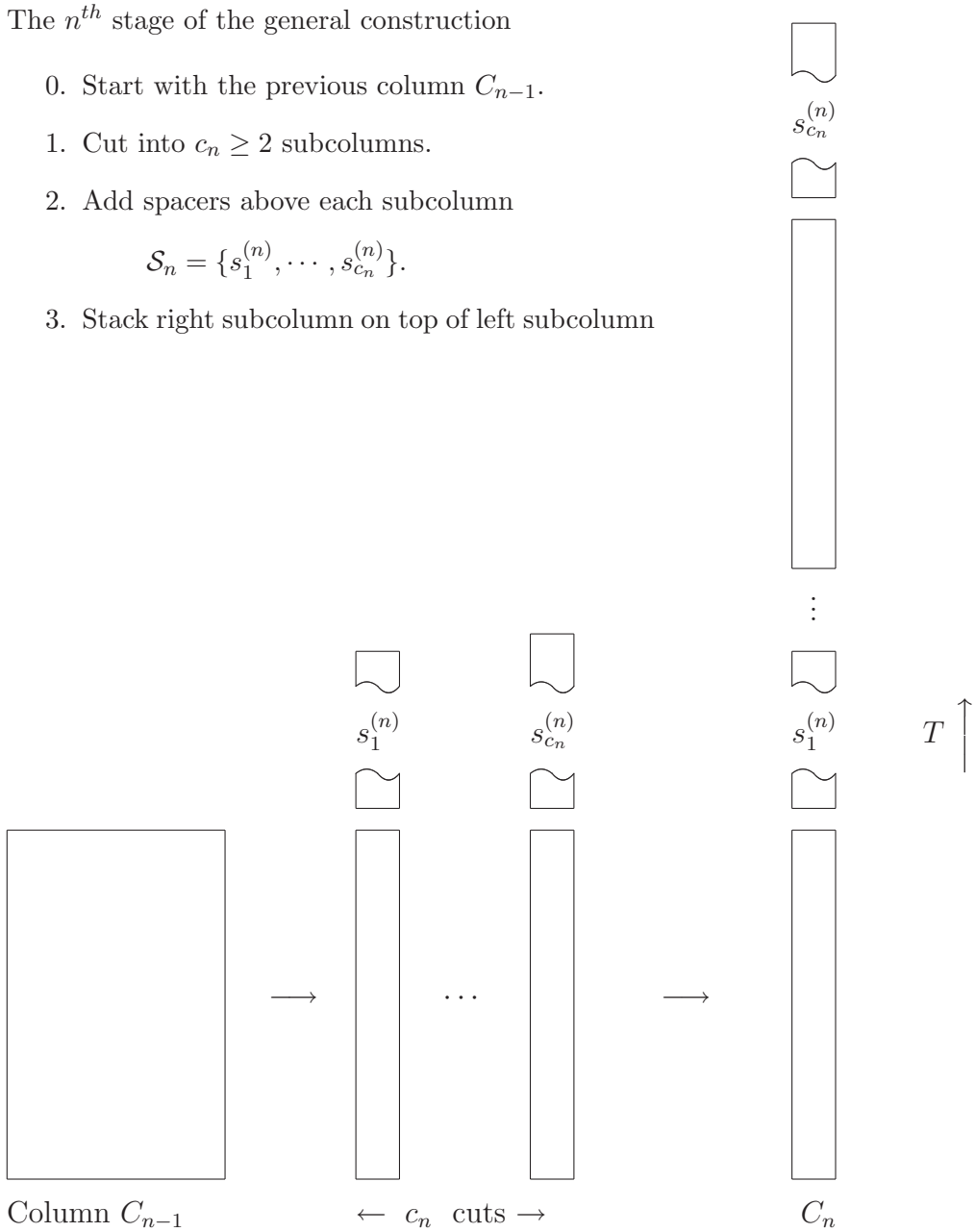

Figure 1: The nth stage of general construction

3.1 Controlling for α-type

In this section, we present two theorems which show how to restrict the general rank-one construction in order to control the resulting α-type.

Proving a transformation is of 0-type is simpler than proving it is of α-type for $\alpha > 0$. This is because for 0-type it is only necessary to show $\limsup \mu(T^n A \cap A) = 0$ for a single set of positive measure, and this can be easily shown for the base set W_0. The idea is then to choose the size of the spacers so that the base set W_0 returns only from a single subcolumn at any given time. This can be seen in condition 2) of the following theorem. The detailed proofs will be given in Section 5.2.

Theorem 1 *Suppose in the general rank-one, cutting and stacking construction for the transformation T (as described in Section 3) that the cuts c_k and spacers $\mathcal{S}_k = \{s_i^{(k)}\}$ satisfy the following two conditions*

1. $c_k > c_{k-1}$,
2. $s_{n+1}^{(k)} \geq \sum_{i=1}^{n} s_i^{(k)} + n \cdot h_{k-1}$, $n \geq 1$

Then T is of 0-type.

The first spacer $s_1^{(k)} \geq 0$ is arbitrary - in practice it will usually be set to 0.

The previous theorem is easily modified to obtain transformations of α-type. Viewing the return times of the base set W_0 to itself, the first spacers are set to 0 allowing a certain fraction of W_0 to return to itself. The latter spacers are again increasing so that nothing else returns at the same time.

Different α types are obtained by removing earlier spacers.

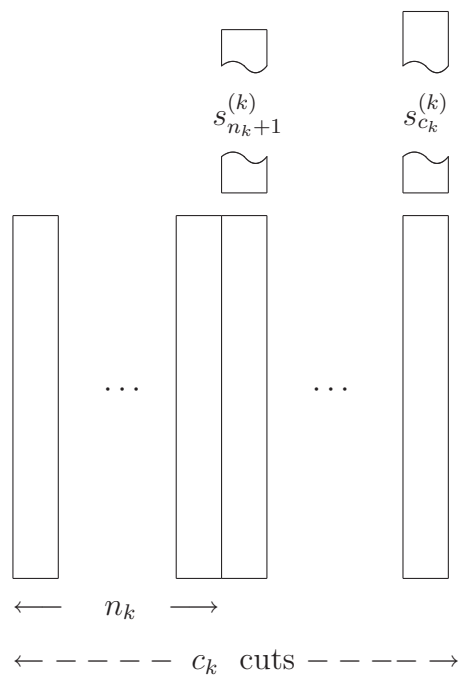

Figure 2: α-Type

The proof however, requires more work as it is not enough to just work with the base set. The result for W_0, is extended to the proof for sets which are finite unions of levels of some C_k, and then extended to arbitrary sets.

Theorem 2 *Let $\alpha \in [0,1]$ and suppose $n_k < c_k$ with $\lim \frac{n_k}{c_k} \to \alpha$. Assume in the general rank-one, cutting and stacking construction for the transformation T that the cuts c_k and spacers $\mathcal{S}_k = \{s_i^{(k)}\}$ satisfy the following conditions*

1. $c_k > c_{k-1}$
2. $n_k < c_k$ *such that* $\lim_{k\to\infty} \frac{n_k}{c_k} = \alpha$
3. $s_i^{(k)} = 0$ *for* $1 \le i \le n_k$

4. $s_{j+1}^{(k)} \geq \sum_{i=1}^{j} s_i^{(k)} + j \cdot h_{k-1}$, $j \geq n_k$

Then T is of α-type

Remark 1 *For α rational, it is not necessary to have the cuts c_k increasing to infinity.*

Remark 2 *For a given α there are many sequences $\lim \frac{n_k}{c_k} \to \alpha$. Using this, it is possible to obtain nonisomorphic transformations of the same α-type.*

3.2 Controlling for Exhaustive Weakly Wandering Sequences

We now show how to modify the general rank one construction in such a way that we can determine an exhaustive weakly wandering sequence. The restriction is surprisingly simple. We restrict the spacers to be multiples of a certain block size (defined below).

As usual, $h_0 = 1$ and $W_0 = [0, 1)$ is the unit interval. As before c_k denotes the cuts, h_k denotes the heights and $\mathcal{S}_k$ denotes the set of spacers. Define the block sizes $b_k = c_k \cdot h_{k-1}$. That is, at stage $k-1$ we have a column of height h_{k-1}. This will be cut into c_k subcolumns, giving a total of $b_k = c_k \cdot h_{k-1}$ intervals. Spacers will now be restricted to be multiples of the block size b_k.

Theorem 3 *In the general rank one construction, suppose the spacers are added in multiples of the block size defined as*

$$b_k = c_k \cdot h_{k-1}$$

That is the spacers are $s_i^{(k)} = s_{k,i} \cdot b_k$. *Let* $s^{(k)}$ *denote the total number of blocks added as spacers,* $s^{(k)} = \sum_{i=1}^{c_k} s_{k,i}$.

Then, the set W *is an exhaustive weakly wandering set under the direct sum sequence*

$$\mathbb{E} = \oplus_{k=1}^{\infty} \{0, b_k, 2b_k, 3b_k, \cdots, s^{(k)} b_k\}$$

The proof depends on properties of factorizations of finite cyclic groups. The full details of the proof will be given in Section 5.3.

3.3 Direct Sums

Not all exhaustive weakly wandering sequence have any structure to them, let alone a "nice" structure. In this section, we derive two theorems which follow

from the direct sum structure of $\mathbb{E}$. In particular, it allows the determination of other exhaustive weakly wandering sequences and other exhaustive weakly wandering sets.

Theorem 4 *Let T be an ergodic, infinite measure preserving transformation. Assume T has an exhaustive weakly wandering set W, which is exhaustive weakly wandering under the direct sum sequence*

$$\mathbb{E} = \oplus_{k=1}^{\infty} \mathbb{E}_k$$

where each $\mathbb{E}_k = \{0, e_{k,1}, e_{k,2}, \cdots, e_{k,n_k}\}$ is a finite set of integers containing at least one non-zero element.

Then the set $W' = \cup_{e \in \oplus_{k=1}^{n} \mathbb{E}_k} T^e W$ is exhaustive weakly wandering under the sequence $\oplus_{k=n+1}^{\infty} \mathbb{E}_k$.

Proof. This follows from the direct sum structure of $\mathbb{E}$.

Notice, that we are not assuming the transformation T was constructed via a rank-one construction. When it is, the set W is necessarily of finite measure and in turn the sets W' are also of finite measure. However, even in this case we can use the direct sum structure to obtain exhaustive weakly wandering sets of infinite measure.

Let I, J be a partition of $\{1, 2, \cdots\}$ where both $|I| = \infty = |J|$.

Theorem 5 *The set $W' = \cup_{e \in \oplus_{i \in I} \mathbb{E}_i} T^e W$ has infinite measure and is exhaustive weakly wandering under the sequence $\oplus_{j \in J} \mathbb{E}_j$.*

Proof. The set W' is a countable union of sets of positive measure and hence must have infinite measure.

Remark 3 *The theorem shows that the constructed transformations always have exhaustive weakly wandering sets with infinite measure. It is known that there exist transformations which only admit exhaustive weakly wandering sets of infinite measure. The transformations that are constructed using Theorem 3 have a property which is dual to the direct sum structure of the exhaustive weakly wandering sequence. This will be shown in Section 3.5.*

3.4 A Special Sequence $\mathbb{B}$

In this section, we present a specific sequence of integers, $\mathbb{B}$. The sequence is exhaustive weakly wandering for a large family of transformations. This family includes transformations of every α-type for $\alpha \in [0, 1]$. Furthermore, for each $\alpha \in [0, 1]$ it is possible to construct infinitely many non-isomorphic transformations of α-type.

Remark 4 *Referring back to Theorem 3, we see those transformations are completely defined by the cuts and placement of spacers. However, the exhaustive weakly wandering sequence only depends upon the cuts and the total number of blocks of spacers added. A simpler example of this is given in Section 4.3.*

Theorem 6 *There exists an increasing sequence of positive integers*

$$\mathbb{B} = \{0 = b_0 < b_1 < b_2 < \cdots\}$$

and there exist ergodic infinite measure preserving maps T_α, $0 \leq \alpha \leq 1$ so that each T_α is of α-type and $\mathbb{B}$ is an exhaustive weakly wandering sequence for T_α.

Proof. The proof is a direct application of Theorems 1, 2 and 3. Essentially the ideas are as follows. Before beginning any construction, preset the number of cuts and the total number of spacers. This will, by Theorem 3, fix the exhaustive weakly wandering sequence, before the transformation has been constructed. Then chose an α. Using Theorems 1 and 2, the distribution of the blocks of spacers is determined. The point being, that the total number of spacers is large allowing a wide range of distributions.

Sequence $\mathbb{B}$

The Sequence $\mathbb{B}$ in Theorem 6, can be written explicitly and is partially given below. It has the direct sum structure which arises from Theorem 3,

$$\begin{aligned} \mathbb{B} &= \oplus_{n=1}^{\infty} \mathbb{B}_n \\ &= \{0, 2, 16, 18, 32, 34 \cdots\} \end{aligned} \tag{1}$$

where

$$\begin{aligned} \mathbb{B}_1 &= \{0, 2\} \\ \mathbb{B}_2 &= \{0, 16, 32, 48, 64, 80, 96, 112\} \\ \mathbb{B}_3 &= \{0, 1024, 2048, 3072, 4096, 5120 \cdots 130048\} \\ &\vdots \\ \mathbb{B}_k &= \{0, b_k,\ 2 \cdot b_k,\ 3 \cdot b_k, \cdots,\ (2^{2^k-1} - 1) \cdot b_k\} \\ &\vdots \end{aligned} \tag{2}$$

and with

$$b_k = 2^k \cdot \prod_{j=0}^{k-1} 2^{2^j+j-1}$$

3.4.1 Restrictions for $\mathbb{B}$

These numbers are obtained by presetting and fixing the number of cuts and the total number of blocks of spacers. That is, before constructing a transformation we set some restrictions for all of them

1. Preset the cuts to $c_k = 2^k$.

2. Preset the total number of blocks of spacers to $s^{(k)} = 2^{2^k-1} - 1$.

This presets the total spacers added at stage k to

$$|\mathcal{S}_k| = \left(2^{2^k-1} - 1\right) \cdot 2^k \cdot \prod_{j=0}^{k-1} 2^{2^j+j-1}$$

3. Add the spacers in multiples of the block size, $b_k = c_k \cdot h_{k-1}$.

From these restrictions, the height of column C_k is also preset, $h_k = \prod_{j=0}^{k} 2^{2^j+j-1}$.

Theorem 7 *Let T be any rank-one constructed transformation satisfying the above three restrictions. Then $\mathbb{B}$ is an exhaustive weakly wandering sequence for T.*

Proof. Follows from Theorem 3.

3.4.2 Distributing the spacers: C_1

Since the number of cuts and the total number of spacers are preset, the variations for the transformation come from the alternate distributions of the blocks of spacers over the subcolumns.

Here is an illustration of the first construction stage. Start with $C_0 = W = [0, 1)$ of height $h_0 = 1$. Cut into $c_1 = 2$ subcolumns and add $|\mathcal{S}_1| = (2^{2^1+1-1} - 2^1) \cdot 2^{2^0+0-1} = 2$ spacers. The block size is $b_1 = 2 \cdot 1 = 2$. Hence there is only one block of spacers to add. In this case we can either put the spacers over the left subcolumn or over the right subcolumn.

In either case, when we stack the right subcolumn over the left subcolumn, we end up with a new column C_1 of height $h_1 = 4$. The difference in the two new columns C_1 is the location of the set W.

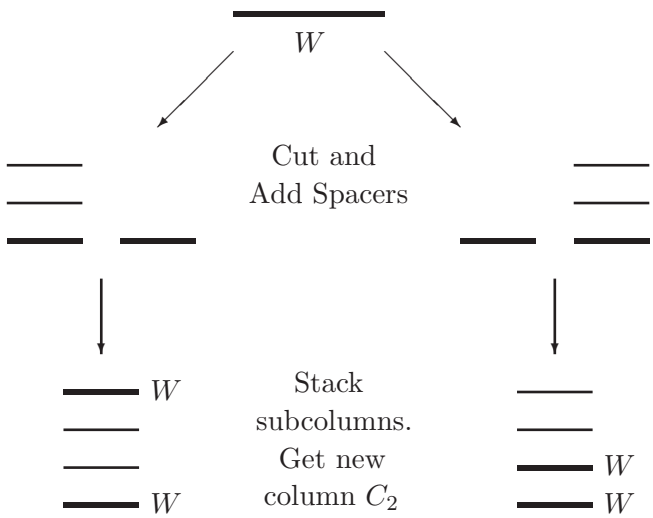

Figure 3: Two ways to add 1 block of spacers

In the case on the right side of the figure, W is the lowest two levels of C_1; in the case on the left side of the figure, W is the lowest and the highest levels in C_1.

As a foreshadowing of the analysis used to show the exhaustiveness, assume the construction stopped here. In this case, we would define a transformation going up the stack and then returning to the bottom level.

Whichever way the spacers are placed, the sets W and T^2W fill up the space disjointly. This is immediate for the case on the right side. In the case on the left in the figure, the top level (which is part of W) "wraps around" to complete the "filling up".

Distributing the spacers: C_2

We continue the analysis. Start with the column C_1, which has a height $h_1 = 4$ (from the previous). Cut this into $c_2 = 2^2 = 4$ subcolumns. The block size is now $b_2 = c_2 \cdot h_1 = 16$. The total number of spacers is $|\mathcal{S}_2| = (2^{2^2-1} - 1) \cdot 2^2 \cdot 2^{2^1+1-1} = 7 \cdot 16 = 7 \cdot b_2$

Hence, at stage 2, there are 7 blocks of spacers of size b_2. These 7 blocks can be distributed over the 4 subcolumns in various ways. We can chose any one of them in a construction. However, in order to control the α-type the blocks of spacers will be placed according to Theorem 2.

3.4.3 Distributing the spacers: C_k

In general, at stage k, there is a total of $|\mathcal{S}_k| = \left(2^{2^k-1} - 1\right) \cdot 2^k \cdot \prod_{j=0}^{k-1} 2^{2^j+j-1}$ spacers. The block size is $b_k = 2^k h_{k-1} = 2^k \prod_{j=0}^{k-1} 2^{2^j+j-1}$. Hence, at stage k, there are $2^{2^k-1} - 1$ blocks of spacers to distribute over the 2^k subcolumns.

We let $\mathscr{T}$ denote transformations satisfying the above restrictions. That is a transformation $T \in \mathscr{T}$ satisfies the three restrictions, meaning that at each stage k of the construction one of the possible distributions for the spacers is chosen. Hence the collection $\mathscr{T}$ contains many more transformations than just the α-types.

3.5 Induced Adding Machines

The exhaustive weakly wandering sequence derived in Theorem 3 has a direct sum structure. In this section, we examine a dual structure. We show that the induced transformation, T_W, is isomorphic to an adding machine transformation.

It was shown in Section 3.3 that the direct sum structure of the exhaustive weakly wandering sequence allows additional exhaustive weakly wandering sets and sequences to be determined. Specifically, Theorem 4 shows that as the sequence "shrinks" the set "enlarges", while Theorem 5 allows us to obtain exhaustive weakly wandering sets of infinite measure. Naturally, of course, one cannot have exhaustive weakly wandering sets of zero measure, but the adding machine "structure" for the the induced map T_W gives the following.

Theorem 8 *Let T be a rank-one transformation constructed under the restrictions of Theorem 3. Then for any $\epsilon > 0$, there exists an exhaustive weakly wandering set W' with measure $\mu(W') \leq \epsilon$.*

We begin with the following.

Theorem 9 *Let T be a transformation satisfying the conditions of Theorem 3 with cuts c_k. Then the induced transformations T_W is isomorphic to the adding machine map $T_{\{c_k\}}$.*

Proof: The proof is straightforward.

First we introduce some notation. Consider the set W in the rank-one construction. At Stage 1, this is cut into c_1 subintervals (of the same size). Denote these as $W = W_{1,j}$, for $j = 0, \cdots, c_1 - 1$.

At Stage 2, the set $W_{1,0}$ is now the base of column C_2. This is cut into c_2 subintervals. We denote these as $W_{2,j}$ for $j = 0, \cdots, c_2 - 1$. Continuing in this manner, the set $W_{k,0}$ is the base of column C_k. It is cut into c_{k+1} subintervals denoted $W_{k+1,j}$, for $j = 0, \cdots c_{k+1} - 1$.

Examining Figure 1, it is evident that, independent of the number of spacers above $W_{1,0}$, when the set returns to W it returns to $W_{1,1}$. Likewise it is evident that $T_W : W_{1,j} \to W_{1,j+1}$ for $j < c_1 - 1$. To analyze T_W on W_{1,c_1-1} we look at the column C_2 and apply the same reasoning. This gives us that T_W permutes the sets $\{W_{1,0}, \cdots, W_{1,c_1-1}\}$.

By the same argument we obtain $T_W^{c_1}$ permutes the sets $\{W_{2,0}, \cdots, W_{2,c_2-1}\}$, $T_W^{c_1 c_2}$ permutes $\{W_{3,0}, \cdots, W_{3,c_3-1}\}$ and so on for the higher terms.

Proof of Theorem 8

By assumption, the set W is exhaustive weakly wandering under the sequence $\mathbb{E}$.

From the above discussion, it is clear that the set $W_{1,0}$ is exhaustive weakly wandering under the sequence $\oplus_{i=0}^{c_1-1}(\mathbb{E} + i)$.

By the same argument $W_{2,0}$ and then $W_{3,0}$, etc, can be made exhaustive weakly wandering. Clearly the measure of the sets $W_{j,0}$ are shrinking to zero.

Remark 5 *The above argument only requires that the induced map be isomorphic to an adding machine map. It does not require that the initial exhaustive weakly wandering sequences have any structure at all.*

4 Examples

In this section, we present some simple examples of transformations where the α-type is is known and an explicit exhaustive weakly wandering sequence can be determined. The proofs may be derived from Theorems 2 and 3, however the examples in this section all have bounded and constant cuts c_k. In addition, the spacers are placed in a simple and orderly manner. This allows proofs in this section to be much easier than the proofs of the full Theorems 2 and 3.

We first generalize and present the classical Hajian-Kakutani example (colloquially known as the 'Ohio-State') in the language and notation of a rank-one construction. Originally, the example was presented as a skyscraper construction over the von Neumann transformation. In the rank-one construction, the von Neumann transformation can be recovered as the induced transformation, T_W, on the exhaustive weakly wandering set $W = [0, 1)$.

The von Neumann transformation is isomorphic to the binary adding machine map. In the same way, for the generalizations, the map induced on the exhaustive weakly wandering set W is isomorphic to a finite measure preserving adding machine map, Theorem 9. Extending this, the generalizations themselves are isomorphic to infinite measure preserving adding machine maps.

4.1 Generalized Hajian Kakutani Example

Fix an integer $p > 1$. Define the transformation V_p as follows. Initialize $h_0 = 1$ and column $C_0 = W = [0, 1)$. Set the cuts $c_k = p$. Hence, at each stage there will be p subcolumns. The block sizes will be $b_k = c_k \cdot h_{k-1}$. In this case, we will add only one block of spacers and we will add it to the right-most subcolumn. Thus the spacers are $\mathcal{S}_k = \{0, \cdots, 0, b_k\}$. The

subcolumns, as usual, are then stacked right on top of left (see Figures 4 and 5).

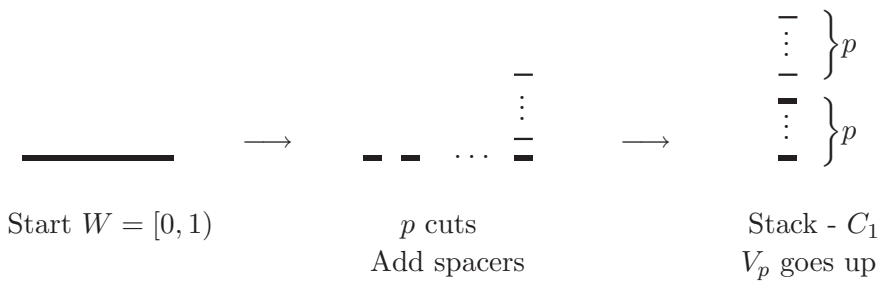

Figure 4: Stage 1 for construction of V_p

From the above we can calculate the block sizes b_k and the heights h_k. We obtain $b_1 = p$ and $h_1 = 2p$; $b_2 = 2p^2$ and $h_2 = 2^2p^2$; and in general $b_k = 2^{k-1}p^k$ and $h_k = 2^kp^k$.

Theorem 10 *The transformation $V = V_p$ is ergodic, infinite measure preserving. It is of α-type with $\alpha = \frac{p-1}{p}$. The set W is exhaustive weakly wandering under the sequence $\oplus_{k=1}^{\infty}\{0, b_k\} = \oplus_{k=1}^{\infty}\{0, 2^{k-1}p^k\}$.*

Proof. Ergodicity and infinite measure preserving are obvious. The proof that V_p is α-type and the sequence is exhaustive weakly wandering sequence follow from Theorems 2 and 3. However, as the cuts c_k are bounded and constant it is possible to give a direct proof. This illustrates many of the ideas which will be used in later proofs.

First we observe that the set W will be exhaustive weakly wandering under the direct sum sequence of the block sizes. We can already "see" part of this in Figure 4.

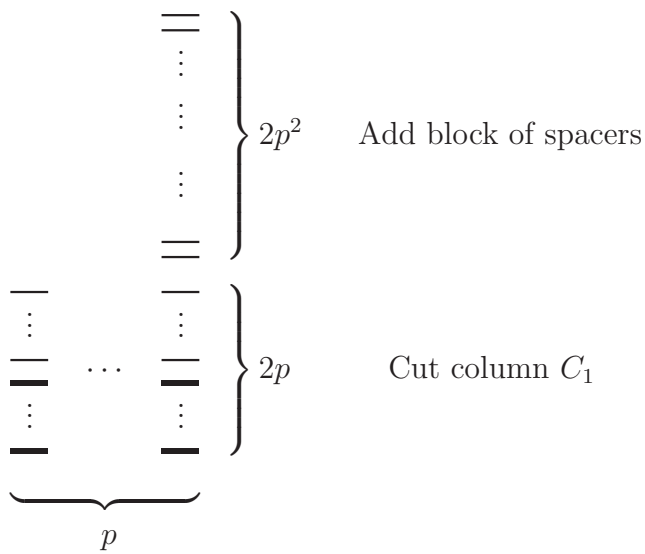

Figure 5: Stage 2 for construction of V_p

Lemma 2 *Letting C_1 denote the column at Stage 1 in the construction of the transformation V_p, we have*

1. $W \cap V^p W = \emptyset$
2. $C_1 = W \cup T^p W$

Proof. Column C_1 is on the right in Figure 4 and the results are immediate from the figure.

Similarly from Figure 5 we can 'see' the following.

Lemma 3 *Letting C_2 denote the column at Stage 2 in the construction of the transformation V_p, we have*

1. $C_1 \cap V^{2p^2} C_1 = \emptyset$

2. $C_2 = C_1 \cup T^{2p^2} C_1$

Combining these two lemmas we obtain

Lemma 4

1. *The four sets* $W,\ T^p W,\ T^{2p^2} W,\ T^{p+2p^2} W$ *are pairwise disjoint.*

2. $C_2 = W \cup T^p W \cup T^{2p^2} W \cup T^{p+2p^2} W$

To prove Theorem 10 we briefly proceed as follows.

Proof of exhaustiveness. First we show the set W is exhaustive weakly wandering under the infinite direct sum sequence $\oplus_{k=1}^{\infty}\{0, 2^{k-1}p^k\}$.

As before, Column C_1 consists of $2p$ levels. The lower p levels are subintervals of W and the upper p levels are spacers. It is immediate that $C_1 = W \cup V_p^p W$ disjointly. By the same analysis $C_2 = C_1 \cup V_p^{2p^2} C_1$ disjointly. This gives

$$C_2 = W \cup V_p^p W \cup V_p^{2p^2} W \cup V_p^{p+2p^2} W$$

The powers of V_p in the expression are $\{0, p\} \oplus \{0, 2p^2\}$. At the kth stage we have $C_k = C_{k-1} \cup V_p^{2^{k-1}p^k} W$ disjointly. By Induction, W is exhaustive weakly wandering under the direct sum sequence $\oplus_{k=1}^{\infty}\{0, 2^{k-1}p^k\}$.

We now outline the proof V_p is of α-type with $\alpha = \frac{p-1}{p}$. Specifically, we will show this just for the set W. In general, this is not enough to determine the α-type. However it is enough in this case because (1) the cuts are constant, (2) the spacers are always placed over the last (right-most subcolumn) and (3) in a rank-one constructions the levels of the columns generate the sigma algebra.

Proof that $\alpha = (p-1)/p$. We give the argument just for Stages 1 and 2. Because the cuts and spacer placements are always the same the same argument goes through for all Stages. We proceed in steps, analyzing the intersections $V_p^j W \cap W$ for $j = 1, \cdots, h_1$. This requires only looking at Stage 1 and Stage 2 of the construction.

1. $\mu(W \cap V_p W) = (p-1)/p$

   Because exactly $1/p$ of W has moved up into the spacers

2. $\mu(W \cap V_p^i W) < (p-1)/p$, $1 < i < p$

   Because more of W has moved up into the spacers but nothing else has returned.

3. $\mu(W \cap V_p^p W) = 0$

   All of W has moved into the spacers.

   For $V_p^{p+i} W$ we see part of W returning to itself in Stage 2.

4. $\mu(W \cap V_p^{p+i} W) < (p-1)/p$, $1 \leq i < 2p = b_1$

   Because, $1/p$ is moving up into the spacers at stage 2. Also since, not all of the rest ((p-1)/p) has returned

   So now we want to examine starting at $V_p^{h_1} W \cap W$. $(h_1 = 2b_1)$

5. $\mu(W \cap V_p^{h_1} W) = (p-1)/p$

   Because exactly $1/p$ is up in the spacers (at stage 2) and the rest has "moved over one-subcolumn".

   We are now in the situation of examining Stage 2 going into Stage 3 - which repeats the same arguments as before.

This completes the proof of Theorem 10

4.2 Direct Sums and the Hajian-Kakutani Example

The actual Hajian-Kakutani example uses $p = 2$. The advantage of such a low number is that it allows us to "see" the direct sum nature of the exhaustive weakly wandering sequence imposed on the columns, C_k, of the transformation's construction.

Note that the Hajian Kakutani example has a very orderly exhaustive weakly wandering property. Specifically, the columns C_k are "filled" exactly with finite images of the exhaustively weakly wandering set. This does not occur in general, and that is one of the reasons that the later proofs are more complicated.

To define the Hajian Kakutani example, set $p = 2$. This gives the heights and blocks $h_0 = 1$, $b_1 = 2$, $h_1 = 4$, $b_2 = 8$ and in general $h_k = 2^{2k}$ and $b_k = 2^{2k-1}$.

Theorem 11 *The Hajian-Kakutani transformation V_2 is ergodic and infinite measure preserving. It is of α-type with $\alpha = \frac{1}{2}$. The set W is exhaustive weakly wandering under the sequence*

$$\mathcal{A} = \oplus_{k=1}^{\infty}\{0,\ 2^{2k-1}\} = \{0,\ 2,\ 8,\ 10,\ 32,\ 34,\ \cdots\}$$

Proof. This of course can be proved by referring to Theorems 2 and 3. However, the low number, $p = 2$, allows this to be proved directly.

Consider the first stage of the construction of V_2 in Figure 4.2.

It is immediate that column $C_1 = W \cup T^2W$ disjointly.

Now consider the second stage of the construction of V_2. Start with column C_2. Cut it into 2 subcolumns. Add $b_2 = 8$ spacers on the right subcolumn and stack.

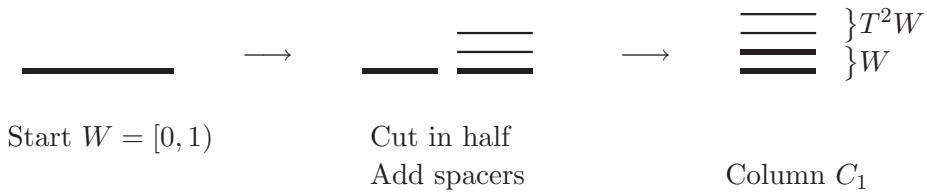

Figure 6: Stage 1 of Construction of V_2

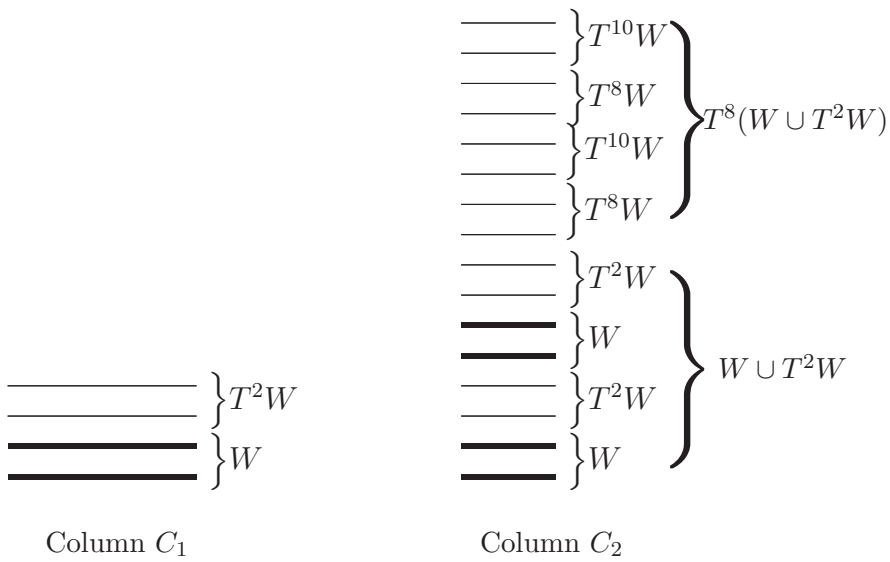

Figure 7: Stage2 of Construction of V_2

It is immediate that $C_2 = C_1 \cup T^8 C_1$ disjointly. This gives

$$C_2 = (W \cup T^2W) \cup T^8(W \cup T^2W)$$

We can summarize this.

Lemma 5 *At stage 2, in the above construction, the set C_2 is is covered by the direct sum, finite exhaustive weakly wandering sequence*

$$\{0,2\} \oplus \{0,8\} = \{0,b_1\} \oplus \{0,b_2\}$$

4.3 A Variation to get $\alpha = 1/3$

In this section, we show how to get a simple transformation with $\alpha = \frac{1}{3}$ for which an explicit exhaustive weakly wandering sequence can be determined. Theorems 2 and 3 can of course produce such an example but it will be more complicated.

We will also present two non-isomorphic transformations with the same α-type but different exhaustive weakly wandering sequences, and two non-isomorphic transformations with the same exhaustive weakly wandering sequences but different α-types.

Define the transformation $V_{3,3}$ as follows. Initialize $h_0 = 1$ and $C_0 = W = [0, 1)$. Set the cuts $c_k = 3$ for all k. Set the spacers $\mathcal{S}_k = \{0, 0, 3 \cdot b_k\}$. That is, at Stage k, the column is cut into 3 subcolumns. No spacers are placed over the left and middle subcolumn. Over the last (right-most) subcolumn 3 blocks of spacers are placed, the block size is $b_k = 3 \cdot h_{k-1}$ (see Figure 8).

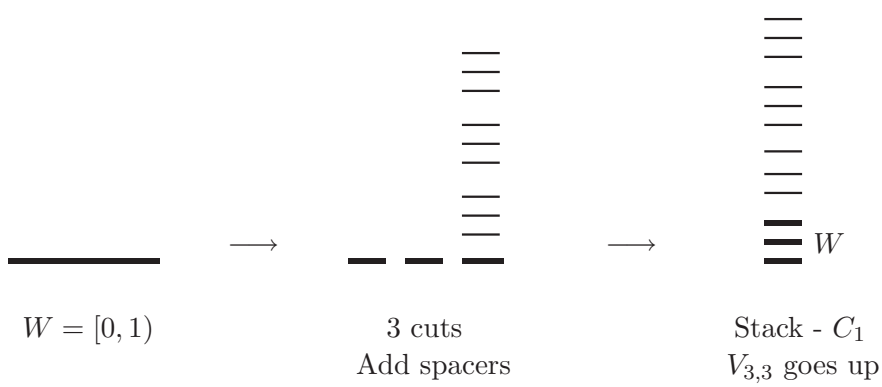

Figure 8: Stage 1 for construction of $V_{3,3}$

Theorem 12 *The transformation $V_{3,3}$ is of α-type with $\alpha = \frac{2}{3}$. The set W is exhaustive weakly wandering under the sequence*

$$\oplus_{k=1}^{\infty}\{0,\ 4^{k-1}3^k,\ 2\cdot 4^{k-1}3^k,\ 3\cdot 4^{k-1}3^k\}$$

Proof. The proof that $\alpha = \frac{2}{3}$ is similar to that of V_3. The exhaustive weakly wandering sequence can be determined directly or by referring to Theorem 3.

Corollary 1 *The two transformations V_3 and $V_{3,3}$ are non-isomorphic but have the same α-type and different exhaustive weakly wandering sequences.*

Now we modify this transformation's construction to get a transformation with $\alpha = \frac{1}{3}$.

Significantly, the only change in the construction is the distribution of the blocks of spacers.

Again, $h_0 = 1$, $C_0 = W = [0, 1)$ and the cuts are $c_k = 3$. Again we will add 3 blocks of spacers but now we set $\mathcal{S}_k = \{0, b_k, 2\cdot b_k\}$. We denote this transformation $V_{3,1,2}$.

Theorem 13 *The transformation $V_{3,1,2}$ is of α-type with $\alpha = \frac{1}{3}$. The set W is exhaustive weakly wandering under the sequence*

$$\oplus_{k=1}^{\infty}\{0,\ 4^{k-1}3^k,\ 2\cdot 4^{k-1}3^k,\ 3\cdot 4^{k-1}3^k\}$$

Proof. The results follow from Theorems 2 and 3. A direct proof for $\alpha = \frac{1}{3}$ is not significantly different than the direct proof for the transformations V_p.

However, the proof concerning the exhaustive weakly wandering sequence requires more tools. The issue is that the images of the set W do not "evenly" fill up the columns C_k at each stage. Examining the column C_1 (see Figure 9),

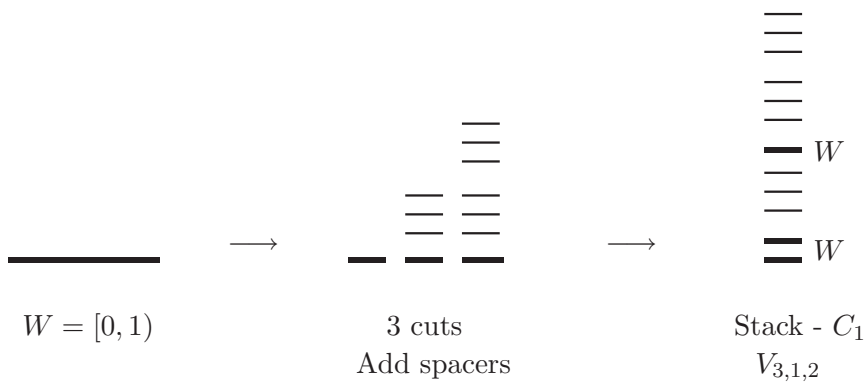

Figure 9: Stage 1 for construction of $V_{3,1,2}$

we see that W moves up into the next spacers of column C_2 before completely filling in the column C_1.

In the sequel, we address this issue with an analysis of factorizations of finite cyclic groups.

Corollary 2 *The two transformations $V_{3,3}$ and $V_{3,1,2}$ are non-isomorphic but have the same exhaustive weakly wandering sequence and are of different α-types.*

5 Proofs

In this section, we present detailed proofs of the longer theorems. The first section will prove that we can build a class of rank-one transformations that are α-type for all α. The second section will prove the theorem from 3.2 which shows we can get the exhaustive weakly wandering sequence for a class of transformations. It is shown in sections 3.3 to 3.5 how this theorem can be used to get a sequence that is exhaustive weakly wandering for a family of $\alpha-$type transformations. This culminates in Theorem 6.

5.1 Proof of Theorem 1

In this section, we prove Theorem 1 which gives a condition for a rank-one constructed transformation to be of 0-type.

The following lemma is fundamental to the proof.

Lemma 6 *Let T be an ergodic measure preserving transformation of the infinite measure set X. If there exists a set $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ and $\lim_{n\to\infty} \mu(T^n A \cap A) = 0$ then $\lim_{n\to\infty} \mu(T^n B \cap B) = 0$ for all sets $B \in \mathcal{B}$ with $\mu(B) < \infty$*

Proof : Let $B \in \mathcal{B}$ with $\mu(B) < \infty$. Since T is ergodic, for each ϵ there exists a positive integer N and a set $C \in \mathcal{B}$ so that

$$B \subseteq \bigcup_{i=0}^{N} T^i A \cup C$$

and $\mu(C) < \epsilon$.

This says, for all n:

$$
\begin{aligned}
\mu(B \cap T^n(B) \leq\ & \mu\Bigg(\Big(\bigcup_{i=0}^{N} T^i A \cup C\Big) \cap \Big(\bigcup_{i=n}^{N+n} T^i A \cup T^n(C)\Big)\Bigg) \\
\leq\ & \mu\Big(\bigcup_{i=0}^{N} T^i A) \cap \bigcup_{i=n}^{N+n} T^i A\Big) + \mu\Big(\bigcup_{i=0}^{N} T^i A) \cap T^n C\Big) \\
& \qquad + \mu\Big(C \cap \bigcup_{i=n}^{N+n} T^i A\Big) + \mu(C \cap T^n(C)) \\
\leq\ & \mu\Big(\bigcup_{i=0}^{N} T^i A) \cap \bigcup_{i=n}^{N+n} T^i A\Big) + 3\epsilon \\
\leq\ & \sum_{i=0}^{N} \sum_{j=0}^{N} \mu(T^i(A) \cap T^{j+n}(A)) + 3\epsilon
\end{aligned}
$$

By assumption, each of the terms in the sum goes to zero as $n \to \infty$, so $\lim_{n\to\infty} \mu(T^n B \cap B) = 0$. Thus the lemma is proved.

Proof of Theorem 1:

We now assume T satisfies the assumption of the theorem. By the lemma, it suffices to show $\lim_{n\to\infty} \mu(T^n A \cap A) = 0$ for the base set $A = W$.

Let $A = \cup_{i=1}^{c_k} A_i$ where A_i is the part of A in subcolumn i after the c_k cuts.

For the first stage, $T^{j+\sum_{n=i}^{i+j-1} s_n^{(1)}} A_i = A_{i+j}$ for i=1 to $c_1 - j$ and $A \cap T^n A = \emptyset$ else. Therefore, $\mu(A \cap T^n A) \leq \frac{1}{c_1}$ for $1 \leq n < 1 + s_{c_1}^{(1)}$ (notice that $1 + s_{c_1}^{(1)} \geq \frac{1}{2} h_1$).

At stage k of the construction, the set A is a union of disjoint levels in column C_k. The height of the highest level of A in column C_k will be less than $\frac{1}{2} h_{k-1}$. This is because $s_{c_{k-1}}^{(k-1)} \geq \sum_{i=1}^{c_{k-1}-1} s_i^{(k-1)} + c_{k-1} \cdot h_{k-2}$.

We now examine how the pieces of A_i move from its initial subcolumn to another subcolumn under powers of T (see Figure 1).

Note: The height of subcolumn i is: $h_{k-1}+s_i^{(k)}$. Since we assume the highest level of A_i is less than $\frac{1}{2}h_{k-1}$, all the pieces of T^nA_i are in subcolumn i for $n < \frac{1}{2}h_{k-1} + s_i^{(k)}$ and $T^nA_i \cap A_i = \emptyset$ for $\frac{1}{2}h_{k-1} \leq n < h_k - \frac{1}{2}h_{k-1}$.

For $\frac{1}{2}h_{k-1} \leq n < h_{k-1} + s_1^{(k)}$, $\mu(A \cap T^nA) < \frac{1}{c_k}$ since all the elements of A except A_1 will be in the spacers from stage k-1 or in the spacers from stage k above their subcolumn since $s_i^{(k)} \geq 2h_{k-1} + s_1^{(k)}$ for $i \geq 2$ by assumption. A_1 moves from subcolumn 1 to 2 during this interval.

$\mu(A \cap T^{h_{k-1}+s_1^{(k)}}A) = \mu(A_2) = \frac{1}{c_k}$, with A_i for i=2 to c_k in the spacers above their subcolumn

For $h_{k-1} + s_1^{(k)} < n < \frac{3}{2}h_{k-1} + s_1^{(k)}$, $\mu(A \cap T^nA) < \frac{1}{c_k}$ since A_2 to A_{c_k} will still be in spacers above their subcolumn. A_1 moves into the spacers above subcolumn 2 during this interval.

For $\frac{3}{2}h_{k-1} + s_1^{(k)} \leq n < 2h_{k-1} + s_1^{(k)}$, $\mu(A \cap T^nA) = 0$ since A_1 will be entirely in spacers above subcolumn 2 and A_2 to A_{c_k} will still be in spacers above their subcolumn.

For $2h_{k-1}+s_1^{(k)} \leq n < h_{k-1}+s_2^{(k)}$, $\mu(A \cap T^nA) < \frac{1}{c_k}$ A_1 is in spacers above subcolumn 2 and A_3 to A_{c_k} will still be in spacers above their subcolumn. A_2 moves into subcolumn 3 during this interval.

$\mu(A \cap T^{h_{k-1}+s_2^{(k)}}A) = \mu(A_3) = \frac{1}{c_k}$

For $h_{k-1} + s_2^{(k)} < n < \frac{3}{2}h_{k-1} + s_2^{(k)}$, $\mu(A \cap T^nA) < \frac{1}{c_k}$ since A_1 is in spacers above subcolumn 2 and A_3 to A_{c_k} will still be in spacers above their subcolumn. A_2 moves into spacers above subcolumn 3 during this interval.

For $\frac{3}{2}h_{k-1} + s_2^{(k)} \leq n < 2h_{k-1} + s_1^{(k)} + s_2^{(k)}$, $\mu(A \cap T^n A) < \frac{1}{c_k}$ since A_1 moves to column 3 during this interval, while A_2 and A_3 are in spacers above subcolumn 3 and the rest are still in spacers above their column.

$\mu(A \cap T^{2h_{k-1}+s_1^{(k)}+s_2^{(k)}} A) = \mu(A_3) = \frac{1}{c_k}$. A_1 intersects A_3.

For $2h_{k-1} + s_1^{(k)} + s_2^{(k)} < n < h_{k-1} + s_3^{(k)}$, A_1 and A_2 are in spacers above subcolumn 3, while A_4 to A_{c_k} will still be in spacers above their subcolumn. A_3 moves into subcolumn 4 during this interval.

$\mu(A \cap T^{h_{k-1}+s_3^{(k)}} A) = \mu(A_4) = \frac{1}{c_k}$

At this point, the pattern has emerged: the levels of A_i do not move to subcolumn i+1 until all the levels of A_j, for $j < i$, are in the spacers above subcolumn i. This is because $s_{n+1}^{(k)} \geq \sum_{i=1}^{n} s_i^{(k)} + n \cdot h_{k-1}$

Note: When A_m intersects A_n, A_i will be in the spacers above the (n-1)th column for $i < m$, above the nth column if $m < i < n$ and above their own column else.

Below are the times when $T^n A_i = A_j$ (you can see this in Figure 1):

- $T^{h_{k-1}+s_i^{(k)}} A_i = A_{i+1}$ for i=1 to $c_k - 1$
- $T^{2h_{k-1}+s_i^{(k)}+s_{i+1}^{(k)}} A_i = A_{i+2}$ for i=1 to $c_k - 2$
- $T^{jh_{k-1}+\sum_{n=i}^{i+j-1} s_n^{(k)}} A_i = A_{i+j}$ for i=1 to $c_k - j$.

Note: Since the 'height' of $A_i < \frac{1}{2}h_{k-1}$, $T^m A \cap A \subset A_{i+j}$ for $jh_{k-1} + \sum_{n=i}^{i+j-1} s_n^{(k)} - \frac{1}{2}h_{k-1} < m < jh_{k-1} + \sum_{n=i}^{i+j-1} s_n^{(k)} + \frac{1}{2}h_{k-1}$

The first time that A_{c_k} could intersect with A under T^n is when

$n \geq \frac{1}{2}h_{k-1} + s_{c_k}^{(k)} > \frac{1}{2}h_k.$

Therefore for $\frac{1}{2}h_{k-1} \leq n \leq \frac{1}{2}h_k$, $\mu(A \cap T^n A) \leq \frac{1}{c_k}$ for all k.

Since c_k is increasing this means that $\lim_{n\to\infty} \mu(A \cap T^n A) = 0$ and T is 0-type.

5.2 Proof of Theorem 2

We now prove Theorem 2, that T satisfying the assumptions of the theorem is of α-type. The issue is that, in general, it is not enough to prove $\limsup_{n\to\infty} \mu(T^n A \cap A) = \alpha \cdot \mu(A)$ for a single set. However, the reasoning for the 0-type case can be reused because the transformation is a rank-one construction and the levels of the columns C_k generate the sigma-algebra. That is, an arbitrary set is approximated by a set D consisting of a disjoint union of levels of some column C_{M-1}. Each level can be analyzed in the same manner, resulting in a proof for the set D and by approximation to the initial set.

For simplicity, assume that $\frac{n_k}{c_k} \leq \alpha$ for all large enough n. A similar argument would hold for $\frac{n_k}{c_k} \geq \alpha$.

Let $A \in \mathcal{B}$ with $0 \leq \mu(A) < \infty$. Since the set of intervals in C_n, for all n, forms a subalgebra $\mathcal{R}$ that generates $\mathcal{B}$, there is a $D \in \mathcal{R}$ so that $\mu(A \triangle D) < \epsilon$ for all ϵ.

Since $D \in \mathcal{R}$, there is an N so that D is a union of levels of C_N. Given an $\epsilon > 0$ there exists an $M > N$ so that $\alpha - \dfrac{n_M}{c_M} < \epsilon$ for all $n \geq M$.

Let $D = \bigcup_{i \in \mathcal{D}} D_i$ where the D_i are levels in C_{M-1} and let $D_{i,j}$ for $1 \leq j \leq c_M$ be the parts of D_i after the c_M cuts.

We want to examine how the levels at Stage $M-1$ appear in the levels of Stage M (see Figure 10).

Let L denote a level of column C_{M-1}, and let L_i denote the piece of L in the ith subcolumn when C_{M-1} has been cut.

We have $L \cap T^{h_{M-1}} L_i = L_{i+1}$ for $1 \leq i \leq n_M$ and $L \cap T^{h_{M-1}} L_i = \emptyset$ for $n_M + 1 \leq i \leq c_M$. Thus $\mu(L \cap T^{h_{M-1}} L) = \dfrac{n_M}{c_M} \mu(L)$ for all M.

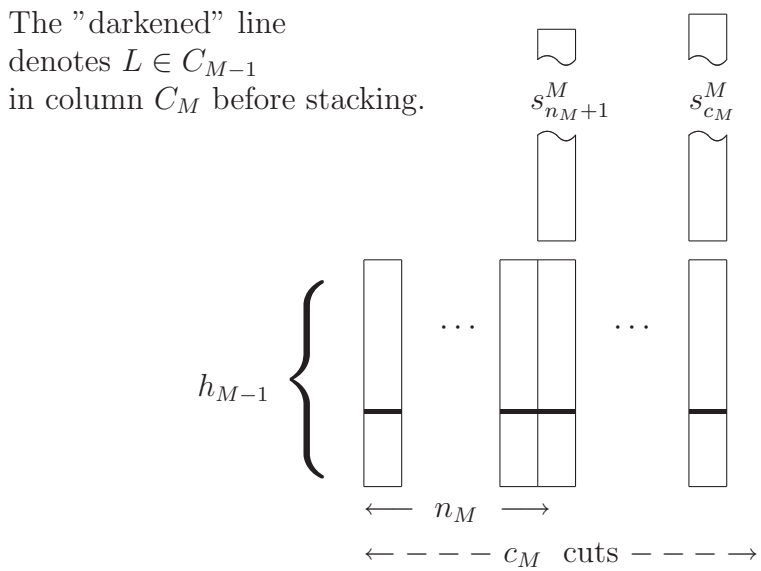

Figure 10: $M-1$ Stage in M Stage

Since this holds for any level of C_{M-1}, it also holds for D, a finite union of levels. We therefore conclude that $\lim_{k\to\infty} \mu(D \cap T^{h_k} D) = \frac{n_k}{c_k}\mu(D) = \alpha\mu(D)$ and Hence $\limsup_{n\to\infty} \mu(D \cap T^n D) \geq \alpha\mu(D)$.

Next we demonstrate the other inequality, $\limsup_{n\to\infty} \mu(D \cap T^n D) \leq \alpha\mu(D)$.

Again we are assume D is a union of levels of C_{M-1}, so $D = \bigcup_{i\in\mathcal{D}} D_i$ where the D_i are levels in C_{M-1} and let $D_{i,j}$ for $1 \leq j \leq c_M$ be the parts of D_i after the c_M cuts. Note that, we may assume that the levels of D in C_{M-1} are in the bottom half of the column. If not, then the levels of D are in the bottom half of C_M and we re-index the columns.

For $\frac{1}{2}h_{M-1} \leq n < h_{M-1}$ under T^n parts of D_i might be in column i and parts might be in i+1 for $1 \leq i \leq n_M$ and for $n_M < n \leq c_M$ D_i will be in the spacers (either from stage M-1 or M) above their column. Thus

$\mu(T^n D \cap D) \leq \frac{n_M}{c_M}\mu(D)$.

For $h_{M-1} \leq n < h_{M-1} + \frac{1}{2}h_{M-1}$ under T^n D_i will be in column i+1 for $1 \leq i \leq n_M$ and for $n_M < n \leq c_M$ D_i will be in the spacers above their column. Thus $\mu(T^n D \cap D) \leq \frac{n_M}{c_M}\mu(D)$.

For $\frac{3}{2}h_{M-1} \leq n < 2h_{M-1}$ under T^n parts of D_i might be in column i+1 and parts might be in i+2 for $1 \leq i \leq n_M - 1$, D_{n_M} will be in spacers above column $n_M + 1$, and for $n_M < n \leq c_M$ D_i will be in the spacers above their column. Thus $\mu(T^n D \cap D) \leq \frac{n_M - 1}{c_M}\mu(D)$.

For $2h_{M-1} \leq n < 2h_{M-1} + \frac{1}{2}h_{M-1}$ under T^n D_i will be in column i+2 for $1 \leq i \leq n_M - 1$, D_{n_M} will be in spacers above column $n_M + 1$, and for $n_M < n \leq c_M$ D_i will be in the spacers above their column. Thus $\mu(T^n D \cap D) \leq \frac{n_M - 1}{c_M}\mu(D)$.

This generalizes to (leaving out the first part):

For $jh_{M-1} \leq n < jh_{M-1} + \frac{1}{2}h_{M-1}$ under T^n D_i will be in column i+j for $1 \leq i \leq n_M + 1 - j$, in spacers above column $n_M + 1$ for $n_M - j < i \leq n_M$, and for $n_M < n \leq c_M$ D_i will be in the spacers above their column. Thus $\mu(T^n D \cap D) \leq \frac{n_M + 1 - j}{c_M}\mu(D)$.

This will continue until $n = n_M h_{M-1}$ at which point the parts of D in the first $n_M + 1$ columns will be in column $n_M + 1$ while the other parts will still be in their original column (since the number of spacers in each column with spacers is more than $n_M h_{M-1}$). From here until $n = \frac{1}{2}h_M$ (at which point, parts of $D_{c_{M-1}}$ might move past T defined under T_{C_M}), D_i for $i > n_M$ will not move to the next column until all the previous columns are in its column (since the number of spacers above its column is more

than all the levels, including previous spacers, that come before it). Thus $\mu(T^n D \cap D) \leq \frac{1}{c_M}\mu(D)$.

We have shown that $\mu(T^n D \cap D) \leq \frac{n_M}{c_M}\mu(D) < (\alpha - \epsilon)\mu(D)$ for $\frac{1}{2}h_{M-1} \leq n < \frac{1}{2}h_M$. Since the same argument works for all stages after M we have $\mu(T^n D \cap D) \leq \frac{n_k}{c_k}\mu(D) \leq (\alpha - \epsilon)\mu(D)$ for all $n > \frac{1}{2}h_{M-1}$, and thus $\limsup_{n\to\infty} \mu(T^n D \cap D) \leq (\alpha)\mu(D)$. Since $\lim\limits_{k\to\infty} \mu(D \cap T^{h_k} D) = \alpha\mu(D)$, this proves the case for D.

For the original set A, we use a standard approximation argument.

Let $\epsilon > 0$ with $\mu(A \triangle D) < \epsilon$. Since $\limsup_{n\to\infty} \mu(D \cap T^n D) = \alpha\mu(D)$, for all ϵ there exists an L so that if $n > L$ then $\sup_{n>L} |\mu(D \cap T^n D) - \alpha\mu(D)| < \epsilon$. Therefore

$$\begin{aligned}\sup_{n>L} |\mu(A \cap T^n A) - \alpha\mu(A)| \leq \sup_{n>L} &|\mu(A \cap T^n A) - \mu(D \cap T^n D)| \\ &+ \sup_{n>L} |\mu(D \cap T^n D) - \alpha\mu(D)| \\ &+ |\alpha\mu(D) - \alpha\mu(A)| \ < \ 4\epsilon\end{aligned}$$

The first term on the right is less than 2ϵ since

$$\begin{aligned}|\mu(A \cap T^n A) - \mu(D \cap T^n D)| \leq &|\mu(A \cap T^n A) - \mu(D \cap T^n A)| \\ &+ |\mu(D \cap T^n A) - \mu(D \cap T^n D)|\end{aligned}$$

Because if $\mu(A \triangle D) < \epsilon$ and T is measure preserving it follows that $\mu((A \cap C) \triangle (D \cap C)) < \epsilon$ and $\mu(T^n A \triangle T^n D) < \epsilon$.

The second term is less than ϵ by the proof above and the third term is less than ϵ by assumption (and since α must be ≤ 1). Therefore, $\limsup_{n\to\infty} \mu(A \cap T^n A) = \alpha\mu(A)$.

This completes the proof that T is α-type.

Note that the first n_k spacers need only be equal. We set them to 0 for simplicity.

5.3 Proof of Theorem 3

In this section, we prove Theorem 3. This theorem gives a simple condition that allows us to determine an explicit exhaustive weakly wandering sequence. As the proof progresses, it also reveals the direct sum structure of the exhaustive weakly wandering sequence. The proof is involved and uses results concerning factorizations of finite cyclic groups. The necessity arises because under the different distributions of the spacers, the columns C_k are not being "filled orderly" as was the case with the Hajian-Kakutani examples in Section 4.

In order to simplify the proof and avoid the overuse of parameters, we will discuss the proof in terms of the special sequence $\mathbb{B}$ given in Section 3.4.

As usual, we begin with $h_0 = 1$ and $C_0 = W = [0, 1)$. Then at Stage 1 we cut the column C_0 into two subcolumns and we place one block of two spacers on either the left or right subcolumn (see Figure 3). Next, we stack the right side subcolumn over the left side subcolumn. This results in column C_1 of height $h_1 = 4$.

At this stage we examine the transformation T_{C_1} which goes up the column C_1 and returns to the bottom level periodically. Clearly, if the spacers have been placed on the right side subcolumn then we have $C_1 = W \dot{\cup} T^2_{C_1} W$.

If the spacers have been placed on the left side subcolumn then we also have $C_1 = W \dot{\cup} T^2_{C_1} W$. However, the "top" half of W needed to "wrap around" to the bottom level for this to occur.

The significance of this becomes evident when we examine the movement of the set W under the map T_{C_2}, (recall Figure 11). In the case of the spacers placed over the right side subcolumn, we still have $C_1 = W \dot{\cup} T^2_{C_2} W$. But in the case of the spacers placed over the left side subcolumn we do not have this. The issue is the top half of W, *i.e.* the piece on the 4th level of C_1 does

not completely return to the base - some portion of it goes into the spacers which are placed at Stage 2

We continue the analysis by looking at Stage 2. Column C_1 has height $h_1 = 4$. The next cut is $c_2 = 4$. This means the block size $b_2 = 16$ and the total number of blocks of spacers we will add is 7 (we are adding a total of 112 spacers). As an illustration, we will place the spacers over three of the subcolumns in sizes 16, 32 and 64 - see Figure 11.

Observe, the level marked with "*" in Column C_2 on the right of the figure. This is part of the set W (the bottom W in Column C_1 of the figure). At Stage 1, this moves to $T^2(*) \subset L_2$ which is part of Column C_1. But at Stage 2, $T^2(*)$ is in the block of spacers (of size 16). Hence, it is no longer "covering" the set it had been covering. However, when we are at Stage 2, there are additional images of the level $*$. In particular, $T^{2+16}(*)$ is again in Column C_1. The tools to make this precise are factorizations of finite cyclic groups.

5.3.1 A Finite Cyclic Group and Subgroup

There are finite cyclic groups and subgroups behind the rank-one construction of the sequence $\mathbb{B}$.

Recall from Section 3.4, the definition

$$\mathbb{B}_k = \{0, b_k,\ 2 \cdot b_k,\ 3 \cdot b_k, \cdots,\ (2^{2^k-1} - 1) \cdot b_k\}$$

where $b_k = 2^k \cdot \prod_{j=0}^{k-1} 2^{2^j+j-1} = c_k \cdot h_{k-1}$, $h_{k-1} = \prod_{j=0}^{k-1} 2^{2^j+j-1}$ and the height of column C_k is $h_k = \prod_{j=0}^{k} 2^{2^j+j-1}$.

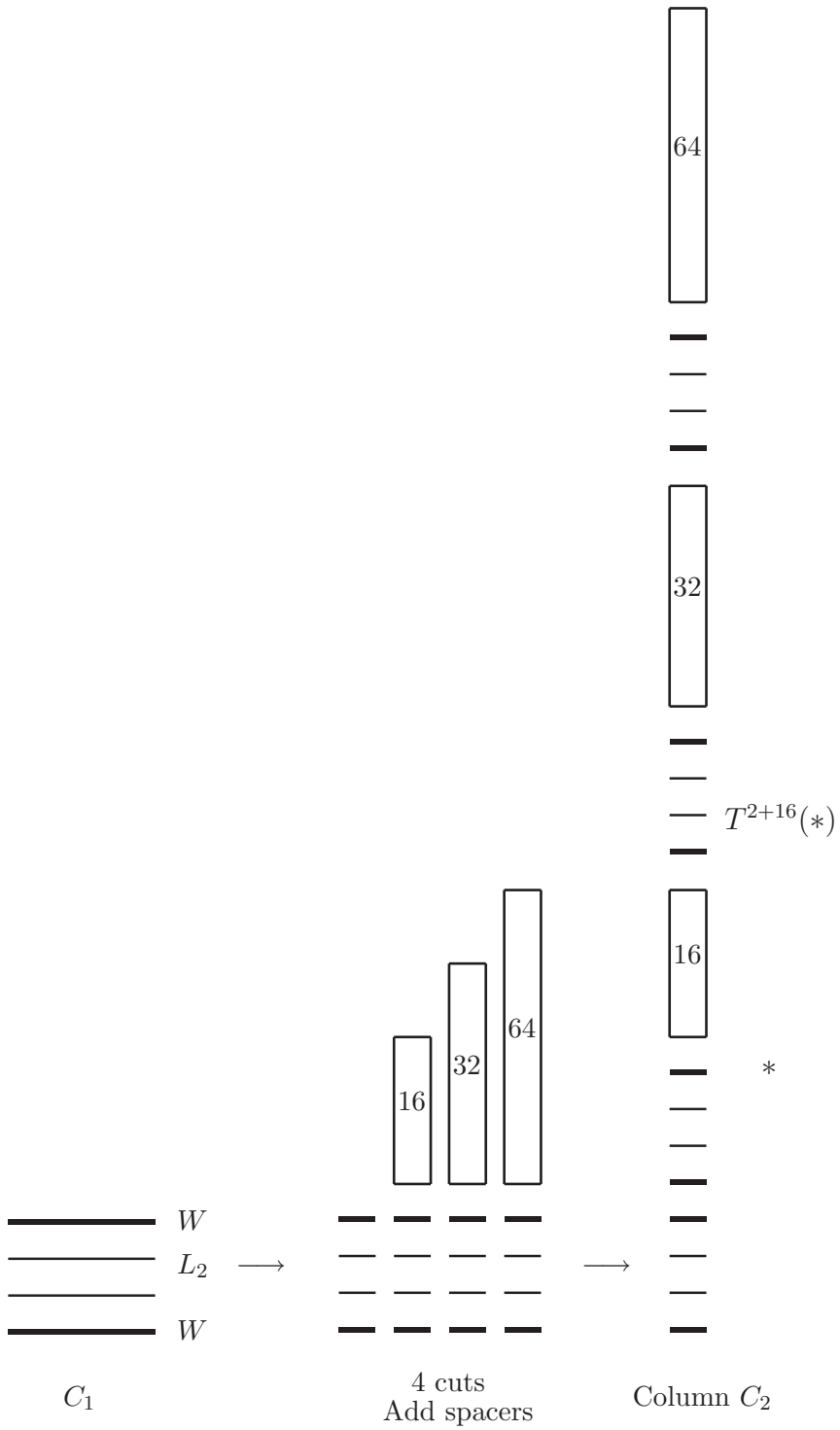

Figure 11: Stage 1 to Stage 2

Then it is easy to factor out the term h_{k-1} obtaining

$$\begin{aligned}\mathbb{B}_k = h_{k-1} \cdot B_k &= \{0, c_k, 2 \cdot c_k, \cdots, (2^{2^k-1} - 1) \cdot c_k\} \\ &= \{0, 2^k, 2 \cdot 2^k, \cdots, (2^{2^k-1} - 1) \cdot 2^k\} \\ &= \{0, 2^k, 2 \cdot 2^k, \cdots, (2^{2^k+k-1} - 2^k)\}\end{aligned}$$

Similarly $h_k = h_{k-1} \cdot 2^{2^k+k-1} = h_{k-1} \cdot N_k$.

Define the finite cyclic group $G_k = \{0, 1, \cdots, 2^{2^k+k-1} - 1\}$ with addition modulo 2^{2^k+k-1}.

Lemma 7 *B_k is a subgroup of the finite cyclic group G_k.*

Factoring out h_{k-1} can also be realized on the columns. In Figure 11, this corresponds to collapsing each block of size $h_1 = 4$, see Figure 12. That is, we start with an interval $[0, 1)$, cut it into four pieces and place 4 spacers over the second subcolumn, 8 spacers over the third subcolumn and 16 spacers over the last subcolumn. The left most interval returns to W on iterates T^0, T^1, T^6 and T^{15}

5.3.2 Factorizations of Finite Cyclic Groups

In this section, we show how factorizations of finite cyclic groups are connected to the exhaustive weakly wandering sequences which are constructed in Theorem 3.

We begin by describing the "generic" (and inductive) situation. At the end of Stage k we have a column C_k of height h_k. The levels of the column are labeled $\{L_0^k, \cdots, L_{h_k-1}^k\}$. The transformation, T_{C_k} is a permutation on

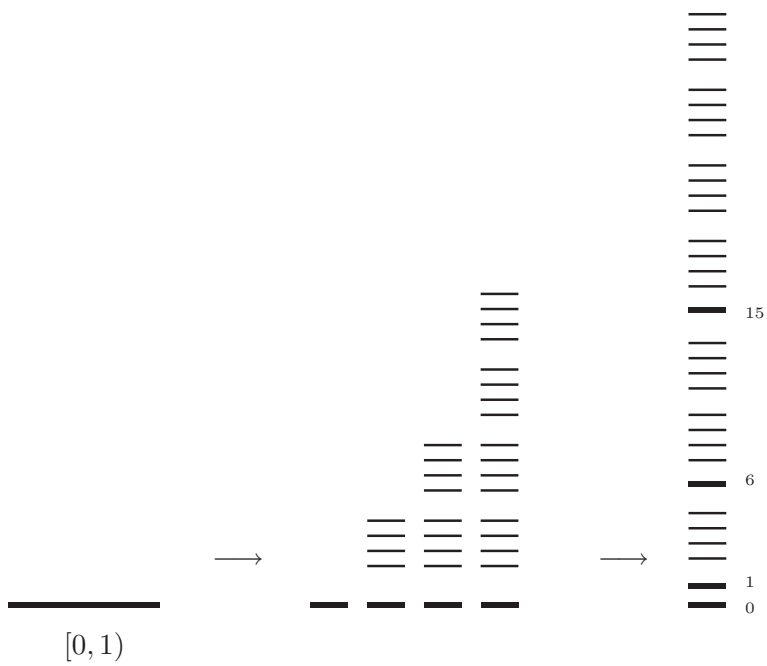

Figure 12: Factoring by $h_1 = 4$

these levels. In addition, there is the set W, which consists of a finite union of the levels of C_k, and a finite set of integers, denotes as B_k so that

$$C_k = \bigcup_{b \in B_k} T^b_{C_k} W, \ (disjoint)$$

Essentially, the idea is that the column C_k can be represented by the integers $\{0, 1, \cdots, h_k - 1\}$ and the transformation T_{C_k} can be represented as adding one modulo h_k. The set W is then represented by the indices of levels and we denote these indices as the set of integers A_k. Thus at this stage we have

$$A_k \oplus B_k = \{0, 1, \cdots, h_k - 1\} \ Mod \ h_k$$

Definition 3 *For an integer $N > 2$, we call finite sets A and B* **comple-**

ments Mod N *(or complements when N is understood) if*

$$A \oplus B = \{0, 1, \cdots, N-1\} \ mod \ N$$

The symbol $\oplus$ means that each sum $a+b$ is unique. Alternately, $a+b = a'+b'$ if and only if $a = a'$ and $b = b'$.

We make the following standing assumption, $A \cap B = \{0\}$. This corresponds to the fact that W always contains the bottom level of every column C_k. To avoid trivial cases, we assume $|A| \geq 2$ and $|B| \geq 2$.

An important point is that a set B may have more than one complement in N. In addition, when B is a subgroup of the finite cyclic group $\{0, \cdots, N-1\}$ modulo N, the complements of B are nicely organized. We give a simple illustration of this simultaneously making the connection to the rank-one construction.

Set $N = 32$ (this will be N_2 in a latter construction). The finite cyclic group is $G = \{0, 1, 2, \cdots, 31\}$ with "addition mod 32". Put

$$B = \{0, 4, 8, 12, 16, 20, 24, 28\}$$

which is a subgroup of G.

The simplest complement of B is $A = \{0, 1, 2, 3\}$. In the rank-one construction, this corresponds to the case where spacers are added only over the right hand (last) column. (The number 32, will be seen to be connected to Stage 2 of the construction).

Here is the "addition table" that shows A and B are complements.

28	28	29	30	31
24	24	25	26	27
20	20	21	22	23
16	16	17	18	19
12	12	13	14	15
8	8	9	10	11
4	4	5	6	7
0	0	1	2	3
+	0	1	2	3

A second complement is $A = \{0, 1, 6, 15\}$. This corresponds to the case where spacers are added over the last three columns in the second stage of construction, see Figure 12.

28	28	29	2	11
24	24	25	30	7
20	20	21	26	3
16	16	17	22	31
12	12	13	18	27
8	8	9	14	23
4	4	5	10	19
0	0	1	6	15
+	0	1	6	15

Remark 6 *Observe that each "column" is invariant under addition by multiples of* 4. *(Addition is modulo 32.)*

This can be summarized as follows.

Lemma 8 *Let B be as above with $N = 32$. For any nonnegative integers $\epsilon_1, \epsilon_2, \epsilon_3$, define*

$$A = \{0 + 4\epsilon_1, 1 + 4\epsilon_2, 2 + 4\epsilon_3, 3 + 4\epsilon_4\}$$

Then $A \oplus B = G$ mod 32

Connecting this to the construction we have the following.

Lemma 9 *Let G_k be the cyclic group and $B_k = \{0, c_k, 2c_k, 3c_k, \ldots, (2^{2^k+k-1}) \cdot c_k\}$ as above. Let $A'_k = \{0 + c_k \cdot \epsilon_1, 1 + c_k \cdot \epsilon_2, 2 + c_k \cdot \epsilon_3, \ldots, c_k - 1 + c_k \cdot \epsilon_{c_k}\}$ where $\epsilon_1, \epsilon_2, \ldots \epsilon_{k-1}$ are nonnegative integers. Then $A'_k \oplus B_k = G_k$.*

Note: The A'_k in this lemma is more general than the A_k in the construction of the transformations. The first entry for A_k is always 0.

5.3.3 Getting Factorizations from the construction

Assume we have a transformation T which has been constructed according to the rules of Theorem 3. In particular, the spacers are multiples of the block size. Each stage of the construction defines a single factorization of a finite cyclic group.

Lemma 10 *Assume at the kth stage of a rank-one construction there are c_k cuts and the spacers are $\mathcal{S}_k = \{s_{k,1} \cdot b_k,\ s_{k,2} \cdot b_k, \cdots,\ s_{k,c_k} \cdot b_k\}$. ($b_k$ is the block size and $s_{k,i}$ is the number of blocks placed over the ith subcolumn.)*

Put

$$N_k = c_k \cdot (1 + \textstyle\sum_{i=1}^{c_k} s_{k,i})$$

$$A_k = \{0, 1 + s_{k,1} \cdot c_k, 2 + (s_{k,1} + s_{k,2}) \cdot c_k, \ldots, c_k - 1 + \textstyle\sum_{i=1}^{c_k - 1} s_{k,i} \cdot c_k\}$$

$$B_k = \{0, c_k, 2 \cdot c_k, \cdots, \sum_{i=1}^{c_k} s_{k,i} \cdot c_k\}$$

Then

$$A_k \oplus B_k = \{0, 1, \cdots, N_k - 1\} \ Mod \ N_k$$

Proof Follows from Lemma 9.

Factorizations can be put together to get new factorizations.

Lemma 11 *Assume we have two factorizations.*

$$A \oplus B = \{0, 1, \cdots, N - 1\} \ Mod \ N$$

$$C \oplus D = \{0, 1, \cdots, M - 1\} \ Mod \ M$$

Then

$$(A \oplus N \cdot C) \oplus (B \oplus N \cdot D) = \{0, 1, \cdots, MN - 1\} \ Mod \ MN$$

Proof. Elementary

We combine the previous results below. The first theorem corresponds to the case when all spacers are placed over the last (rightmost) subcolumn. c_k is the number of cuts at the kth stage and s_k denotes the total number of blocks of spacers being added.

Theorem 14 *Given integers $c_k, s_k > 1$. Let*

$$N_k = c_k \cdot s_k$$

$$G_k = \{0, 1, \cdots, N_k - 1\}$$

$$A_k = \{0, 1, \cdots, c_k - 1\}$$

$$B_k = \{0, c_k, 2 \cdot c_k, \cdots, (s_k - 1) \cdot c_k\}$$

$$\mathbb{A}_k = \oplus_{i=1}^{k} N_{k-1} \cdot A_k, \quad \mathbb{A} = \oplus_{i=1}^{\infty} N_{k-1} \cdot A_k$$

$$\mathbb{B}_k = \oplus_{i=1}^{k} N_{k-1} \cdot B_k \quad \mathbb{B} = \oplus_{i=1}^{\infty} N_{k-1} \cdot B_k$$

Then

1. $\mathbb{A}_k \subset \mathbb{A}_{k+1}$
2. $\mathbb{B}_k \subset \mathbb{B}_{k+1}$
3. $G_k = \mathbb{A}_k \oplus \mathbb{B}_k$
4. $\mathbb{N} = \mathbb{A} \oplus \mathbb{B}$

When the spacers are placed arbitrarily, the set A_k changes.

Theorem 15 *Given $c_k > 1$ and $\mathcal{S}_k = \{s_{k,1}, \cdots, s_{k,c_k}\}$. Let*

$$s_k = \sum_{i=1}^{c_k} s_{k,i}$$

$$A_k = \{0, 1 + s_{k,1} \cdot c_k, 2 + (s_{k,1} + s_{k,2}) \cdot c_k, \ldots, c_k - 1 + c_k \cdot \sum_{i=1}^{c_k - 1} s_{k,i}\}$$

$$N_k = c_k \cdot s_k$$

$$G_k = \{0, 1, \cdots, N_k - 1\}$$

$$B_k = \{0, c_k, 2 \cdot c_k, \cdots, (s_k - 1) \cdot c_k\}$$

$$\mathbb{A}_k = \oplus_{i=1}^{k} N_{k-1} \cdot A_k, \quad \mathbb{A} = \oplus_{i=1}^{\infty} N_{k-1} \cdot A_k$$

$$\mathbb{B}_k = \oplus_{i=1}^{k} N_{k-1} \cdot B_k \quad \mathbb{B} = \oplus_{i=1}^{\infty} N_{k-1} \cdot B_k$$

Then

1. $\mathbb{A}_k \subset \mathbb{A}_{k+1}$

2. $\mathbb{B}_k \subset \mathbb{B}_{k+1}$

3. $G_k = \mathbb{A}_k \oplus \mathbb{B}_k \ Mod \ \prod_{i=1}^{k} N_i$

4. $\mathbb{N} \supset \mathbb{A} \oplus \mathbb{B}$

The point is that condition 4 of the first theorem makes the proof of exhaustiveness straightforward, but condition 4 of the second theorem isn't enough. That is why, it is necessary to use condition 3 and an induction argument to prove exhaustiveness in the general case of arbitrary placement of the blocks of spacers.

Completing the proof

Remark 7 *Observe that, by construction* $s_i^{(k)} \in \mathbb{B}_k$. $h_k = \prod_{j=0}^{k} N_j$.

The proof is examined in stages.

Spacers only over the last column: this is the simple case. It is straightforward, but the analysis will clarify the notation and some of the later arguments.

Stage 1 At the end of Stage 1 there are $h_1 = N_1$ levels of the column C_1. This is represented by $\{0, 1, 2, \ldots, h_1 - 1\}$. The set W is represented by the first c_1 levels (indices), that is $\mathbb{A}_1 = \{0, 1, \cdots, c_1 - 1\}$. The powers of T needed to fill C_1 are $B_1 = \{0, c_1, 2c_1, \ldots, s^{(1)} \cdot c_1\}$. We have, without using modular arithmetic

$$\mathbb{A}_1 \oplus \mathbb{B}_1 = \{0, 1, 2, \ldots, h_1 - 1\}$$

In other words $W \cup T^{c_1} W \ldots \cup T^{s^{(1)} \cdot c_1} W = C_1$.

Stage 2 At the end of stage 2 there are h_2 levels of the column C_2. This is represented by $\{0, 1, 2, \ldots, h_2 - 1\}$. Set $A_2 = \{0, 1, \cdots, c_2 - 1\}$ and $B_2 = \{0, c_2, 2c_2, \ldots, s^{(2)} \cdot c_2\}$. Then $A_2 \oplus B_2 = \{0, 1, \cdots, N_2 - 1\}$. This gives

$$\begin{aligned}
\mathbb{A}_2 \oplus \mathbb{B}_2 &= (A_1 \oplus h_1 \cdot A_2) \oplus (B_1 \oplus h_1 \cdot B_2) \\
&= (A_1 \oplus B_1) \oplus h_1 \cdot (A_2 \oplus B_2) \\
&= \{0, 1, \ldots h_2 - 1\}
\end{aligned}$$

As before the set $\mathbb{B}_2$ corresponds to the powers of T necessary to fill the column, and the set $\mathbb{A}_2$ represents the levels containing the set W.

The proof continues by induction. The point is that, in this case, it's easy to see that C_k is filled and that showing $A_k \oplus B_k = \{0, 1, \ldots N_k - 1\}$ for all k will show that T is exhaustive.

In the general case, the spacers are added in multiples of the block size. We know the total number of blocks of spacers - but we do not know over which subcolumn the spacers have been placed.

In terms of the factorization of the finite cyclic groups this means that

$$A_k \oplus B_k = \{0, 1, \ldots N_k - 1\} \; Mod \; N_k$$

The modular arithmetic for the groups, corresponds to subintervals of W wrapping around the top of a column in order to cover a level lower down the column.

As an illustration, suppose for stage 1: $c_1 = 2$ and $\mathcal{S}_1 = \{0, b_1\}$ (which gives $A_1 = \{0, 1\}$, $B_1 = \{0, 2\}$ and $h_1 = N_1 = 4$); for stage 2: $c_2 = 4$ and $\mathcal{S}_2 = \{0, b_2, 2b_2, 4b_2\}$ (see Figure 12).

This gives: $B_2 = \{0, 4, 8, 12, 16, 20, 24, 28\}$, $A_2 = \{0, 1, 6, 15\}$, $N_2 = 32$, and $h_2 = 128$. Which in turn gives $A_2 \oplus B_2 = \{0, 1, \ldots 31\}$ but only mod 32. That means that not all the levels of C_2 are covered when viewed from Stage 2. Specifically, examining the numbers we see that $6 + 28 = 34 = 2$ mod 32. Since these are multiplied by h_1 =4, this represents $T^{112}L_{24}$ which is not defined at stage 2. In other words, it means that level 2 will not be filled at the end of stage 2. However, T^{112} will enter at Stage 3. This means that a portion of level 2 will be covered. Continuing in this way, all of level 2 will be covered. The point is to use the factorizations to show that eventually, at some later stage more and more subintervals of level 2 will be covered by images of W under T.

General Case: The proof is by induction

Stage 1 At the end of stage 1 there are again $h_1 = N_1$ levels in the column C_1. In this case, the set W is represented by $A_1 = \{0, 1 + s_{1,1} \cdot c_1, 2 + (s_{1,1} + s_{1,2}) \cdot c_1, \ldots, c_1 - 1 + c_1 \cdot \sum_{i=1}^{c_1-1} s_{1,i}\}$. B_1 is still $\{0, c_1, 2c_1, \ldots, s^{(1)} \cdot c_1\}$ and $\mathbb{A}_1 \oplus \mathbb{B}_1 = \{0, 1, 2, \ldots, h_1 - 1\}$, but now only mod N_1 and so some levels at stage 1 will not be filled in at this stage.

Suppose that level L_i is not filled and suppose that $a + b = i$ mod N_1 (with $a \in A_1$ and $b \in B_1$). We have to look at later stages in the construction to see what happens to L_a. Note, that this means that the level L_a maps to L_i after wrapping around the top of the column C_1.

In stage 2, L_a and L_i are broken into c_2 pieces, denoted $L_{a,j}$ and $L_{i,j}$, $j = 1, \cdots, c_2$, where:

- $T^{b+s_l^{(2)}} L_{a,l} = L_{i,l+1}$ for $l = 1, 2, \ldots, c_2 - 1$ with $s_l^{(2)}$ the number of spacers over column l

- $T^{b+s^{(2)}_{c_2}} L_{a,c_2}$ would be $= L_{i,1}$ if no spacers were added at later stages

Notice that $b + s^{(2)}_l \in \mathbb{B}$ since $b \in B_1$ by assumption and $s^{(2)}_l \in \mathbb{B}_2$ by construction.

To see what happens to $T^{b+s^{(2)}_{c_2}} L_{a,c_2}$ we have to look at stage 3. The set up will be similar to stage 2, where L_{a,c_2} and $L_{i,1}$ are broken into 16 pieces where:

- $T^{b+s^{(2)}_{c_2}+s^{(3)}_l} L_{a,c_2,l} = L_{i,1,l+1}$ for $l = 1, 2, \ldots c_3 - 1$
- $T^{b+s^{(2)}_{c_2}+s^{(3)}_l} L_{a,c_2,c_3}$ would be $= L_{i,1,1}$ if no spacers were added at later stages

Notice that $4b + s^{(2)}_{c_2} + s^{(3)}_l \in \mathbb{B}$.

Again we have to go to the next stage to see what happens to the right-most piece of L_{a,c_2}.

The process continues and will eventually fill up all of L_i (after stage 2, $\frac{c_2-1}{c_2}$ of it is filled and after stage 3, $\frac{c_2-1}{c_2} + \frac{c_3-1}{c_3} \cdot \frac{1}{c_2}$ is filled, ...).

It is an induction argument. If we assume that all of L_i is filled except $L_{i,1,\ldots,1}$ (with m ones) then the above argument will show that $L_{i,1,\ldots,1,i}$ (with m ones) is filled for $i > 1$.

Stage M

We assume that the levels at stage M-1 are filled by the construction.

This assumption says that each level of the form $a \cdot h_{M-1}$ represents a block of length h_{M-1} (so level 0 represents levels {0,1,..., h_{M-1}), which means if we show that all levels of the form $a \cdot h_{M-1}$ are filled then we will have shown that C_M is filled.

Therefore we only have to show that G_M is the direct sum of A_M and B_M mod N_M and that the levels missed because they are equal to the sum mod N_M are filled at later stages.

For stage M, W is "represented" by $A_M = \{0, 1 + s_{M,1} \cdot c_M, 2 + (s_{M,1} + s_{M,2}) \cdot c_M, \ldots, c_M - 1 + c_M \cdot \sum_{i=1}^{c_M - 1} s_{M,i}\}$, and B_M is $\{0, c_M, 2c_M, \ldots, s^{(M)} \cdot c_M\}$. We then have $A_M \oplus B_M = \{0, 1, 2, \ldots, h_1 - 1\}$ mod N_M. Again this means some levels at stage M will are not filled in at this stage.

Suppose that level $L_{i \cdot h_{M-1}}$ is not filled and suppose that $a + b = i$ mod N_M (with $a \in A_M$ and $b \in B_M$). We again study the later stages in the construction to see what happens to the level L_a.

In stage M+1, $L_{a \cdot h_{M-1}}$ and $L_{i \cdot h_{M-1}}$ are broken into c_{M+1} pieces, $L_{a \cdot h_{M-1}, j}$ and $L_{i \cdot h_{M-1}, j}$, where:

- $T^{b \cdot h_{M-1} + s_l^{(M+1)}} L_{a \cdot h_{M-1}, l} = L_{i \cdot h_{M-1}, l+1}$ for $l = 1, 2, \ldots, c_{M+1} - 1$
- $T^{b \cdot h_{M-1} + s_{c_{M+1}}^{(M+1)}} L_{a \cdot h_{M-1}, c_{M+1}}$ would be $= L_{i \cdot h_{M-1}, 1}$ if no spacers were added at later stages

As before, we $b \cdot h_{M-1} + s_l^{(M+1)} \in \mathbb{B}$.

To see what happens to $T^{b \cdot h_{M-1} + s_{c_{M+1}}^{(M+1)}} L_{a \cdot h_{M-1}, c_{M+1}}$ we have to go to stage $M + 2$. The set up will be similar to stage $M + 1$, where $L_{a \cdot h_{M-1}, c_{M+1}}$ and $L_{i \cdot h_{M-1}, 1}$ are broken into c_{M+2} pieces where:

- $T^{b \cdot h_{M-1} + s_{c_{M+1}}^{(M+1)} + s_l^{(M+2)}} L_{h_{M-1} a, c_{M+1}, l} = L_{h_{M-1} i, 1, l+1}$ for $l = 1, \ldots c_{M+2} - 1$
- $T^{h_{M-1} b_j + s_{c_{M+1}}^{(M+1)} + s_{c_{M+2}}^{(M+2)}} L_{h_{M-1} \cdot a, c_{M+1}, c_{M+2}}$ would be $= L_{h_{M-1} \cdot i, 1, 1}$

Once more we have, $h_{M-1} + s_{c_{M+1}}^{(M+1)} + s_l^{(M+2)} \in \mathbb{B}$.

As with stage 2, the process continues and will eventually fill up all of $L_{h_{M-1} \cdot i}$ (after stage $M+1$, $\frac{c_{M+1}-1}{c_{M+1}}$ of it is filled).

By induction, the proof is complete.

References

[1] J. Aaronson, *An Introduction to Infinite Ergodic Theory* AMS Mathematical Surveys and Monographs 50, American Mathematical Society (1997).

[2] S. Eigen and A. Hajian, *Sequences of Integers and Ergodic Transformations* Advances in Math 73 (1989), 256-262.

[3] S. Eigen and A. Hajian, *A Characterization of Exhaustive Weakly Wandering Sequences for Nonsingular Transformations* Comment. Math. Univ. St. Paul. 36 (1987), no. 2, 227–233.

[4] S. Eigen, A. Hajian, and K. Halverson *Multiple Recurrence and Infinite Measure Preserving Odometers* Israel Journal of Math 108 (1998), 37-44.

[5] S. Eigen, A. Hajian, and Y. Ito *Ergodic Measure Preserving Transformations of Finite Type* Tokyo Journal of Math 11 (1988), 459-470.

[6] S. Eigen, A. Hajian, and S. Kalikow *Ergodic Transformations and Sequences of Integers* Israel Journal of Math 75 (1991), 119-128.

[7] S. Eigen, A. Hajian, and V.S. Prasad *Universal Skyscraper Templates for Infinite Measure Preserving Transformations* Discrete Contin.Dyn. Sys. 16 (2006), 343-360.

[8] M.H. Ellis and N. Friedman, "On Eventually Weakly Wandering Sequences" *Advances in Math.* Supplementary Studies, 2, Probability and Ergodic Theory 1978, 185-194.

[9] N. Friedman *Introduction to Ergodic Theory* Van Nostrand Reinhold, New York, 1970.

[10] A. Hajian, *On Ergodic Measure-Preserving Transformations Defined On An Infinite Measure Space* Proc. Math. Soc. 16, No. 1 (Feb. 1965), 45-48.

[11] A. Hajian and Y. Ito, *Weakly Wandering and Related Sequences* Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 8 (1967), 315-324.

[12] A. Hajian and Y. Ito, *Transformations That Do Not Accept a Finite Invariant Measure* Bulletin of the American Math. Soc. Vol. 84, Number 3 (May 1978), 417-427.

[13] A. Hajian, Y. Ito, and S. Kakutani *Invariant Measures and Orbits of Dissipative Transformations* Advances in Math., Number 9 (1972), 52-65.

[14] A. Hajian and S. Kakutani, *Weakly Wandering Sets and Invariant Measures* Amer. Math. Soc. 110 (1964), 136-151.

[15] A. Hajian and S. Kakutani, *Example of an Ergodic Measure Preserving Transformation on an Infinite Space* Lecture Notes in Math 160 (1970), 45-52.

[16] L.K. Jones and U. Krengel, *On Transformations Without Finite Invariant Measure* Advances in Math. 12 (1974), 275-295.

[17] M. Osikawa and T. Hamachi, *On Zero Type and Positive Type Transformations with Infinite Invariant Measures* Mem. Fac. Sci., Kyushu Ser. A Vol. 25 (1971), 280-295.