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Various parameters of subgraphs and supergraphs of the hypercube

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Various Parameters of Subgraphs and Supergraphs
of the Hypercube

A dissertation presented

by

Elizabeth A. Donovan

to

The Department of Mathematics

in partial fulfillment of the requirements

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in the field of

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ABSTRACT OF DISSERTATION

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Abstract

The hypercube, in its various forms, has been greatly studied over the last sixty years. Our research involving this graph primarily uses a concept of “levels” in this graph, where the i^{th} level of the n -cube is defined as the set of all vertices of weight i . When examining more than one level, we are, in fact, just viewing the induced subgraph whose vertex set consists of those vertices of desired weights.

We begin by considering the vertices of any two consecutive levels on the lower half of the cube sorted in colexicographic order and obtain a perfect matching of initial segments of maximum size. This leads to the long-standing open question of finding a Hamiltonian cycle in the middle two levels of the n -cube, where n is odd. For, if we could find two such disjoint complete matchings on these levels, we might be able to extend the results to a Hamiltonian cycle. We attempted to answer this question by considering the Johnson graph whose vertices are those from one level and edges are formed between vertices of distance two. We then form a Hamiltonian cycle where adjacent vertices in the cycle are at distance two from one another in the middle levels subgraph. The chapter concludes with determining the automorphism group of the Johnson graph.

We then develop a type of greedy matching in which the vertices of any two consecutive levels are sorted, independently, in colexicographic order. Some specific matching results are given, as well as a second method of generating a matching, the reverse greedy algorithm. We switch our view to that of lexicographic orderings with the original algorithm and prove that, regardless of the method or ordering used, all three matchings produced are equal. The key result follows from this work: a complete

saturation of the middle two levels of an odd cube, again independent of the algorithm or ordering chosen.

The structure of the automorphism group is the basis for the next section on transitivity, whose starting point is the following question: given two m -sets of vertices whose pairwise distances in the first set are the same as those in the second set, respectively, is there an automorphism that maps each member of the first set to a designated member of the second set? While this answer is affirmative for $m = 1, 2, 3$, we produce a counterexample to show that the n -cube is not distance m -transitive for $m > 3$. We then define two new type of transitivity, intersection m -transitivity and Δ m -transitivity, and show, not only that these new parameters are equivalent, but also that Q_n has these properties. After finding the automorphism group of the middle two levels of an odd dimensional cube, we extend these transitivity definitions to this restricted level graph and draw similar results to those that hold for the whole cube.

We conclude our paper with the definition of a supergraph of the n -dimensional hypercube, Q_n^* . This new graph maintains the full structure of the underlying hypercube, with the addition of complementary edges: edges that connect a vertex v to its complement, v^c . Many parameters for this supergraph are established, including the size of its diameter, maximum connectivity and colorability. We also note how the same conclusions for the three types of transitivity discussed previously still hold for this supergraph.

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Chapter 1

Introduction

1.1 Preliminaries

Let Q_n denote the n -dimensional hypercube. That is, Q_n is the graph whose vertex set $V(Q_n)$ consists of the 2^n binary n -strings with edge set

$$E(Q_n) = \{\langle u, v \rangle \mid u, v \text{ differ in 1 position}\}$$

This position in which u and v differ is called the edge direction of $e = \langle u, v \rangle$.

We may construct Q_n inductively: take two copies of Q_{n-1} , augment the binary strings of one copy with zeros, the other copy with ones. The edge set is the union of the edge sets of the two copies of Q_{n-1} and any new pairs of vertices, one from each copy of Q_{n-1} , which differ only in the n^{th} position. From this, we can see that Q_n can not only be viewed as $Q_{n-1} \times Q_1$ but also $Q_m \times Q_{n-m}$, $m \leq n$.

The Hamming distance between two vertices u and v , $d(u, v)$, is defined as the number of bits in which these vertices differ. The weight of v , $\text{wt}(v)$, is the number of ones in its binary representation. If the weight of a vertex is even, it is said to have even parity, and if the weight is odd, its parity is odd.

Using this, we may separate the vertex set of Q_n into two sets, those with even parity and those with odd. Since every edge of Q_n is adjacent to one vertex in this first set and one in the second, Q_n is bipartite.

Consider the natural 1-1 correspondence between binary strings of length n and subsets of $[n] = \{1, 2, \dots, n\}$ in which there exists a 1 in the i^{th} position in the binary string if and only if i is in the corresponding set. Then we can view Q_n as a lattice of subsets of $[n]$, where each vertex is a subset of $[n]$ and each edge e is a pair $\langle x, y \rangle = \langle x, x \Delta \{i\} \rangle$, $i \in [n]$. Note that this subset form elicits the following clarifications: $\text{wt}(x) = |x|$ and $d(x, y) = |x \Delta y| = \text{wt}(x \Delta y)$. Also, when we denote an element x of the vertex set as x_1, x_2, \dots, x_i , it is assumed that $x_1 < x_2 < \dots < x_i$.

Any notation not explicitly defined can be found in [12].

1.2 Structure of $\text{Aut}(Q_n)$

We briefly review the structure of the automorphism group of Q_n , $\text{Aut}(Q_n)$ (also known as the octahedral group). Details can be found in [4] and [9].

For $x \in Q_n$, define $\sigma_A(x) = x \Delta A$ and $\Sigma_n = \{\sigma_A | A \in \mathcal{P}([n])\}$. Then $\Sigma_n \simeq \mathbb{Z}_2^n$ and we can view this as a subgroup of the automorphism group of Q_n , $\text{Aut}(Q_n)$. These automorphisms are commonly referred to as translations. Let \mathcal{S}_n denote the group of permutations of $[n]$. For $\theta \in \mathcal{S}_n$, define $\rho_\theta(x) = \rho_\theta(x_1 x_2 \dots x_m) = x_{\theta(1)} x_{\theta(2)} \dots x_{\theta(m)}$. ρ_θ , often called a rotation of the n -cube, is an element of $\text{Aut}(Q_n)$, and so $\{\rho_\theta | \theta \in \mathcal{S}_n\}$ is a subgroup of $\text{Aut}(Q_n)$. It is easily seen that $\rho_\theta \sigma_A = \sigma_{\theta(A)} \rho_\theta$. Any automorphism of Q_n can be expressed as $\sigma_A \rho_\theta$ and $\text{Aut}(Q_n) = \langle \sigma_A, \rho_\theta | A \subset [n], \theta \in \mathcal{S}_n \rangle$.

Chapter 2

Perfect Initial Colex Matchings

Throughout this chapter we assume $n = 2k + 1$ and all matchings are perfect.

2.1 Introduction

Definition 2.1.1. For two sets A and B , $|B| = 1$, define $A+B = \{a \cup b \mid a \in A, b \in B\}$.

Definition 2.1.2. Let $Q_n(X)$ denote the induced subgraph of Q_n with vertex set X .

So, for $0 \leq i \leq n$, we denote by $Q_n(L_i)$ (or just L_i when there is no danger of confusion) the set of vertices of cardinality (or “weight”) i , and we refer to L_i as the i^{th} level of Q_n .

Definition 2.1.3. For two elements $A = \{a_1, a_2, \dots, a_i\}$ and $B = \{b_1, b_2, \dots, b_i\}$ in L_i , $1 \leq i \leq n - 1$, each sorted in increasing order, A is less than B in colex order, if $a_r < b_r$ for some r , $1 \leq r \leq i$, and $a_s = b_s$ for all $s > r$.

Remark 2.1.4. If $Q_n(L_i)$ is written in colex order, then $Q_n(L_i)$ can, for any j , $i \leq j \leq n - 1$, be expressed as $Q_{j-1}(L_{j-1}) + \{j\}$ followed by $Q_j(L_{j-1}) + \{j + 1\}$, then $Q_{j+1}(L_{j-1}) + \{j + 2\}$, and so on, through to $Q_{n-1}(L_{j-1}) + \{n\}$, where each segment is itself arranged in colex order.

Recall the famous Marriage Lemma of P. Hall [7]:

Lemma 2.1.5. *For $k > 0$, every k -regular bipartite graph has a perfect matching.*

2.2 Maximum Matching Between Initial Segments of L_k and L_{k-1}

Lemma 2.2.1. *The maximum size of a matching between an initial segment of $Q_n(L_{k-1})$ in colex order and an initial segment of $Q_n(L_k)$ in colex order is $\sum_{j=1}^k \binom{n-2j}{k-j}$.*

Proof. Let $k = 1$. Then $n = 3$ and we must consider $L_0 = \{\emptyset\}$ and $L_1 = \{1, 2, 3\}$. Clearly, a maximum matching is achieved by selecting \emptyset from L_0 and $\{1\}$ from L_1 . Thus, the maximum size of a matching between L_0 and L_1 in Q_3 is $1 = \binom{1}{0} = \binom{3-2}{1-1}$.

Now let us assume that the proposition hold for $k - 1$. That is, the maximum size of a matching between an initial segment of $Q_{n-2}(L_{k-2})$ in colex order and an initial segment of $Q_{n-2}(L_{k-1})$ in colex order is $\sum_{j=1}^{k-1} \binom{(n-2)-2j}{(k-1)-j}$.

Consider L_k and L_{k-1} in Q_n and assume below that each segment of these 2 levels is sorted in colex order. We can write each of these two levels as in Remark 2.1.4. For L_k , the first $n - k - 1$ segments, $Q_{k-1}(L_{k-1}) + \{k\}, \dots, Q_{n-3}(L_{k-1}) + \{k\}$, are, by the remark, just $Q_{n-2}(L_k)$ in colex order. So, L_k may be viewed more succinctly as $Q_{n-2}(L_k)$ followed by $Q_{n-2}(L_{k-1}) + \{n - 1\}$ and, finally, $Q_{n-1}(L_{k-1}) + \{n\}$. Similarly, we may express L_{k-1} in colex order by combining the first $n - k$ segments in L_{k-1} in order to write this as $Q_{n-2}(L_{k-1})$ followed by $Q_{n-2}(L_{k-2}) + \{n - 1\}$ and, finally, $Q_{n-1}(L_{k-2}) + \{n\}$.

Therefore, L_k contains $\binom{n-2}{k-1}$ members of $[n - 1]$ that contain $\{n - 1\}$, while in L_{k-1} there are $\binom{n-2}{k-2}$ such members. Since the size of $Q_{n-2}(L_{k-1})$ is $\binom{n-2}{k-1}$ and $\binom{n-2}{k-2} < \binom{n-2}{k-1}$, we must have that the maximum possible size of the initial segment in

L_{k-1} that yields a matching is less than or equal to $\binom{n-2}{k-1} + \binom{n-2}{k-2}$. (For, if not, the next member of L_{k-1} would contain n and none of the members of an initial segment of L_k of the same length would contain this element. Thus, it would be unmatched.)

In order to form a matching, we must have that every member of $Q_n(L_{k-1})$ that contains $\{n-1\}$ must be matched to a member of $Q_n(L_k)$ that also contains $\{n-1\}$. That is, in the matching, members from $Q_{n-2}(L_{k-2}) + \{n-1\}$ in L_{k-1} must be paired with members of $Q_{n-2}(L_{k-1}) + \{n-1\}$ in L_k . Since $Q_{n-2}(L_k)$ in L_k and $Q_{n-2}(L_{k-1})$ in L_{k-1} form a k -regular bipartite graph, by Lemma 2.1.5, we have that there exists a matching between these two initial segments. Thus, we can just restrict our view to finding a maximum matching between $Q_{n-2}(L_{k-2}) + \{n-1\}$ and $Q_{n-2}(L_{k-1}) + \{n-1\}$.

But this restriction is equivalent to determining the maximum size of a matching between an initial segment of $Q_{n-2}(L_{k-2})$ in colex order and an initial segment of $Q_{n-2}(L_{k-1})$ in colex order. By the induction hypothesis, this value is $\sum_{j=1}^{k-1} \binom{(n-2)-2j}{(k-1)-j}$.

Thus, the maximum size of a matching between an initial segment of $Q_n(L_{k-1})$ in colex order and an initial segment of $Q_n(L_k)$ in colex order is

$$\begin{aligned} \binom{n-2}{k-1} + \sum_{j=1}^{k-1} \binom{(n-2)-2j}{(k-1)-j} &= \binom{n-2}{k-1} + \sum_{j=2}^k \binom{n-2j}{k-j} \\ &= \sum_{j=1}^k \binom{n-2j}{k-j} \end{aligned}$$

□

Corollary 2.2.2. *The maximum size of a matching between an initial segment of $Q_{n-1}(L_{k-1})$ in colex order and an initial segment of $Q_{n-1}(L_k)$ in colex order is $\sum_{j=1}^k \binom{n-2j}{k-j}$.*

Proof. By the proof of Lemma 2.2.1, we have that the maximum size of the matching

of initial segments of colex order of L_{k-1} and L_k in Q_n is $\sum_{j=1}^k \binom{n-2j}{k-j} < \binom{n-2}{k-1} + \binom{n-2}{k-2} = \binom{n-1}{k-1}$. But $\binom{n-1}{k-1}$ is exactly the size of the smaller of the two levels to be matched in Q_{n-1} , namely L_{k-1} . Thus, by the properties of colex order, the same bound still holds in Q_{n-1} . \square

2.3 Maximum Matching Between Initial Segments of L_{i+1} and L_i

In this section we generalize the results of Section 2.2.

Lemma 2.3.1. *Let $0 \leq i \leq k-1$. Then, the maximum size of a matching between an initial segment of $Q_n(L_i)$ in colex order and an initial segment of $Q_n(L_{i+1})$ in colex order is $\sum_{j=0}^i \binom{2i-2j+1}{i-j}$.*

Proof. Consider L_i and L_{i+1} in Q_n and assume, herein, that each segment of these 2 levels is sorted in colex order. We can write each of these two levels as in Remark 2.1.4. For L_i , the first $i+1$ segments, $Q_{i-1}(L_{i-1}) + \{i\}, \dots, Q_{2i}(L_{i-1}) + \{2i+1\}$, are just $Q_{2i+1}(L_i)$. So, L_i may be viewed more compactly as $Q_{2i+1}(L_i)$ followed by $Q_{2i+1}(L_{i-1}) + \{2i+2\}$, then $Q_{2i+2}(L_{i-1}) + \{2i+3\}$ and so on until $Q_{n-1}(L_{i-1}) + \{n\}$. Similarly, we may combine the first i segments in L_{i+1} in order to write this level as $Q_{2i+1}(L_{i+1})$ followed by $Q_{2i+1}(L_i) + \{2i+2\}$ and so on, until $Q_{n-1}(L_i) + \{n\}$.

Following the proof of Lemma 2.2.1, we can see that the maximum size of an initial segment from L_i is $\binom{2i+1}{i} + \binom{2i+1}{i-1}$, and that, to achieve a maximum matching, every member of $Q_{2i+1}(L_{i-1}) + \{2i+2\}$ from L_i must be paired with a member of $Q_{2i+1}(L_i) + \{2i+2\}$ from L_{i+1} .

Again, since $Q_{2i+1}(L_i)$ in L_i and $Q_{2i+1}(L_{i+1})$ in L_{i+1} form a $(i+1)$ -regular bipartite graph, by Lemma 2.1.5, we have that there exists a matching between these two

initial segments. Thus, we can just restrict our view to finding a maximum matching between $Q_{2i+1}(L_{i-1}) + \{2i+2\}$ and $Q_{2i+1}(L_i) + \{2i+2\}$. That is, we can just restrict our view to finding a maximum matching between $Q_{2i+1}(L_{i-1})$ and $Q_{2i+1}(L_i)$. By Lemma 2.2.1 with $k = i$ and $n = 2i + 1$, we have that this value is $\sum_{j=1}^i \binom{(2i+1)-2j}{i-j}$. Thus, the maximum size of a matching between an initial segment of $Q_n(L_i)$ in colex order and an initial segment of $Q_n(L_{i+1})$ in colex order is

$$\binom{2i+1}{i} + \sum_{j=1}^i \binom{(2i+1)-2j}{i-j} = \sum_{j=0}^i \binom{2i-2j+1}{i-j}$$

□

Corollary 2.3.2. *Let $0 \leq i \leq k-1$. Then, the maximum size of a matching between an initial segment of $Q_{n-1}(L_i)$ in colex order and an initial segment of $Q_{n-1}(L_{i+1})$ in colex order is $\sum_{j=0}^i \binom{2i-2j+1}{i-j}$.*

Proof. By the proof of Lemma 2.3.1, we have that the maximum size of the matching of initial segments of colex order of L_i and L_{i+1} in Q_n is $\sum_{j=1}^i \binom{2i-2j+1}{i-j}$.

Now,

$$\begin{aligned} \sum_{j=1}^i \binom{2i-2j+1}{i-j} &< \binom{2i+1}{i} + \binom{2i+1}{i-1} \\ &= \binom{2i+2}{i} \\ &\leq \binom{2k}{i} \end{aligned}$$

But $\binom{2k}{i} = \binom{n-1}{i}$ is the size of L_i , the smaller of the two levels to be matched in Q_{n-1} . Therefore, by the properties of colex order, the same bound still holds in Q_{n-1} .

□

Chapter 3

A Hamiltonian Cycle in $J(n, i)$

3.1 Introduction

This chapter, as well as Chapter 4, is motivated by the following question:

Question 1. For $n = 2k + 1$, does there exist a Hamiltonian cycle in $Q_n(L_k, L_{k+1})$?

Since any Hamiltonian cycle of even length can be decomposed into two disjoint perfect matchings, determining different methods of matching L_k and L_{k+1} , and other two-sets of levels, could aid in the solution of this problem. Initially, our interest in such algorithmic matchings was motivated by the work of Shields and Savage in [11], though [6] casts doubt that our approach will not solve this long-standing problem.

Now $Q_n(L_k, L_{k+1})$ is bipartite, so in order for the answer to be affirmative, we must be able to find a spanning cycle such that every other vertex lies in the same level. That is, a vertex would be distance 2 from the next, and the previous, vertex of the same weight in the cycle. Let us restrict our view to one such level and consider a graph with this distance property.

3.2 Colex Order and $J(n, i)$

The Johnson graph, $J(n, i)$, consists of those vertices of weight i from Q_n , with x adjacent to y if and only if $d(x, y) = 2$. Various results for the Johnson graph can be found in [2], [5], and [8].

Notation. For L_i in Q_n , $n \geq 3$, $2 \leq i \leq n - 1$, let $s_{n,i}$ be the smallest member of L_i in colex order that contains a 1, and let $t_{n,i}$ be the largest such member. That is, $s_{n,i} = 123 \dots i$ and $t_{n,i} = 1(n - i + 2)(n - i + 3) \dots n$.

The largest element of $s_{n,i}$ is i and the largest element of $t_{n,i}$ is n . Since $2 \leq n - 1$, $s_{n,i}$ and $t_{n,i}$ are distinct.

Proposition 3.2.1. For $j \geq 3$, $2 \leq i \leq j - 1$, $d(s_{j,i}; s_{j+1,i}) = 0$ and $d(t_{j,i}; t_{j+1,i}) = 2$.

Proof. Since $s_{j,i} = 12 \dots i = s_{j+1,i}$, we have the distance between these two sets is 0.

Now, $t_{j,i} = 1(j - i + 2)(j - i + 3) \dots (j)$ and

$$t_{j+1,i} = 1(j + 1 - i + 2)(j + 1 - i + 3) \dots (j)(j + 1) = 1(j - i + 3)(j - i + 4) \dots (j)(j + 1)$$

so $t_{j,i} \Delta t_{j+1,i} = (j - i + 2)(j + 1)$. Thus, $d(t_{j,i}, t_{j+1,i}) = 2$. \square

Now, $Q_n(L_i)$ is the vertex set of $J(n, i)$, so by Remark 2.1.4, if $Q_n(L_i)$ is written in colex order, we can decompose $Q_n(L_i)$ into smaller segments of colex order.

Corollary 3.2.2. Let $Q_n(L_i)$ be written in colex order. Then, under the decomposition given above, in $J(n, i)$, $s_{j,i-1} \cup \{j + 1\}$ is adjacent to $s_{j+1,i-1} \cup \{j + 2\}$ and $t_{j,i-1} \cup \{j + 1\}$ is adjacent to $t_{j+1,i-1} \cup \{j + 2\}$, for $i - 1 \leq j \leq n - 2$. That is, for any two adjoining segments in $Q_n(L_i)$, the first element containing a one in the first segment is adjacent to the first element containing a one in the second segment, and similarly for the last elements containing a 1.

Proof. By Proposition 3.2.1, we have that $s_{j,i} = s_{j+1,i}$ and so,

$$(s_{j,i-1} \cup \{j+1\})\Delta(s_{j+1,i-1} \cup \{j+2\}) = (j+1)(j+2)$$

Thus, the distance between these vertices is 2 and they are adjacent in $J(n, i)$.

Using the proof of Proposition 3.2.1, we have that $t_{j,i}\Delta t_{j+1,i} = (j-i+2)(j+1)$, which gives

$$(t_{j,i} \cup \{j+1\})\Delta(t_{j+1,i} \cup \{j+2\}) = (j-i+2)(j+2)$$

Therefore, $t_{j,i-1} \cup \{j+1\}$ and $t_{j+1,i-1} \cup \{j+2\}$ are adjacent in $J(n, i)$. \square

3.3 A Hamiltonian Cycle in $J(n, i)$

Notation. Let $\mathcal{P}_{n,i}$ denote a Hamiltonian path in $J(n, i)$ that starts at $s_{n,i}$ and ends at $t_{n,i}$.

Proposition 3.3.1. *For $n \geq 2$, $\mathcal{P}_{n,2}$ exists.*

Proof. For all $x, y \in L_1$ in Q_n , $x \neq y$, $d(x, y) = 2$. Therefore, since for some v_1 and v_2 in some segment $Q_j(L_i) + \{j+1\}$, $v_1 = x(j+1)$ and $v_2 = y(j+1)$, when $Q_n(L_2)$ is decomposed into segments of colex order, elements within each segment are adjacent in $J(n, i)$. Also, the first element in a segment $Q_j(L_i) + \{j+1\}$ is adjacent to the last element in the segment $Q_{j+1}(L_i) + \{j+2\}$. (The first element of $Q_j(L_i) + \{j+1\}$ is $1(j+1)$ and the last element of $Q_{j+1}(L_i) + \{j+2\}$ is $(j+1)(j+2)$.)

Form a Hamiltonian path from $s_{n,2} = 12$ to $t_{n,2} = 1n$ in the following manner:

From the decomposition of $Q_n(L_i)$, take each segment, starting from the first, $Q_{i-1}(L_1) + \{i\}$, and continuing to the last, $Q_{n-1}(L_1) + \{n\}$, and reverse the order

within the segment. (That is, for a segment w, x, y, z , write the segment as z, y, x, w .) Then, the first element of one segment is adjacent to the last element of the next segment in this ordering while adjacency is preserved within the segment. Also, the path begins at the first element 12 and ends at the first element of the last segment, $1n$. Therefore, we have constructed the necessary Hamiltonian path. \square

Lemma 3.3.2. *Let $n \geq 4$. For $3 \leq i \leq n - 1$, $\mathcal{P}_{n,i}$ exists.*

Proof. (Induction on n) Let $n = 4$. Then $Q_4(L_3) = \{123, 124, 134, 234\}$ and we have that $123, 124, 234, 134$ is a Hamiltonian path that starts at $s_{4,3} = 123$ and ends at $t_{4,3} = 134$.

Now, let us assume that the proposition holds for all values less n . That is, for all m , $4 \leq m \leq n - 1$ and for any i , $3 \leq i \leq m - 1$, $\mathcal{P}_{m,i}$ exists.

Consider $J(n, i)$, and write the members of $Q_n(L_i)$ in colex order. Then, as in the remark, we can decompose $Q_n(L_i)$ into the $n - i + 1$ segments of colex order $Q_{i-1}(L_{i-1}) + \{i\}$ followed by $Q_i(L_{i-1}) + \{i + 1\}$, then $Q_{i+1}(L_{i-1}) + \{i + 2\}$, and on to $Q_{n-1}(L_{i-1}) + \{n\}$. By the induction hypothesis and Proposition 3.3.1, each of these segments, $Q_j(L_{i-1})$, $2 \leq j \leq n - 1$, contains a Hamiltonian path that starts at $s_{j,i-1}$ and ends at $t_{j,i-1}$.

Form a Hamiltonian path in $J(n, i)$ using these orderings, noting that the last segment must be listed in reverse, in order to end with $t_{n,i}$. That is, in the odd case ($n - i + 1$ odd), the second segment must be in the forward direction and then we alternate directions with the remaining segments. In the even case, the second segment will be in the reverse direction, with the remaining segments alternating directions as we add them.

Specifically, if $n - i + 1$ is odd, start with the first segment $Q_{i-1}(L_{i-1}) + \{i\}$, followed by $Q_i(L_{i-1}) + \{i + 1\}$ in reverse, then $Q_{i+1}(L_{i-1}) + \{i + 2\}$, and $Q_{i+2}(L_{i-1}) + \{i + 3\}$

in reverse, and so on until $Q_{n-1}(L_{i-1}) + \{n\}$.

Similarly, if $n - i + 1$ is even, start with the first segment $Q_{i-1}(L_{i-1}) + \{i\}$ in reverse, followed by $Q_i(L_{i-1}) + \{i + 1\}$, then $Q_{i+1}(L_{i-1}) + \{i + 2\}$ in reverse, and so on, alternating the direction of each segment until $Q_{n-1}(L_{i-1}) + \{n\}$.

By Corollary 3.2.2, we have that both of these form a path. Since $Q_{i-1}(L_{i-1}) + \{i\} = 123 \dots (i - 1)$ and $Q_{n-1}(L_{i-1}) + \{n\}$ is traversed in order, each path starts at $s_{n,i}$ and ends at $t_{n,i}$. \square

Corollary 3.3.3. *For $n \geq 1$ and for all i , $0 \leq i \leq n$, there is a Hamiltonian path in $J(n, i)$.*

Proof. By Proposition 3.3.1 and Lemma 3.3.2, we have a Hamiltonian path exists for any i , $2 \leq i \leq n - 1$. Since $|Q_n(L_0)| = |Q_n(L_n)| = 1$, then this vertex itself forms a Hamiltonian path.

For $i = 1$, for all $x, y \in Q_n(L_1)$, $x \neq y$, $d(x, y) = 2$ and so $\langle x, y \rangle \in E(J(n, 1))$. Therefore, any ordering of these vertices yields a Hamiltonian path in $J(n, 1)$. \square

Theorem 3.3.4. *Let $n \geq 1$. For all i , $0 \leq i \leq n$, there exists a Hamiltonian cycle in $J(n, i)$.*

Proof. For $i = 0$ and $i = n$, this is clear. Since the distance between any two members of $Q_n(L_1)$ is 2, any ordering of these elements yields a Hamiltonian cycle. By Proposition 3.3.1, we have a Hamiltonian path between 12 and $1n$ in $Q_n(L_2)$. But $d(12, 1n) = 2$, so, we, in fact, have a Hamiltonian cycle in $J(n, 2)$.

Decomposing $Q_n(L_i)$ into $n - i + 1$ segments of colex order and using Lemma 3.3.2, we can form a Hamiltonian cycle in the following way:

If $n - i + 1$ is odd, start with the segment $Q_{i-1}(L_{i-1}) + \{i\}$ in reverse, followed by $Q_i(L_{i-1}) + \{i + 1\}$, then $Q_{i+1}(L_{i-1}) + \{i + 2\}$ in reverse, and $Q_{i+2}(L_{i-1}) + \{i + 3\}$, and

so on, alternating the direction of each segment until $Q_{n-1}(L_{i-1}) + \{n\}$, traversed in reverse.

If $n - i + 1$ is even, start with the first segment $Q_{i-1}(L_{i-1}) + \{i\}$, followed by $Q_i(L_{i-1}) + \{i + 1\}$ in reverse, then $Q_{i+1}(L_{i-1}) + \{i + 2\}$, and so on, again alternating the direction of each segment until $Q_{n-1}(L_{i-1}) + \{n\}$, in reverse.

By Corollary 3.2.2, both of these constructions form a path. Also, $Q_{n-1}(L_{i-1}) + \{n\}$ is reversed, so the final member of this path is $s_{n-1,i-1} \cup \{n\} = 123 \dots (i-1)(n)$, which is adjacent to $123 \dots (i-1)(i) = s_{i-1,i-1} \cup \{i\}$, the initial element in this path.

Therefore, we have constructed a Hamiltonian cycle in $J(n, i)$ for all i . \square

3.4 $\text{Aut}(J(n, i))$

Notation. Denote x adjacent to y by $x \leftrightarrow y$.

Theorem 3.4.1. *For $n \neq 2i$, the automorphism group of $J(n, i)$ is*

$$\text{Aut}(J(n, i)) = \{\rho_\theta \mid \theta \in \mathcal{S}_n\}$$

Proof. Clearly, for $i = 1$, $\text{Aut}(J(n, 1)) = \{\rho_\theta \mid \theta \in \mathcal{S}_n\}$, since $J(n, 1) \cong K_n$.

For $i \geq 2$, let $x \in Q_n(L_i)$ and let $\mathcal{L}_j = \{y \mid d_J(x, y) = j\}$. Since the group $\{\rho_\theta \mid \theta \in \mathcal{S}_n\}$ acts transitively on $J(n, i)$, we may assume $x = [i]$. For $1 \leq p \leq i$, $i + 1 \leq q \leq n$, define $y_{p,q} \in Q_n(L_i)$ by $y_{p,q} = x \setminus \{p\} \cup \{q\}$, $Y_p(x) = Y_p = \{y_{p,q} \mid i + 1 \leq q \leq n\}$ and $Z_q(x) = Z_q = \{y_{p,q} \mid 1 \leq p \leq i\}$. Let $\overline{Y}_p = Y_p \cup \{x\}$ and $\overline{Z}_q = Z_q \cup \{x\}$. We need the following lemma.

Lemma 3.4.2. *The maximal cliques in $J(n, i)$ that contain x are $\{\overline{Y}_p \mid 1 \leq p \leq i\}$, and $\{\overline{Z}_q \mid i + 1 \leq q \leq n\}$, $|\overline{Y}_p| = n - i + 1$ and $|\overline{Z}_q| = i + 1$.*

Proof. Using the definitions give above, we have

$$\bigcup_{p=1}^i Y_p = \bigcup_{\substack{1 \leq p \leq i \\ i+1 \leq q \leq n}} y_{p,q} = \mathcal{L}_1 = N(x)$$

and

$$\bigcup_{q=i+1}^n Z_q = \bigcup_{\substack{i+1 \leq q \leq n \\ 1 \leq p \leq i}} y_{p,q} = \mathcal{L}_1 = N(x)$$

Consider 2 distinct vertices, y_{p_1, q_1} and y_{p_2, q_2} . Now, $y_{p_1, q_1} \Delta y_{p_2, q_2} = \{\{p_1, q_1\} \Delta \{p_2, q_2\}\}$. If $p_1 \neq p_2$ and $q_1 \neq q_2$, then $|y_{p_1, q_1} \Delta y_{p_2, q_2}| = 4$ and these two vertices are not adjacent in $J(n, i)$. If either $p_1 = p_2$ or $q_1 = q_2$ (but not both, since the 2 members are distinct), then $|y_{p_1, q_1} \Delta y_{p_2, q_2}| = 2$ and we have y_{p_1, q_1} is adjacent to y_{p_2, q_2} .

Form the matrix $M = [y_{p,q}]$, $1 \leq p \leq i$, $i+1 \leq q \leq n$. Then the entries of M are exactly the neighbors of x and any two of them are adjacent if and only if they lie in the same row or the same column. (Note that each row is a Y_p , $1 \leq p \leq i$, and each column is a Z_q , $i+1 \leq q \leq n$). Thus, any clique containing x whose size is at least 2 is either a subset of $\overline{Y_p}$, $1 \leq p \leq i$, or a subset of $\overline{Z_q}$, $i+1 \leq q \leq n$. Hence, $\{\overline{Y_p} | 1 \leq p \leq i\}$ is the set of maximal cliques of size $n - i + 1$ that contain x , while $\{\overline{Z_q} | i+1 \leq q \leq n\}$ is the set of maximal cliques of size $i + 1$ that contain x . \square

Let $f \in \text{Stab}_J(x)$. Since f preserves adjacency, f acts as a permutation on $\{\overline{Y_p} | 1 \leq p \leq i\}$ as well as on $\{\overline{Z_q} | i+1 \leq q \leq n\}$. So, for each p , $1 \leq p \leq i$, $f(\overline{Y_p}) = \overline{Y_{\theta(p)}}$, for a unique $\theta(p)$, $1 \leq \theta(p) \leq i$. Similarly, for each q , $i+1 \leq q \leq n$, $f(\overline{Z_q}) = \overline{Z_{\theta(q)}}$ for a unique $\theta(q)$, $i+1 \leq \theta(q) \leq n$. Thus, $\theta \in \mathcal{S}_n$ and θ is defined on all elements of $[n]$.

By construction, θ maps $x = [i]$ to itself, so $\rho_\theta(x) = x$ and ρ_θ agrees with f on $\mathcal{L}_0 = \{x\}$. Now, for any $p \in [i]$ and $q \in \{i+1, i+2, \dots, n\}$, $Y_p \cap Z_q = \{y_{p,q}\}$, so

$$\{f(y_{p,q})\} = f(Y_p) \cap f(Z_q) = Y_{\theta(p)} \cap Z_{\theta(q)} = \{y_{\theta(p),\theta(q)}\}$$

and, thus,

$$\begin{aligned} f(y_{p,q}) &= y_{\theta(p),\theta(q)} = x \setminus \{\theta(p)\} \cup \{\theta(q)\} = \rho_\theta(x) \setminus \{\theta(p)\} \cup \{\theta(q)\} \\ &= \rho_\theta(x \setminus \{p\} \cup \{q\}) = \rho_\theta(y_{p,q}) \end{aligned}$$

Therefore, ρ_θ agrees with f on \mathcal{L}_1 as well as \mathcal{L}_0 .

Define $f_1 = \rho_\theta^{-1}f$. Then f_1 acts as the identity on \mathcal{L}_0 and \mathcal{L}_1 . We need the following lemma.

Lemma 3.4.3. *Let $j \geq 1$. For $z \in \mathcal{L}_{j+1}$, if $v \in \mathcal{L}_{j+1}$, and $v \in \bigcap_{\substack{w \in \mathcal{L}_j \\ w \leftrightarrow z}} N(w)$, then $v = z$.*

Proof. By definition, $z \in \bigcap_{\substack{w \in \mathcal{L}_j \\ w \leftrightarrow z}} N(w)$. Let $v \in \bigcap_{\substack{w \in \mathcal{L}_j \\ w \leftrightarrow z}} N(w)$, $v \neq z$. Suppose $l \notin v$, for some $l \in z$. Let $w_{q,p_1} \in \mathcal{L}_j$, $w_{q,p_1} = z \setminus \{q\} \cup \{p_1\}$, $p_1 \in x \setminus z$, $q \in z \setminus x$, $q \neq l$. Since z is adjacent to w_{q,p_1} , v is also adjacent to w_{q,p_1} and, so, $|v \cap w_{q,p_1}| = i - 1$. If $p_1 \notin v$, then $v \cap (z \setminus \{q\} \cup \{p_1\}) = z \setminus \{q\}$ and $z \setminus \{q\} \subset v$. But $q \neq l$, which implies $l \in v$. Contradiction. Thus, $p_1 \in v$ and we have $|v \cap z \setminus \{q\}| = i - 2$. Now, $|v| = i$ and $|z \setminus \{q\}| = i - 1$ and $l \in z \setminus \{q\}$ while $l \notin v$. Therefore, $z \setminus \{l, q\} \subset v$ and, moreover, $z \setminus \{l, q\} \cup \{p_1\} \subset v$.

Now, consider another neighbor of z (and, also, v), from \mathcal{L}_j : $w_{q,p_2} = z \setminus \{q\} \cup \{p_2\}$, $p_2 \in x \setminus z$, $p_2 \neq p_1$. Again, $|v \cap w_{q,p_2}| = i - 1$ and $z \setminus \{l, q\} \cup \{p_2\} \subset v$. Therefore, $v = z \setminus \{l, q\} \cup \{p_1, p_2\}$.

Since $v, z \in \mathcal{L}_{j+1}$, both v and z each contain exactly $j + 1$ elements from $[n] \setminus x$. Thus, $|x \cap v| = |x \cap z| = i - (j + 1) = i - j - 1$. Recall that $q \in z \setminus x$ and $p_1, p_2 \in x \setminus z$. If $l \notin x$, then $|x \cap (z \setminus \{l, q\})| = i - j - 1$ and

$$|x \cap v| = |x \cap (z \setminus \{l, q\} \cup \{p_1, p_2\})| = (i - j - 1) + 2 = i - j + 1 = i - (j - 1)$$

giving $v \in \mathcal{L}_{j-1}$. Contradiction. Similarly, if $l \in x$, $|x \cap (z \setminus \{l, q\})| = (i - j - 1) - 1 = i - j - 2$ and we have

$$|x \cap v| = |x \cap (z \setminus \{l, q\} \cup \{p_1, p_2\})| = (i - j - 2) + 2 = i - j$$

again contradicting $v \in \mathcal{L}_{j+1}$.

Therefore, $l \in v$ for all $l \in z$. Thus, $v = z$ and we have obtained the desired result. \square

Proposition 3.4.4. *If f_1 is the identity on \mathcal{L}_0 and \mathcal{L}_1 , then f_1 acts as the identity on all \mathcal{L}_j , $0 \leq j \leq i$.*

Proof. Proceed by induction on j , $j \geq 1$. For $j = 1$, this is clear by the statement of the proposition.

Assume the proposition holds true for all $l \leq j$. So, f_1 is the identity on $\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_j$. Let us show that f_1 acts as the identity on \mathcal{L}_{j+1} . Let $z \in \mathcal{L}_{j+1}$ and consider those $w \in \mathcal{L}_j$ with $z \in N(w)$. Now, $z \in \bigcap_{z \leftrightarrow w} N(w)$ and, since $f_1(z) \in \mathcal{L}_{j+1}$, $f_1(z) \in \bigcap_{z \leftrightarrow w} N(f_1(w))$. But f_1 acts as the identity on \mathcal{L}_j , so $f_1(z) \in \bigcap_{z \leftrightarrow w} N(w)$. By Lemma 3.4.3, we have $z = f_1(z)$ and f_1 acts as the identity on \mathcal{L}_{j+1} .

Therefore, f_1 is the identity on all \mathcal{L}_j , $0 \leq j \leq i$. \square

Returning to the proof of Theorem 3.4.1, by Proposition 3.4.4, we see that

$$id = f_1 = \rho_\theta^{-1} f$$

and so, for $f \in \text{Stab}_J(x)$, $f = \rho_\theta$ where $|\text{Stab}_J(x)| = i!(n-i)!$. Since $\{\rho_\theta \mid \theta \in \mathcal{S}_n\} \subset \text{Aut}(J(n, i))$ acts transitively on the vertex set of $J(n, i)$,

$$\begin{aligned} |\text{Aut}(J(n, i))| &= |Q_n(L_i)| |\text{Stab}_J(x)| \\ &= \binom{n}{i} i!(n-i)! \\ &= n! \\ &= |\{\rho_\theta \mid \theta \in \mathcal{S}_n\}| \end{aligned}$$

Thus, $\text{Aut}(J(n, i)) = \{\rho_\theta \mid \theta \in \mathcal{S}_n\}$. □

Chapter 4

Greedy Algorithms

4.1 Introduction

For any n , the vertices in L_i have exactly $n - i$ neighbors in L_{i+1} , and the vertices in L_{i+1} have exactly $i + 1$ neighbors in L_i . Thus the (bipartite) subgraph of Q_n induced by $L_i \cup L_{i+1}$, $Q_n(L_i, L_{i+1})$, is what, in [10], is called semiregular and has a matching which saturates L_i for $i < n/2$. This, like Hall's Marriage Lemma [7] for bipartite regular graphs, follows from a simple counting argument to show that Hall's necessary and sufficient condition for the existence of a complete matching is satisfied.

In the next section we describe and prove the efficacy of a simple greedy algorithm for constructing one such complete matching between two consecutive levels.

After proving this algorithm, we discovered a similar result in [1]. In this paper, Aigner stated his result in terms of lex order, but we show in Corollary 4.4.4 that these results are equivalent to those using colex order when matching.

4.2 The Greedy Matching Algorithm

Algorithm 4.2.1 (Greedy Algorithm (GA)).

List the members of L_i in colex order, and do the same for the members of L_{i+1} .

Step 1. Match $12 \cdots i$ with $12 \cdots i(i+1)$.

Step 2. While there is an unmatched i -set:

DO: choose the first such in colex order, and match it to the first neighbor in L_{i+1} (in colex order) which is as yet unmatched.

Notation. Let $M_{i,n}$ be the matching produced by the GA between $Q_n(L_i)$ and $Q_n(L_{i+1})$.

Example 4.2.2. In $Q_5(L_2, L_3)$, the colex orderings of L_2 and L_3 are:

$$L_2 = 12, 13, 23, 14, 24, 34, 15, 25, 35, 45$$

$$L_3 = 123, 124, 134, 234, 125, 135, 235, 145, 245, 345$$

The matching produced by the greedy algorithm, in the order in which it is generated, is:

$$\begin{aligned} M_{2,5} = & \{ \langle 12, 123 \rangle, \langle 13, 134 \rangle, \langle 23, 234 \rangle, \langle 14, 124 \rangle, \langle 24, 245 \rangle, \langle 34, 345 \rangle, \\ & \langle 15, 125 \rangle, \langle 25, 235 \rangle, \langle 35, 135 \rangle, \langle 45, 145 \rangle \} \end{aligned}$$

The following proposition and lemmas are immediate consequences of the definition of the GA.

Proposition 4.2.3. For any $n \geq 3$, the GA produces the following matchings: $M_{0,n} = \{ \langle \emptyset, \{1\} \rangle \}$ and $M_{1,n} = \{ \langle 1, 12 \rangle, \langle 2, 23 \rangle \} \cup \{ \langle j, 1j \rangle \mid 3 \leq j \leq n \}$.

Lemma 4.2.4. Let $n \geq 3$. Suppose that $x \in Q_n(L_i)$ and suppose that y is the least element of $Q_n(L_{i+1})$ (in colex order) with the following properties:

(1) $x \subset y$

(2) there is no $z \in Q_n(L_i)$ with $z < x$ such that the GA matches z to y .

Then the GA matches x to y .

Lemma 4.2.5. *Let $n \geq 3$. Suppose that $y \in Q_n(L_{i+1})$ and suppose that x is the least element of $Q_n(L_i)$ such that:*

(1) $x \subset y$

(2) there is no $w \in Q_n(L_{i+1})$ with $w < y$ such that the GA matches x to w .

Then the GA matches x to y .

Lemma 4.2.6. *For $n \geq 3$ and for all $0 \leq i \leq n$, the GA produces a matching $\widetilde{M}_{i,n}$ between $Q_n(L_i)$ and $Q_{n+1}(L_{i+1})$ which saturates $Q_n(L_i)$.*

Proof. Let $n = 3$. $\widetilde{M}_{0,3} = \{\langle \emptyset, \{1\} \rangle\}$, $\widetilde{M}_{1,3} = \{\langle 1, 12 \rangle, \langle 2, 23 \rangle, \langle 3, 13 \rangle\}$.
 $\widetilde{M}_{2,3} = \{\langle 12, 123 \rangle, \langle 13, 134 \rangle, \langle 23, 234 \rangle\}$. $\widetilde{M}_{3,3} = \{\langle 123, 1234 \rangle\}$.

Now let $n \geq 4$ and let $x = x_1x_2 \cdots x_i$. First suppose that $x_i < n$. Let $y = x_1x_2 \cdots x_i(x_i + 1) \in Q_n(L_{i+1})$. Since $x_i + 1 > x_j$ for all $j \leq i$, $x_i + 1 \notin x$. Hence, if $w \in Q_n(L_i)$ and $w \leq x$, then $x_i + 1 \notin w$. So, if $w \subset y$, $w = x_1x_2 \cdots x_i = x$ and no predecessor of x can be matched to y . Thus, there exist elements $y^* \in Q_n(L_{i+1})$, $y^* \supset x$, satisfying properties (1) and (2) of Lemma 4.2.4. Hence the least such element of $Q_n(L_{i+1})$ is matched by the GA to x .

Now suppose $x_i = n$. With x as above, let $y = x_1x_2 \cdots x_{i-1}n(n+1) \in Q_{n+1}(L_{i+1})$. By the same argument as before, no predecessor of x can be matched to y by the GA. Thus there exist (as yet) unmatched elements of $Q_{n+1}(L_{i+1})$, $\leq y$, which are supersets of x . Hence there is a least such, and this is the element of $Q_{n+1}(L_{i+1})$ to which the GA matches x . \square

The next lemma deals with the special case in which $n \in x$ and $n - 1 \notin x$.

Lemma 4.2.7. *Let $n \geq 4$, $1 \leq i < n/2$, and let $x^* \in Q_{n-2}(L_i)$ and $y^* \in Q_{n-2}(L_{i+1})$. Let $x = x^* \cup \{n\}$ and $y = y^* \cup \{n\}$. Then the GA for Q_{n-2} matches x^* to y^* if and only if the GA for Q_n matches x to y .*

Proof. Define $f : L_i \times L_{i+1} \rightarrow \mathbb{Z}_{\geq 2}$ by $f(x^*, y^*) = \text{ord}(x^*) + \text{ord}(y^*)$, where $\text{ord}(z^*)$ is the rank (position of) z^* in colex order. We argue by induction on $f(x^*, y^*)$. For the basis step we have $f(x^*, y^*) = 2$ (since $\text{ord}(x^*) = \text{ord}(y^*) = 1$) and so $x^* = 12 \cdots i$ and $y^* = 12 \cdots i(i+1) = x^* \cup \{i+1\}$. Since x^* and y^* are the initial elements of $Q_{n-2}(L_i)$ and $Q_{n-2}(L_{i+1})$, respectively, and $x^* \subset y^*$, the GA matches x^* to y^* . We must show that $x = x^* \cup \{n\} \in Q_n(L_{i+1})$ is matched to $y = y^* \cup \{n\} = x^* \cup \{i+1\} \cup \{n\} = x \cup \{i+1\}$. Clearly $y = x \cup \{i+1\}$ is the first element of $Q_n(L_{i+2})$ which is a superset of x . Next we must show that if $z \in Q_n(L_{i+1})$ and $z < x$ then the GA does *not* match z to y . If z were matched to y then $z = y \setminus \{q\}$ for some $q \in y$. If $q = n$, then $z = y^* = 12 \cdots (i+1)$. By the argument above, y^* is matched to $12 \cdots (i+1)(i+2) = w$. Since $i+2 < n$, $n \notin w$, and so $w \neq y$, and z is not matched to y . Now if $q \neq n$, then $q \leq i+1$. If $q = i+1$ then $z = y \setminus \{i+1\} = \{1, 2, \dots, i+1, n\} \setminus \{i+1\} = \{1, 2, \dots, i\} \cup \{n\} = x$. Contradiction, since $z < x$. Therefore $q \leq i$. Then $z = (\{1, 2, \dots, i+1\} \setminus \{q\}) \cup \{n\}$. Hence $z > x = \{1, 2, \dots, i\} \cup \{n\}$. Contradiction. Hence by Lemma 4.2.4, x is matched to y .

Now suppose $f(x^*, y^*) > 2$, and the lemma is true for all pairs (z^*, w^*) with $f(z^*, w^*) < f(x^*, y^*)$. First suppose x^* is matched to y^* . We want to show that $x = x^* \cup \{n\}$ is matched to $y = y^* \cup \{n\}$. Suppose x is matched to some $w \in Q_n(L_{i+2})$, with $w < y$. Note that since $x \subset w$, $n \in w$. Hence $w = w^* \cup \{n\}$ for some $w^* \in Q_{n-1}(L_{i+1})$. Since $w < y$ and $n \in w \cap y$, we have $w^* < y^*$, so $w^* \in Q_{n-2}(L_{i+1})$.

Thus $f(x^*, w^*) < f(x^*, y^*)$. By the induction hypothesis $x = x^* \cup \{n\}$ is matched to $w = w^* \cup \{n\}$ if and only if x^* is matched to w^* . So if x is matched to w , then x^* is matched to $w^* < y^*$, contradicting the assumption that x^* is matched to y^* . Hence x is not matched to any $w < y$.

Next, suppose there is some $z \in Q_n(L_{i+1})$ such that $z < x$ and z is matched to y . Since $z \subset y$ and $|y| - |z| = 1$, if $n \notin z$ then $z = y \setminus \{n\} = y^* \in Q_{n-2}(L_{i+1})$. Thus $n - 1 \notin z$ and by Lemma 4.2.6, z is matched to some element of $Q_{n-1}(L_{i+2})$. But $y \notin Q_{n-1}(L_{i+2})$, so this contradicts the assumption that z is matched to y . Now suppose that $n \in z$. Then $z = z^* \cup \{n\}$ for some $z^* \in Q_{n-1}(L_i)$. Since $z < x = x^* \cup \{n\}$, $z^* < x^*$, so $z^* \in Q_{n-2}(L_i)$. So $f(z^*, y^*) < f(x^*, y^*)$. Thus, by our induction hypothesis, since z is matched to y , z^* is matched to y^* . But $z^* < x^*$ and by hypothesis x^* is matched to y^* . So we have a contradiction, and therefore no $z < x$ is matched to y . Hence, by Lemma 4.2.4, x is matched to y .

Now for the reverse implication, assume that x is matched to y . We must show that x^* is matched to y^* . First suppose x^* is matched to some $w^* < y^*$. Then by the implication just proven, $x^* \cup \{n\}$ is matched to $w^* \cup \{n\} < y^* \cup \{n\} = y$. This contradicts the assumption that x is matched to y . So x^* is not matched to any predecessor of y^* .

Next, suppose that there is some $z^* \in Q_{n-2}(L_i)$, $z^* < x^*$, such that z^* is matched to y^* . Again, by the implication proven previously, we have $z^* \cup \{n\}$ matched to $y^* \cup \{n\} = y$. But $z^* \cup \{n\} < x^* \cup \{n\} = x$, and so this contradicts the assumption that x is matched to y . Hence no predecessor of x^* is matched to y^* . Thus by Lemma 4.2.6, x^* is matched to y^* . This completes the induction step and the proof of the lemma. \square

Lemma 4.2.8. *Let $x^* = [i]$ and $y^* = [i + 1]$, so that x^* is matched to y^* . Then:*

- (1) if $n \geq i + 3$ then $x^* \cup \{n\}$ is matched to $y^* \cup \{n\}$.
(2) $x^* \cup \{i + 2\}$ is matched to $x^* \cup \{i + 2\} \cup \{i + 3\}$.

Proof. (1) First we show that no predecessor of $y^* \cup \{n\}$ can be matched to $x^* \cup \{n\}$. For if $x^* \cup \{n\}$ were matched to $w < y^* \cup \{n\}$, then $x^* \cup \{n\} \subset w$, so $w = w^* \cup \{n\}$. Hence $w^* < y^*$, which is impossible since y^* is the initial element of $L_{i+1}(Q_n)$.

Next we show that $y^* \cup \{n\} \in L_{i+2}$ is not matched to any predecessor z of $x^* \cup \{n\} \in L_{i+1}$. Suppose $z < x^* \cup \{n\}$ and z is matched to $y^* \cup \{n\}$. Assume $n \notin z$. Since $z \subset y^* \cup \{n\} = y$, and $|y| - |z| = 1$, if $n \notin z$ then $z = y^* = 12 \cdots (i + 1)$. But $y^* = [i + 1]$ is matched to $[i + 2]$ by Step 1 of Algorithm 4.2.1, and $[i + 2] \neq y^* \cup \{n\}$. Contradiction. If $n \in z$, then $z = z^* \cup \{n\}$ for some $z^* < x^*$. This is impossible, since x^* is the least element of L_i .

Thus by Lemma 4.2.4, $x^* \cup \{n\}$ is matched to $y^* \cup \{n\}$.

(2) Let $x = x^* \cup \{i + 2\}$. We want to prove that x is matched to $y = x \cup \{i + 3\}$. First suppose that x is matched to some $w < y$. Then $x \subset w$, so $w = \{1, 2, \dots, i, i + 2\} \cup \{j\}$, and since $w < y$, $j < i + 3$. Thus $j = i + 1$, so $w = \{1, 2, \dots, i, i + 1, i + 2\}$. But then $z = \{1, 2, \dots, i + 1\}$ is matched to w , and $z < x$. Thus, no predecessor of y is matched to x .

Next, suppose y is matched to some predecessor z of x , *i.e.* $z < x = x^* \cup \{i + 2\} = [i] \cup \{i + 2\}$, and z is matched to $y = [i] \cup \{i + 2, i + 3\}$. The only $z \in L_{i+1}$ such that $z < [i + 2] \setminus \{i + 1\}$ is $z = [i + 1]$. But by Algorithm 4.2.1 Step 1, $[i + 1]$ is matched to $[i + 2] < y = [i + 3] \setminus \{i + 1\}$. Contradiction.

Hence, by Lemma 4.2.4, x is matched to y . □

Lemma 4.2.9. *Let $n \geq 7$ and $3 \leq i < n/2$. Suppose that $x^* \in Q_{n-1}(L_{i-1})$ and $y^* \in Q_{n-1}(L_i)$. Then x^* is matched to y^* if and only if $x^* \cup \{n\}$ is matched to $y^* \cup \{n\}$.*

Proof. Proceed by induction on $f(x^*, y^*) = \text{ord}(x^*) + \text{ord}(y^*)$, where $\text{ord}(z)$ is defined in the proof of Lemma 4.2.7. The base case is easy: $f(x^*, y^*) = 2$ iff $x^* = [i - 1]$ and $y^* = [i]$. The (\Rightarrow) implication follows from the definition of the GA, and the (\Leftarrow) from Lemma 4.2.8 (1). (If $n \geq 2i + 1$ and $i \geq 3$ then $n - i \geq i + 1 \geq 4$, so $n - i \geq 4$ and hence $n - 1 \geq i + 3$.)

Now assume the lemma is true for all j^* such that $1 \leq j^* \leq j$, and suppose that $f(x^*, y^*) = j$. Suppose that x^* is matched to y^* . If $x = x^* \cup \{n\}$ is not matched to $y = y^* \cup \{n\}$ then either:

Case (1): there exists a $w < y$ such that x is matched to w , or

Case (2): there exists a $z < x$ such that z is matched to y .

For Case (1), suppose x is matched to w , and $w < y$. Then $n \in w$. Let $w^* = w \setminus \{n\}$ with $w^* \in Q_{n-1}(L_i)$. Since $w < y$, $w^* < y^*$. If we show that x^* is matched to w^* we will have a contradiction, since by hypothesis x^* is matched to y^* and $w^* < y^*$. So suppose that x^* is not matched to w^* . Then either:

Subcase (1a): there exists $v^* < w^*$ such that x^* is matched to v^* , or

Subcase (1b): there exists $z^* < x^*$ such that z^* is matched to w^* .

For Subcase (1a), $v^* < w^* < y^*$, contradicting the assumption that x^* is matched to y^* .

For Subcase (1b), $z^* < x^*$ and z^* is matched to w^* . Now since $z^* < x^*$, $\text{ord}(z^*) < \text{ord}(x^*)$, which implies that $f(z^*, w^*) < f(x^*, w^*) < f(x^*, y^*)$ and so by the induction hypothesis, $z^* \cup \{n\}$ is matched to $w^* \cup \{n\}$. But by the assumption for Case (1), x is matched to w , *i.e.* $x^* \cup \{n\}$ is matched to $w^* \cup \{n\}$, which implies that $z^* \cup \{n\} = x^* \cup \{n\}$ and therefore $z^* = x^*$, contradicting the assumption in Case (1b) that $z^* < x^*$. So, x is not matched to any predecessor w of y .

For Case (2): We are assuming that $z < x = x^* \cup \{n\}$ and z is matched to $y = y^* \cup \{n\} \in Q_n(L_{i+1})$. There are 2 subcases.

Subcase (2a): Suppose that $n \in z$. Let $z^* = z \setminus \{n\}$, so that $z = z^* \cup \{n\}$, where $z^* \in Q_{n-1}(L_i)$. Since $z^* \cup \{n\} < x^* \cup \{n\}$, $z^* < x^*$. Now $\text{ord}(z^*) < \text{ord}(x^*)$, so $f(z^*, y^*) < f(x^*, y^*) = j$. Since by assumption $z^* \cup \{n\}$ is matched to $y^* \cup \{n\}$, by the induction hypothesis, z^* is matched to y^* . But $z^* < x^*$ and we are assuming that x^* is matched to y^* . Contradiction.

Subcase (2b): Suppose $n \notin z$. Then $z \in Q_{n-1}(L_i)$, and $z < x$. Since $n \notin z, y \setminus z = \{n\}$, and thus $y = z \cup \{n\}$. Hence $z = y^*$. So $y^* < x^* \cup \{n\}$, y^* is matched to $y^* \cup \{n\}$, and x^* is matched to y^* . Thus, $y^* = x^* \cup \{p\}$ for some $p \notin x^*$, and $x^* \cup \{p\}$ is matched to $x^* \cup \{p\} \cup \{n\}$. Suppose that $p = n-1$. Then $y^* = x^* \cup \{n-1\}$ and $n-1 \notin x^*$. Since $n \notin z, z = z^* \cup \{n\}$ for some $z^* < x^*$. So $z^* \cup \{n\}$ is matched to $(x^* \cup \{n-1\}) \cup \{n\}$. Now since $z^* \cup \{n\} < x^* \cup \{n\}$ by assumption, $z^* < x^*$. Therefore $\text{ord}(z^*) < \text{ord}(x^*)$ and so $f(z^*, y^*) < f(x^*, y^*)$. Then by the induction hypothesis, since $z^* \cup \{n\}$ is matched to $(x^* \cup \{n-1\}) \cup \{n\}$, we have z^* matched to $x^* \cup \{n-1\} = y^*$. But by hypothesis, x^* is matched to y^* , and so $z^* = x^*$, contradicting our assumption that z is matched to y .

If $p \neq n-1$, and $n-1 \notin x^*$, then $n-1 \notin y^*$ either, so that $x^* \in Q_{n-2}(L_{i-1})$ and $y^* \in Q_{n-2}(L_i)$. Then $x^* \cup \{n\}$ is matched to $y^* \cup \{n\}$ by Lemma 4.2.7. Contradiction.

Before continuing with the proof of this lemma, we need the following proposition:

Proposition 4.2.10. *For $k \geq 3$, let $y^* \in Q_{2k}(L_k)$ and $y \in Q_{2k+1}(L_{k+1})$ as in Case (2b) of Lemma 4.2.9. If y^* is matched to y , then y^* was unmatched by the GA for $Q_{2k}(L_{k-1}, L_k)$. That is, there does not exist an $x^* \in Q_{2k}(L_{k-1})$ such that x^* is matched to y^* .*

Proof. Recall $z = y^*$ and proceed by induction on k . For $k = 3$, the proposition is true for $M_{3,7}$, as seen in Section 4.6. Now suppose $k > 3$ and for $\bar{y} \in Q_{2k-2}(L_{k-1})$ and $\bar{y} \cup \{2k-1\} \in Q_{2k-1}(L_k)$, if \bar{y} is matched to $\bar{y} \cup \{2k-1\}$, then \bar{y} is unmatched

by the GA for $Q_{2k-2}(L_{k-2}, L_{k-1})$. Let $y^* = \bar{y} \cup \{2k\}$ and $y = y^* \cup \{2k+1\}$ with y^* matched to y and assume there exists an $x^* \in Q_{2k}(L_{k-1})$ such that x^* is matched to y^* . If, for all $\bar{x} \in Q_{2k-2}(L_{k-2})$, \bar{x} is not matched to \bar{y} , then, by Lemma 4.2.7, $\bar{x} \cup \{2k\}$ is not matched to $\bar{y} \cup \{2k\}$, for all $\bar{x} \cup \{2k\} \in Q_{2k}(L_{k-1})$. That is, y^* is unmatched by the GA for $Q_{2k}(L_{k-1}, L_k)$. Contradiction. \square

Returning to the proof of Lemma 4.2.9 Case (2b), let $n-1 \in x^*$ and proceed by induction on n . The cases for $n = 7, 8$ with $i = 3$ can be seen in Section 4.6. Suppose $n > 8$ and the lemma holds for $n-1$. That is, for all $3 \leq i < (n-1)/2$, \bar{x} is matched to \bar{y} if and only if $\bar{x} \cup \{n-1\}$ is matched to $\bar{y} \cup \{n-1\}$, where $\bar{x} \in Q_{n-2}(L_{i-1})$ and $\bar{y} \in Q_{n-2}(L_i)$. Note that for n even, say $n = 2k$, we must show the lemma holds for $3 \leq i < k$ given that the induction hypothesis is true for $3 \leq i < k-1/2$ *i.e.*, for $3 \leq i < k$. However, if $n = 2k+1$, then we must show the lemma holds for all i , $3 \leq i \leq k$, where the induction hypothesis holds for $3 \leq i < 2k/2 = k$. Also, recall x^* is matched to y^* for $x^*, y^* \in Q_{n-1}$. Assume as before: there exists some $z < x$ such that z is matched to y , $n \notin z$ and $n-1 \in x^*$.

For z to be matched to y , we must have that all possible supersets of z less than y in colex order were already matched. That is, all possible supersets of z appearing before y containing $n-1$ and not n were previously matched by the GA. So, for all $w \in Q_{n-1}(L_{i+1})$ such that $z \subset w$, $n-1 \in w$, $n \notin w$, we have z is not matched to w . Writing $z = z^{**} \cup \{n-1\}$ and $w = w^{**} \cup \{n-1\}$, for all w , $z^{**} \cup \{n-1\}$ is not matched to $w^{**} \cup \{n-1\}$. Now, $z^{**} \in Q_{n-2}(L_{i-1})$ and for all w^{**} , $w^{**} \in Q_{n-2}(L_i)$.

Recall, $z = y^*$. Now by Proposition 4.2.10, in the special case with $n = 2k+1$ and $i = k$, we arrive at a contradiction.

Next let n be odd and $i < n/2 - 1$ or let n be even and $i < n/2$. By the induction hypothesis on n , since $z^{**} \cup \{n-1\}$ is not matched to $w^{**} \cup \{n-1\}$, then

for any $w^{**} \in Q_{n-2}(L_i)$ z^{**} is not matched to w^{**} . That is, z^{**} is not matched to any $z^{**} \cup \{q\}$, $q \leq n-2$, $q \notin z^{**}$. Thus, $z^{**} \in Q_{n-2}(L_{i-1})$ is not matched to an element of $Q_{n-2}(L_i)$. By Lemma 4.2.6, z^{**} must be matched to an element of $Q_{n-1}(L_i)$, namely $z^{**} \cup \{n-1\}$. So z^{**} is matched to $z^{**} \cup \{n-1\} = z$, *i.e.* z^{**} is matched to y^* . But x^* is matched to y^* and $z^{**} < x^*$. Contradiction. Therefore $x^* \cup \{n\}$ is matched to $y^* \cup \{n\}$. This completes the proof of the (\Rightarrow) implication.

For the converse, assume that $x^* \cup \{n\}$ is matched to $y^* \cup \{n\}$. We must prove that x^* is matched to y^* . First, suppose that x^* is matched to some predecessor w^* of y^* . Then $\text{ord}(w^*) < \text{ord}(y^*)$. Therefore $f(x^*, w^*) < f(x^*, y^*)$. By the induction hypothesis, $x^* \cup \{n\}$ is matched to $w^* \cup \{n\}$. So both $y^* \cup \{n\}$ and $w^* \cup \{n\}$ are matched to $x^* \cup \{n\}$ and, therefore, $y^* \cup \{n\} = w^* \cup \{n\}$. Hence $y^* = w^*$. Contradiction.

Finally, suppose that some predecessor z^* of x^* is matched to y^* . Then $\text{ord}(z^*) < \text{ord}(x^*)$. Thus, $f(z^*, y^*) < f(x^*, y^*)$, and so, by the induction hypothesis, $z^* \cup \{n\}$ is matched to $y^* \cup \{n\}$. So both $x^* \cup \{n\}$ and $z^* \cup \{n\}$ are matched to $y^* \cup \{n\}$. Therefore $z^* \cup \{n\} = x^* \cup \{n\}$, and so $x^* = z^*$. Contradiction. \square

Theorem 4.2.11. *For $n \geq 3$ and for $0 \leq i < n/2$, the GA produces a matching from $Q_n(L_i)$ to $Q_n(L_{i+1})$ which saturates $Q_n(L_i)$.*

Proof. We argue by induction on n . The case $n = 3$ is trivial: the only $i < 3/2$ are $i = 0$ and $i = 1$. $M_{0,3} = \{\langle \emptyset, \{1\} \rangle\}$ and the edges belonging to the matching $M_{1,3}$, in the order produced by the algorithm, are $\langle 1, 12 \rangle, \langle 2, 23 \rangle, \langle 3, 13 \rangle$. For $n = 4, 5, 6$ and $i = 0, 1$, Proposition 4.2.3 gives the desired saturation of L_i . For $M_{2,5}$ and $M_{2,6}$, the result can be seen in Section 4.6.

Now let $n \geq 7$. Assume the theorem holds for all $m < n$. Let $i < n/2$ and note that $i < n/2$ implies that $i - 1 < (n - 1)/2$, so by the induction hypothesis the

greedy algorithm, when applied to $Q_{n-1}(L_{i-1}, L_i)$, yields a matching M' saturating $Q_{n-1}(L_{i-1})$. The desired result now follows from Lemma 4.2.6 and Lemma 4.2.9. \square

Corollary 4.2.12. *For $n = 2k + 1, k \geq 1$, the GA yields a perfect matching between the two middle levels of $Q_n, Q_n(L_k)$ and $Q_n(L_{k+1})$.*

Proof. This is an immediate consequence of Theorem 4.2.11 and the fact that $\binom{n}{k} = \binom{n}{k+1}$. \square

Definition 4.2.13. Denote by M_L the matching between $Q_n(L_i)$ and $Q_n(L_{i+1})$ produced by the GA applied to the “lower” level L_i , and M^U the matching produced by the analogous GA applied to the “upper” level L_{i+1} .

By the definition of the GA, Lemmas 4.2.4 and 4.2.5 yield matches in M_L . The next two lemmas are the analogues for M^U of these two results, and follow directly from the GA.

Lemma 4.2.14. *Let $n \geq 3$. Let $x \in Q_n(L_i)$ and suppose that y is the least element of $Q_n(L_{i+1})$ with the following properties:*

- (1) $x \subset y$
- (2) *there is no $z \in Q_n(L_i)$ with $z < x$ such that $\langle z, y \rangle \in M^U$.*

Then $\langle x, y \rangle \in M^U$.

Lemma 4.2.15. *Let $n \geq 3$. Let $y \in Q_n(L_{i+1})$ and suppose that x is the least element of $Q_n(L_i)$ with the following properties:*

- (1) $x \subset y$
- (2) *there is no $w \in Q_n(L_{i+1})$ with $w < y$ such that $\langle x, w \rangle \in M^U$.*

Then $\langle x, y \rangle \in M^U$.

Lemma 4.2.16. For $n \geq 3$ and $0 \leq i \leq n - 1$, $M_L = M^U$.

Proof. Assume there exists some edge in M_L that is not in M^U . Let x be the smallest element of L_i such that $\langle x, y \rangle \in M_L \setminus M^U$. Since both x and y can not be unmatched (for then, the GA would match them) we have the following four cases:

Case (1): x is M^U -unsaturated

Case (2): y is M^U -unsaturated

Case (3): there exists a $z < x$ such that $\langle z, y \rangle \in M^U$

Case (4): there exists a $w < y$ such that $\langle x, w \rangle \in M^U$

For Case (1), if x is M^U -unsaturated, then y is matched in M^U . In fact, y must be matched to some $z < x$ (otherwise $\langle x, y \rangle \in M^U$). This is exactly Case (3). Similarly, Case (2) reduces to Case (4).

For Case (3): $z < x$ and for some $w < y$, $\langle z, w \rangle \in M_L$, since if not, $\langle z, y \rangle \in M_L$, contradicting the fact that $\langle x, y \rangle \in M_L$. So, $\langle z, w \rangle \in M_L \setminus M^U$, with $z < x$, contradicting the minimality of x .

For Case (4), given that $x < w$, $w < y$ and, since $\langle x, y \rangle \in M_L$, $\langle x, w \rangle \notin M_L$, we have that $\langle z, w \rangle \in M_L$ for some $z < x$. But $\langle x, w \rangle \in M^U$ and $z < x$. Therefore $\langle z, w \rangle \notin M^U$, i.e., $\langle z, w \rangle \in M_L \setminus M^U$. since $z < x$, this contradicts the minimality of x .

Thus, $M_L \subset M^U$. Similarly, we can show that $M^U \subset M_L$, and we have arrived at our desired conclusion. \square

4.3 The Reverse Greedy Matching Algorithm

Let $A = \{a_1, a_2, \dots, a_i\}$ and $B = \{b_1, b_2, \dots, b_i\}$ be two elements in L_i , $1 \leq i \leq n - 1$, each sorted in increasing order. A is less than B in colex order, $A <_c B$, if $a_r < b_r$ for some r , $1 \leq r \leq i$, and $a_s = b_s$ for all $s > r$. In reverse colex order, $A <_{rc} B$ if

$a_r > b_r$ for some r , $1 \leq r \leq i$, and $a_s = b_s$ for all $s > r$, i.e. $A <_{rc} B \iff B <_c A$.

Algorithm 4.3.1 (Reverse Greedy Algorithm (RGA)).

List the members of L_i in colex order, and do the same for the members of L_{i+1} .

Step 1. Match $(n - i + 1)(n - i + 2) \dots n$ with $(n - i)(n - i + 1) \dots n$.

Step 2. While there is an unmatched i -set:

DO: Choose the last such in colex order and match it to the last neighbor in L_{i+1} (in colex order) which is as yet unmatched.

Remark 4.3.2. *By the definition of reverse colex order, Algorithm 4.3.1 is equivalent to sorting L_i and L_{i+1} in reverse colex order and matching from the least element to the greatest.*

Example 4.3.3. In $Q_5(L_2, L_3)$, the reverse colex orderings of L_2 and L_3 are:

$$L_2 = 45, 35, 25, 15, 34, 24, 14, 23, 13, 12$$

$$L_3 = 345, 245, 145, 235, 135, 125, 234, 134, 124, 123$$

The matching produced by the reverse greedy algorithm, in the order in which is generated is:

$$M = \{\langle 45, 345 \rangle, \langle 35, 235 \rangle, \langle 25, 245 \rangle, \langle 15, 145 \rangle, \langle 34, 234 \rangle, \langle 24, 124 \rangle, \\ \langle 14, 134 \rangle, \langle 23, 123 \rangle, \langle 13, 135 \rangle, \langle 12, 125 \rangle\}$$

Notation. For $x \subset [n]$, denote by x^c the complement of x in $[n]$, i.e., $x^c = [n] \setminus x$.

Definition 4.3.4. Let M be a matching in $Q_n(L_i, L_{i+1})$. Then a complementary matching in $Q_n(L_{n-i-1}, L_{n-i})$ is $M^c = \{\langle x^c, y^c \rangle \mid \langle x, y \rangle \in M\}$.

Lemma 4.3.5. *For $n \geq 3$ and $0 \leq i \leq n - 1$, let M be the matching obtained from the GA between L_i and L_{i+1} in Q_n and N be the matching obtained from the RGA between L_{n-i-1} and L_{n-i} in Q_n . Then $M = N^c$.*

Proof. First we show that $M \subset N^c$. Assume not. Then there exists some edge in $M \setminus N^c$. Let x be the smallest element of L_i such that for some $y \in L_{i+1}$, $\langle x, y \rangle \in M \setminus N^c$. Note that not both x^c and y^c can be N -unsaturated, for then the RGA would have matched them. We must consider the following four cases:

Case (1): x^c is N -unsaturated

Case (2): y^c is N -unsaturated

Case (3): there exist $z^c > x^c$ such that $\langle z^c, y^c \rangle \in N$

Case (4): there exist $w^c > y^c$ such that $\langle x^c, w^c \rangle \in N$

In Case (1), if x^c is N -unsaturated then y^c is not only N -saturated, but by the RGA, there is some z with $z^c > x^c$ such that $\langle z^c, y^c \rangle \in N$. This is Case (3).

Similarly, Case (2) reduces to Case (4).

For Case (3), if $\langle z^c, y^c \rangle \in N$, then $\langle z, y \rangle \in N^c$, and since $z^c > x^c$, $z < x$. Now, z can not be M -unsaturated, for then the GA would have matched it to y . So for some $w < y$, $\langle z, w \rangle \in M$. Since $w \neq y$, $z \neq x$, $\langle z, y \rangle \in N^c$, and, given that N^c is a matching, $\langle z, w \rangle \notin N^c$. Hence, $\langle z, w \rangle \in M \setminus N^c$. Since $z < x$, this contradicts the minimality of x .

In Case (4), if $\langle x^c, w^c \rangle \in N$, with $w^c > y^c$, then $\langle x, w \rangle \in N^c$ with $w < y$. Now w can not be M -unsaturated, for then the GA would have matched x to w , contradicting the fact that $\langle x, y \rangle \in M$. Since $w \neq y$ and $\langle x, y \rangle \in M$, $\langle x, w \rangle \notin M$, and so $\langle z, w \rangle \in M$ for some z . Hence, $x \neq z$. Suppose $x < z$. Then since $w < y$ and x and y are adjacent, the GA would have selected $\langle x, w \rangle$ for M instead of $\langle z, w \rangle$. Thus, (since colex is a total ordering) $z < x$. Now, since $\langle x, w \rangle \in N^c$, $\langle z, w \rangle \notin N^c$. Hence $\langle z, w \rangle \in M \setminus N^c$,

contradicting the minimality of x .

Therefore $M \subset N^c$. Similarly, we can show that $N \subset M^c$, and, thus, $N^c \subset M$. Hence $M = N^c$, as desired. \square

Remark 4.3.6. Let \overline{M}^U and \overline{M}_L be the analogous matchings from Lemma 4.2.16 for the RGA for $Q_n(L_i, L_{i+1})$. $\overline{M}^U = \overline{M}_L$ follows directly from results given in the next section.

4.4 Lexicographic Order and the GA

Define $\theta(j) = n + 1 - j$ for $1 \leq j \leq n$. That is, θ is the involution $\theta = (1\ n)(2\ (n-1)) \cdots ([n/2]\ [(n+1)/2])$. For an i -set $X = \{x_1, x_2, \dots, x_i\}$, we define the action of φ on X as the action of θ on the individual elements of the set: $\varphi(X) = \{\theta(x_1), \theta(x_2), \dots, \theta(x_i)\}$. It is well known that φ is an automorphism of Q_n , as originally mentioned in [4] and in [9], from which the ρ_θ notation for φ is derived.

Let $A = \{a_1, a_2, \dots, a_i\}$ and $B = \{b_1, b_2, \dots, b_i\}$ be two elements in L_i , $1 \leq i \leq n-1$, each sorted in increasing order. A is less than B in lexicographic (or lex) order if $a_r < b_r$ for some r , $1 \leq r \leq i$, and $a_s = b_s$ for all $s < r$. We denote this inequality in lex order as $A <_l B$.

Proposition 4.4.1. For all $0 \leq i \leq n$ and for all $A, B \in L_i$, $A <_{rc} B$ iff $\varphi(A) <_l \varphi(B)$.

Proof. This is trivial for $i = 0, n$.

Let $1 \leq i \leq n-1$ and let $A, B \in L_i$ in colex order with $A <_{rc} B$. Then, $a_r > b_r$ for some r , $1 \leq r \leq i$, and $a_s = b_s$ for all $s > r$. Thus, in colex order, we have $A >_c B$.

Now,

$$\begin{aligned}
\varphi(A) &= \{\theta(a_1), \theta(a_2), \dots, \theta(a_i)\} \\
&= \{n - a_1 + 1, n - a_2 + 1, \dots, n - a_i + 1\} \\
&= \{a_1^*, a_2^*, \dots, a_i^*\}
\end{aligned}$$

So, we have $a_j^* > a_{j+1}^*$, $1 \leq j < i$. Similarly, defining $b_1^*, b_2^*, \dots, b_i^*$ by $b_j^* = \theta(b_j)$, we have $b_j^* > b_{j+1}^*$. Since $a_s = b_s$ for all $s > r$, we have $a_s^* = b_s^*$, while $a_r > b_r$ gives $a_r^* < b_r^*$. Writing the elements of $\varphi(A)$ and $\varphi(B)$ in increasing order, we have $\varphi(A) = a_i^*, \dots, a_2^*, a_1^*$ and $\varphi(B) = b_i^*, \dots, b_2^*, b_1^*$. Therefore, these two sets satisfy the inequality $\varphi(A) <_l \varphi(B)$. \square

Unless otherwise specified, $A < B$ implies the use of colex ordering.

Lemma 4.4.2. *Let $1 \leq i < n/2$ and let $x \in Q_n(L_i)$ and $y \in Q_n(L_{i+1})$. Then the GA matches x to y if and only if the RGA matches $\varphi(x)$ to $\varphi(y)$.*

Proof. Proceed by induction on $f(x, y) = \text{ord}(x) + \text{ord}(y)$. For the base case, we have $f(x, y) = 2$, which occurs exactly when $x = [i]$ and $y = [i + 1]$ and, thus, $\varphi(x) = (n - i + 1)(n - i + 2) \dots n$ and $\varphi(y) = (n - i)(n - i + 1) \dots n$. Since x and y are the initial elements of L_i and L_{i+1} , respectively, and $x \subset y$, the GA matches x to y . Similarly, $\varphi(x)$ and $\varphi(y)$ are the final elements in colex of levels L_i and L_{i+1} , respectively, and $\varphi(x) \subset \varphi(y)$, so the RGA matches $\varphi(x)$ to $\varphi(y)$.

Now let us assume the lemma holds for all values m , $2 \leq m^* < m$, and let $f(x, y) = m$. Let $y = x \cup \{j\}$.

Suppose x is matched to y by the GA. We want to show that $\varphi(x)$ is matched to $\varphi(y)$ by the RGA. Assume $\varphi(y)$ is matched to some $\varphi(z)$ by the RGA, where $\varphi(z) > \varphi(x)$. Since φ is order-reversing, $z < x$, $\text{ord}(z) < \text{ord}(x)$, and $f(z, y) < f(x, y)$.

By the induction hypothesis, if $\varphi(z)$ is matched to $\varphi(y)$ then z is matched to y , which contradicts the assumption that x is matched to y by the GA. Hence, $\varphi(y)$ is not matched, under the RGA, to any $\varphi(z) > \varphi(x)$.

Now assume $\varphi(x)$ is matched, by the RGA, to some $\varphi(w) > \varphi(y)$. Since φ is order-reversing, $w < y$ and $\text{ord}(w) < \text{ord}(y)$. So, $f(x, w) < f(x, y)$, and, by the induction hypothesis, we have $\varphi(x)$ is matched to $\varphi(w)$ iff x is matched to w . But this again contradicts our initial assumption that x is matched to y by the GA. Thus, we must have $\varphi(x)$ is matched to $\varphi(y)$ by the RGA if x is matched to y by the GA.

Now suppose $\varphi(x)$ is matched to $\varphi(y)$ by the RGA. If x is not matched to y by the GA, then either there exists a $z < x$ such that z is matched to y by the GA, or there exists a $w < y$ such that w is matched to x by the GA. In the first case, we have $\text{ord}(z) < \text{ord}(x)$ or $f(z, y) < f(x, y)$, which, by the induction hypothesis means that z is matched to y by the GA iff $\varphi(z)$ is matched to $\varphi(y)$ by the RGA. But this implies that $\varphi(x) = \varphi(z)$ while the inequality $z < x$ implies $\varphi(z) > \varphi(x)$. Thus, we have arrived at a contradiction.

Similarly, in the second case, $\text{ord}(w) < \text{ord}(y)$ and $f(x, w) < f(x, y)$ and so x is matched to w by the GA iff $\varphi(x)$ is matched to $\varphi(w)$ by the RGA. Again, this implies $\varphi(w) = \varphi(y)$, or $w = y$, a contradiction to $w < y$. Therefore, x is matched to y by the GA if and only if $\varphi(x)$ is matched to $\varphi(y)$ by the RGA. \square

Since φ is an automorphism of the n -cube, it sends edges to edges and, therefore, matchings to matchings.

Corollary 4.4.3. *Let $1 \leq i < n/2$ and let M_c be the matching between $Q_n(L_i)$ and $Q_n(L_{i+1})$ produced by the GA and M_{rc} be the matching on these two levels produced by the RGA. Then, $M_c = \varphi(M_{rc})$.*

Proof. This follows from Lemma 4.4.2 and the fact that φ preserves adjacency and

is an involution. □

Corollary 4.4.4. *Let $1 \leq i < n/2$ and M_c be defined as in Corollary 4.4.3. If $Q_n(L_i)$ and $Q_n(L_{i+1})$ are sorted in lex order, and M_l is the matching between them by the GA, then $M_c = M_l$.*

Proof. This follows directly from Proposition 4.4.1 and Lemma 4.4.2. □

4.5 Saturation Results for All of Q_n

Theorem 4.5.1. *For $n \geq 3$ and for $0 \leq i < n/2$, the RGA produces a matching from $Q_n(L_i)$ to $Q_n(L_{i+1})$ which saturates $Q_n(L_i)$.*

Proof. Since $\varphi^2 = id$, this follows directly from Corollary 4.4.3 and Theorem 4.2.11. □

Corollary 4.5.2. *For $n = 2k + 1, k \geq 1$, the RGA yields a perfect matching between the two middle levels of Q_n , $Q_n(L_k)$ and $Q_n(L_{k+1})$.*

Proof. This follows from Theorem 4.5.1 and the fact that $\binom{n}{k} = \binom{n}{k+1}$. □

Theorem 4.5.3. *For $n \geq 3$, and for all $i, 0 \leq i \leq n-1$, the GA produces a matching from $Q_n(L_i)$ to $Q_n(L_{i+1})$ which saturates the smaller of the two levels.*

Proof. For $i < n/2$, this is given in Theorem 4.2.11. For $i \geq n/2$, this is an immediate consequence of Theorem 4.5.1 and Lemma 4.3.5. □

Corollary 4.5.4. *For $n \geq 3$, and for all $i, 0 \leq i \leq n-1$, the RGA produces a matching from $Q_n(L_i)$ to $Q_n(L_{i+1})$ which saturates the smaller of the two levels.*

Proof. For $i < n/2$, Theorem 4.5.1 yields the desired result. Since φ is an involution, for $i \geq n/2$, this follows from Theorem 4.2.11 and Lemma 4.3.5. \square

Corollary 4.5.5. *For $n \geq 3$, and for all i , $0 \leq i \leq n-1$, when $Q_n(L_i)$ and $Q_n(L_{i+1})$ are sorted in lex order, the GA produces a matching between them which saturates the smaller of the two levels.*

Proof. This is immediate from Corollary 4.4.4. \square

4.6 GA Examples

Recall that $M_{i,n}$ represents the GA matching between $L_i(Q_n)$ and $L_{i+1}(Q_n)$. Note that $M_{2,5}$, $M_{3,7}$, and $M_{4,9}$ are perfect matchings as specified in Corollary 4.2.12.

$M_{2,5}$:

$\langle 12, 123 \rangle$ $\langle 13, 134 \rangle$ $\langle 23, 234 \rangle$ $\langle 14, 124 \rangle$ $\langle 24, 245 \rangle$ $\langle 34, 345 \rangle$ $\langle 15, 125 \rangle$
 $\langle 25, 235 \rangle$ $\langle 35, 135 \rangle$ $\langle 45, 145 \rangle$

$M_{2,6}$:

$\langle 12, 123 \rangle$ $\langle 13, 134 \rangle$ $\langle 23, 234 \rangle$ $\langle 14, 124 \rangle$ $\langle 24, 245 \rangle$ $\langle 34, 345 \rangle$ $\langle 15, 125 \rangle$
 $\langle 25, 235 \rangle$ $\langle 35, 135 \rangle$ $\langle 45, 145 \rangle$ $\langle 16, 126 \rangle$ $\langle 26, 236 \rangle$ $\langle 36, 136 \rangle$ $\langle 46, 146 \rangle$
 $\langle 56, 156 \rangle$

$M_{2,7}$:

$\langle 12, 123 \rangle$ $\langle 13, 134 \rangle$ $\langle 23, 234 \rangle$ $\langle 14, 124 \rangle$ $\langle 24, 245 \rangle$ $\langle 34, 345 \rangle$ $\langle 15, 125 \rangle$
 $\langle 25, 235 \rangle$ $\langle 35, 135 \rangle$ $\langle 45, 145 \rangle$ $\langle 16, 126 \rangle$ $\langle 26, 236 \rangle$ $\langle 36, 136 \rangle$ $\langle 46, 146 \rangle$
 $\langle 56, 156 \rangle$ $\langle 17, 127 \rangle$ $\langle 27, 237 \rangle$ $\langle 37, 137 \rangle$ $\langle 47, 147 \rangle$ $\langle 57, 157 \rangle$ $\langle 67, 167 \rangle$

$M_{3,7}$:

$\langle 123, 1234 \rangle$ $\langle 124, 1245 \rangle$ $\langle 134, 1345 \rangle$ $\langle 234, 2345 \rangle$ $\langle 125, 1235 \rangle$ $\langle 135, 1356 \rangle$
 $\langle 235, 2356 \rangle$ $\langle 145, 1456 \rangle$ $\langle 245, 2456 \rangle$ $\langle 345, 3456 \rangle$ $\langle 126, 1236 \rangle$ $\langle 136, 1346 \rangle$
 $\langle 236, 2346 \rangle$ $\langle 146, 1246 \rangle$ $\langle 246, 2467 \rangle$ $\langle 346, 3467 \rangle$ $\langle 156, 1256 \rangle$ $\langle 256, 2567 \rangle$
 $\langle 356, 3567 \rangle$ $\langle 456, 4567 \rangle$ $\langle 127, 1237 \rangle$ $\langle 137, 1347 \rangle$ $\langle 237, 2347 \rangle$ $\langle 147, 1247 \rangle$
 $\langle 247, 2457 \rangle$ $\langle 347, 3457 \rangle$ $\langle 157, 1257 \rangle$ $\langle 257, 2357 \rangle$ $\langle 357, 1357 \rangle$ $\langle 457, 1457 \rangle$
 $\langle 167, 1267 \rangle$ $\langle 267, 2367 \rangle$ $\langle 367, 1367 \rangle$ $\langle 467, 1467 \rangle$ $\langle 567, 1567 \rangle$

$M_{3,8}$:

$\langle 123, 1234 \rangle$ $\langle 124, 1245 \rangle$ $\langle 134, 1345 \rangle$ $\langle 234, 2345 \rangle$ $\langle 125, 1235 \rangle$ $\langle 135, 1356 \rangle$
 $\langle 235, 2356 \rangle$ $\langle 145, 1456 \rangle$ $\langle 245, 2456 \rangle$ $\langle 345, 3456 \rangle$ $\langle 126, 1236 \rangle$ $\langle 136, 1346 \rangle$
 $\langle 236, 2346 \rangle$ $\langle 146, 1246 \rangle$ $\langle 246, 2467 \rangle$ $\langle 346, 3467 \rangle$ $\langle 156, 1256 \rangle$ $\langle 256, 2567 \rangle$
 $\langle 356, 3567 \rangle$ $\langle 456, 4567 \rangle$ $\langle 127, 1237 \rangle$ $\langle 137, 1347 \rangle$ $\langle 237, 2347 \rangle$ $\langle 147, 1247 \rangle$
 $\langle 247, 2457 \rangle$ $\langle 347, 3457 \rangle$ $\langle 157, 1257 \rangle$ $\langle 257, 2357 \rangle$ $\langle 357, 1357 \rangle$ $\langle 457, 1457 \rangle$
 $\langle 167, 1267 \rangle$ $\langle 267, 2367 \rangle$ $\langle 367, 1367 \rangle$ $\langle 467, 1467 \rangle$ $\langle 567, 1567 \rangle$ $\langle 128, 1238 \rangle$
 $\langle 138, 1348 \rangle$ $\langle 238, 2348 \rangle$ $\langle 148, 1248 \rangle$ $\langle 248, 2458 \rangle$ $\langle 348, 3458 \rangle$ $\langle 158, 1258 \rangle$
 $\langle 258, 2358 \rangle$ $\langle 358, 1358 \rangle$ $\langle 458, 1458 \rangle$ $\langle 168, 1268 \rangle$ $\langle 268, 2368 \rangle$ $\langle 368, 1368 \rangle$
 $\langle 468, 1468 \rangle$ $\langle 568, 1568 \rangle$ $\langle 178, 1278 \rangle$ $\langle 278, 2378 \rangle$ $\langle 378, 1378 \rangle$ $\langle 478, 1478 \rangle$
 $\langle 578, 1578 \rangle$ $\langle 678, 1678 \rangle$

$M_{4,9}$:

$\langle 1234, 12345 \rangle$ $\langle 1235, 12356 \rangle$ $\langle 1245, 12456 \rangle$ $\langle 1345, 13456 \rangle$ $\langle 2345, 23456 \rangle$
 $\langle 1236, 12346 \rangle$ $\langle 1246, 12467 \rangle$ $\langle 1346, 13467 \rangle$ $\langle 2346, 23467 \rangle$ $\langle 1256, 12567 \rangle$
 $\langle 1356, 13567 \rangle$ $\langle 2356, 23567 \rangle$ $\langle 1456, 14567 \rangle$ $\langle 2456, 24567 \rangle$ $\langle 3456, 34567 \rangle$
 $\langle 1237, 12347 \rangle$ $\langle 1247, 12457 \rangle$ $\langle 1347, 13457 \rangle$ $\langle 2347, 23457 \rangle$ $\langle 1257, 12357 \rangle$
 $\langle 1357, 13578 \rangle$ $\langle 2357, 23578 \rangle$ $\langle 1457, 14578 \rangle$ $\langle 2457, 24578 \rangle$ $\langle 3457, 34578 \rangle$

⟨1267, 12367⟩ ⟨1367, 13678⟩ ⟨2367, 23678⟩ ⟨1467, 14678⟩ ⟨2467, 24678⟩
 ⟨3467, 34678⟩ ⟨1567, 15678⟩ ⟨2567, 25678⟩ ⟨3567, 35678⟩ ⟨4567, 45678⟩
 ⟨1238, 12348⟩ ⟨1248, 12458⟩ ⟨1348, 13458⟩ ⟨2348, 23458⟩ ⟨1258, 12358⟩
 ⟨1358, 13568⟩ ⟨2358, 23568⟩ ⟨1458, 14568⟩ ⟨2458, 24568⟩ ⟨3458, 34568⟩
 ⟨1268, 12368⟩ ⟨1368, 13468⟩ ⟨2368, 23468⟩ ⟨1468, 12468⟩ ⟨2468, 24689⟩
 ⟨3468, 34689⟩ ⟨1568, 12568⟩ ⟨2568, 25689⟩ ⟨3568, 35689⟩ ⟨4568, 45689⟩
 ⟨1278, 12378⟩ ⟨1378, 13478⟩ ⟨2378, 23478⟩ ⟨1478, 12478⟩ ⟨2478, 24789⟩
 ⟨3478, 34789⟩ ⟨1578, 12578⟩ ⟨2578, 25789⟩ ⟨3578, 35789⟩ ⟨4578, 45789⟩
 ⟨1678, 12678⟩ ⟨2678, 26789⟩ ⟨3678, 36789⟩ ⟨4678, 46789⟩ ⟨5678, 56789⟩
 ⟨1239, 12349⟩ ⟨1249, 12459⟩ ⟨1349, 13459⟩ ⟨2349, 23459⟩ ⟨1259, 12359⟩
 ⟨1359, 13569⟩ ⟨2359, 23569⟩ ⟨1459, 14569⟩ ⟨2459, 24569⟩ ⟨3459, 34569⟩
 ⟨1269, 12369⟩ ⟨1369, 13469⟩ ⟨2369, 23469⟩ ⟨1469, 12469⟩ ⟨2469, 24679⟩
 ⟨3469, 34679⟩ ⟨1569, 12569⟩ ⟨2569, 25679⟩ ⟨3569, 35679⟩ ⟨4569, 45679⟩
 ⟨1279, 12379⟩ ⟨1379, 13479⟩ ⟨2379, 23479⟩ ⟨1479, 12479⟩ ⟨2479, 24579⟩
 ⟨3479, 34579⟩ ⟨1579, 12579⟩ ⟨2579, 23579⟩ ⟨3579, 13579⟩ ⟨4579, 14579⟩
 ⟨1679, 12679⟩ ⟨2679, 23679⟩ ⟨3679, 13679⟩ ⟨4679, 14679⟩ ⟨5679, 15679⟩
 ⟨1289, 12389⟩ ⟨1389, 13489⟩ ⟨2389, 23489⟩ ⟨1489, 12489⟩ ⟨2489, 24589⟩
 ⟨3489, 34589⟩ ⟨1589, 12589⟩ ⟨2589, 23589⟩ ⟨3589, 13589⟩ ⟨4589, 14589⟩
 ⟨1689, 12689⟩ ⟨2689, 23689⟩ ⟨3689, 13689⟩ ⟨4689, 14689⟩ ⟨5689, 15689⟩
 ⟨1789, 12789⟩ ⟨2789, 23789⟩ ⟨3789, 13789⟩ ⟨4789, 14789⟩ ⟨5789, 15789⟩
 ⟨6789, 16789⟩

Chapter 5

m -Transitivity

5.1 Introduction

Let $v_i, w_i \in \mathcal{P}([n])$ and define $\mathbf{V}_m = \{v_1, \dots, v_m\}$ and $\mathbf{W}_m = \{w_1, \dots, w_m\}$, $1 \leq m \leq 2^n$, and let us recall some of our previous statements. For any two vertices $u, v \in V(Q_n)$, $d(u, v) = |u\Delta v| = \text{wt}(u\Delta v)$. Also, Q_n is vertex-transitive, *i.e.* for any $u, v \in V(Q_n)$, there is an automorphism ψ of Q_n such that $\psi(u) = v$. For example, if $A = u\Delta v$, $\psi = \sigma_A$. From Section 2.1, we have that for any two sets A and B , $A + B = \{a \cup b \mid a \in A, b \in B\}$.

5.2 Distance m -Transitivity in Q_n

The motivating question (the distance question) for this chapter is:

Question 2. Given two sets \mathbf{V}_m and \mathbf{W}_m with $d(v_i, v_j) = d(w_i, w_j)$ for all i, j , $1 \leq i, j \leq m$, does there exist an automorphism $f \in \text{Aut}(Q_n)$ such that $f(v_i) = w_i$ for all i , $1 \leq i \leq m$?

For $m = 1, 2, 3$, we can find such an f . ($m = 1$ is vertex-transitivity, $m = 2$ is

distance-transitivity).

Lemma 5.2.1. *Q_n is distance 3-transitive. That is, for two 3-sets, \mathbf{V}_3 and \mathbf{W}_3 , subsets of Q_n , with $d(v_i, v_j) = d(w_i, w_j)$ for all i, j , $1 \leq i, j \leq 3$, there is an automorphism $f \in \text{Aut}(Q_n)$ such that $f(v_i) = w_i$ for all i , $1 \leq i \leq 3$.*

Proof. Let us consider the following three cases:

Case (1): $v_1 = w_1 = \emptyset$

Case (2): $v_1 = \emptyset$, no restriction on w_1

Case (3): No restrictions on v_1 and w_1

For Case (1), in order to satisfy the restrictions on f , we must have $f(\emptyset) = f(v_1) = w_1 = \emptyset$, and, so, $f = \rho_\theta$. Since $d(v_i, v_j) = d(w_i, w_j)$ for all i, j , we can see that, not only, $\text{wt}(v_i) = \text{wt}(w_i)$, but also, the number of elements common to $\{v_2, v_3\}$ is equal to the number of elements common to $\{w_2, w_3\}$. Construct θ by mapping a common element of v_2 and v_3 to one common to both w_2 and w_3 . Then extend θ to all elements by mapping each unique element of v_2 to a unique element of w_2 , similarly for v_3 and w_3 . Therefore, we have constructed a θ so that, with $f = \rho_\theta$, f sends v_i to w_i for $1 \leq i \leq 3$.

Case (2): Consider the set $\mathbf{W}_3^* = \{w_1^*, w_2^*, w_3^*\}$, where $w_i^* = \sigma_{w_1}(w_i)$. Then, $\mathbf{W}_3^* = \{\emptyset, w_1 \Delta w_2, w_1 \Delta w_3\}$, and, by Case 1, there exists an $f^* \in \text{Aut}(Q_n)$ such that $f^* : \mathbf{V}_3 \rightarrow \mathbf{W}_3^*$, $f^*(v_i) = w_i^*$, $f^* = \rho_\theta$. Now,

$$f^*(v_i) = \rho_\theta(v_i) = w_i^* = w_1 \Delta w_i = \sigma_{w_1}(w_i)$$

So,

$$\sigma_{w_1}(w_i) = \rho_\theta(v_i)$$

which yields

$$w_i = \sigma_{w_1}^{-1} \rho_\theta(v_i) = \sigma_{w_1} \rho_\theta(v_i)$$

since σ_{w_1} is of order 2. Therefore, by letting $f = \sigma_{w_1} \rho_\theta$, we have $f(v_i) = w_i$ for $1 \leq i \leq 3$.

In Case (3), let us consider the set $\mathbf{V}_3^* = \{v_1^*, v_2^*, v_3^*\}$, where $v_i^* = \sigma_{v_1}(v_i)$. Then, $\mathbf{V}_3^* = \{\emptyset, v_1 \Delta v_2, v_1 \Delta v_3\}$, and, by Case 2, there exists a map $f^* \in \text{Aut}(Q_n)$ such that $f^* : V_3^* \rightarrow W_3$, with $f^*(v_i^*) = w_i$, for all i , and $f^* = \sigma_{w_1} \rho_\theta$. Thus,

$$w_i = f^*(v_i^*) = \sigma_{w_1} \rho_\theta(v_1 \Delta v_i) = \sigma_{w_1} \rho_\theta(\sigma_{v_1}(v_i)) = \sigma_{w_1} \rho_\theta \sigma_{v_1}(v_i) = \sigma_{w_1 \Delta \theta(v_1)} \rho_\theta(v_i)$$

Hence, by letting $f = \sigma_{w_1 \Delta \theta(v_1)} \rho_\theta$, we have $f(v_i) = w_i$, for all i , $1 \leq i \leq 3$.

Therefore, for any $\mathbf{V}_3, \mathbf{W}_3$ with $d(v_i, v_j) = d(w_i, w_j)$, $1 \leq i, j \leq 3$, there exists an $f \in \text{Aut}(Q_n)$ such that $f(v_i) = w_i$, $1 \leq i \leq 3$. \square

The following example illustrates that Q_n is not distance 4-transitive.

Example 5.2.2. Let $V = \{\emptyset, 12, 13, 14\}$ and $W = \{\emptyset, 56, 57, 67\}$. Then, $d(v_i, v_j) = d(w_i, w_j) = 2$ for all i, j , but there is no f such that $f(v_i) = w_i$, $1 \leq i \leq 4$. Since f must map v_1 to w_1 , we have that $\sigma_A = \sigma_\emptyset$, and thus, $f = \rho_\theta$. Consider v_2 . If f sends 1 to 5, then, $f(v_3) = f(14) = \{5, \theta(4)\}$. But $5 \notin w_3$, so this is not a valid permutation. Hence, in v_2 , f must have mapped 1 to 6 and, thus, 2 to 5. But $5 \in w_3$ and $2 \notin v_3$, so, again, this permutation fails to provide the necessary map. Since these are the only two possibilities for mapping v_2 to w_2 , we have shown that θ , and thus f , does not exist.

This example can be extended to any $m \geq 5$ by adding the same set $x \in \{8, \dots, n\}$ to V as to W . Hence, the distance condition is not sufficient.

Since for $m \geq 4$ the distance condition is not sufficient, we shall consider an

alternative: intersections.

5.3 Intersection m -Transitivity in G

For any subgraph G of Q_n , let $V_m = \{v_1, v_2, \dots, v_m\}$ and $W_m = \{w_1, w_2, \dots, w_m\}$ be two sets of vertices, where each member of V_m and W_m is a vertex of G , viewed as a subset of $[n]$.

Condition 5.3.1. *There exist two subsets of $[n]$, A and B , such that, for each k ,*

$$\left| (A \Delta v_{l_1}) \cap \dots \cap (A \Delta v_{l_k}) \right| = \left| (B \Delta w_{l_1}) \cap \dots \cap (B \Delta w_{l_k}) \right|$$

$$1 \leq l_1 < l_2 < \dots < l_k \leq m, 1 \leq k \leq m.$$

Definition 5.3.2. If, for every V_m and W_m that satisfy Condition 5.3.1, there exists an automorphism $f \in \text{Aut}(G)$ such that $f(v_i) = w_i$ for all i , $1 \leq i \leq m$, then G is intersection m -transitive.

One might be curious to determine how distance m -transitivity in G relates to this new idea of intersection m -transitivity in G . Or, more specifically, how does the original distance transitivity condition relate to intersection 2-transitivity in G . This is illustrated in the next proposition.

Proposition 5.3.3. *For any subgraph G of Q_n , if G is intersection 2-transitive, then G is distance transitive.*

Proof. Let V_2 and W_2 be two 2-sets of vertices such that V_2 and W_2 satisfy Condi-

tion 5.3.1. Then V_2 and W_2 satisfy the following 3 conditions:

$$\begin{aligned} |A\Delta v_1| &= |B\Delta w_1| \\ |A\Delta v_2| &= |B\Delta w_2| \\ |(A\Delta v_1) \cap (A\Delta v_2)| &= |(B\Delta w_1) \cap (B\Delta w_2)| \end{aligned}$$

Now, $d(x, y) = |x\Delta y| = |(x\Delta C)\Delta(y\Delta C)| = d(x\Delta C, y\Delta C)$, for all $x, y \in V(G)$, $C \subset [n]$. Therefore,

$$\begin{aligned} d(v_1, v_2) &= d(A\Delta v_1, A\Delta v_2) \\ &= |A\Delta v_1| + |A\Delta v_2| - 2|(A\Delta v_1) \cap (A\Delta v_2)| \\ &= |B\Delta w_1| + |B\Delta w_2| - 2|(B\Delta w_1) \cap (B\Delta w_2)| \\ &= d(B\Delta w_1, B\Delta w_2) \\ &= d(w_1, w_2) \end{aligned}$$

where $d(x, y) = |x\Delta y| = |x| + |y| - 2|x \cap y|$. Thus, the necessary condition for distance transitivity for G holds. Since G is intersection 2-transitive, there exists $f \in \text{Aut}(G)$ such that $f(v_i) = w_i$, $i = 1, 2$. Using the corresponding automorphism for each pair of V_2, W_2 , we have that G is distance transitive. \square

Notice that if G is distance transitive, we do not necessarily know that G is intersection 2-transitive since we can not be guaranteed that the necessary conditions are satisfied.

This proposition gives rise to the following question:

Question 3. Is Condition 5.3.1 equivalent to the corresponding condition with symmetric difference?

Section 5.5 will show the answer to this is affirmative, but first we will determine how Condition 5.3.1 behaves with Q_n .

5.4 Intersection m -Transitivity in Q_n

Taking $A, B = \emptyset$ in Condition 5.3.1, we can define Condition 5.4.1.

Condition 5.4.1. *For each k , $1 \leq k \leq m$, and for all sequences $1 \leq l_1 < l_2 < \dots < l_k \leq m$,*

$$\left| v_{l_1} \cap \dots \cap v_{l_k} \right| = \left| w_{l_1} \cap \dots \cap w_{l_k} \right|$$

Lemma 5.4.2. *Condition 5.4.1 implies there exists a permutation θ of $[n]$ such that for all i , $1 \leq i \leq m$, $\rho_\theta(v_i) = w_i$.*

Proof. The proof is direct when considering induction on n . □

Proposition 5.4.3. *Applying ρ_θ to a set does not affect the size of the intersection. That is, for a subset $X = \{x_1, \dots, x_m\}$ of $V(Q_n)$, and for any k ,*

$$\left| x_{l_1} \cap \dots \cap x_{l_k} \right| = \left| \rho_\theta \left(x_{l_1} \cap \dots \cap x_{l_k} \right) \right| = \left| \rho_\theta \left(x_{l_1} \right) \cap \dots \cap \rho_\theta \left(x_{l_k} \right) \right|$$

$1 \leq l_1 < l_2 < \dots < l_k \leq m$, $1 \leq k \leq m$.

Proof. This follows directly from the fact that ρ_θ preserves both intersections and cardinalities of sets. □

Corollary 5.4.4. *For $V = \{\emptyset, v_2, v_3, \dots, v_m\}$ and $W = \{\emptyset, w_2, w_3, \dots, w_m\}$, V and W satisfy Condition 5.4.1 if and only if there exists an automorphism $f \in \text{Aut}(Q_n)$ such that $f(v_i) = w_i$, $f = \rho_\theta$, θ a permutation on $[n]$, $1 \leq i \leq m$.*

Proof. (\Rightarrow) This is just an extension of Lemma 5.4.2.

(\Leftarrow) Since $f = \rho_\theta$, and letting $X = V$, we can see that this is just an extension of Proposition 5.4.3. \square

Condition 5.4.5. For each k , $1 \leq k \leq m$, and for all sequences $1 \leq l_1 < l_2 < \dots < l_k \leq m$,

$$\left| v_{l_1} \cap \dots \cap v_{l_k} \right| = \left| (w_1 \Delta w_{l_1}) \cap \dots \cap (w_1 \Delta w_{l_k}) \right|$$

Corollary 5.4.6. For $V = \{\emptyset, v_2, v_3, \dots, v_m\}$ and $W = \{w_1, w_2, \dots, w_m\}$, V and W satisfy Condition 5.4.5 if and only if there exists an automorphism $f \in \text{Aut}(Q_n)$ such that $f(v_i) = w_i$, $f = \sigma_{w_1} \rho_\theta$, θ a permutation of $[n]$, $1 \leq i \leq m$.

Proof. (\Rightarrow) By Condition 5.4.5, we have

$$\begin{aligned} \left| v_{l_1} \cap \dots \cap v_{l_k} \right| &= \left| (w_1 \Delta w_{l_1}) \cap \dots \cap (w_1 \Delta w_{l_k}) \right| \\ &= \left| \sigma_{w_1}(w_{l_1}) \cap \dots \cap \sigma_{w_1}(w_{l_k}) \right| \\ &= \left| w_{l_1}^* \cap \dots \cap w_{l_k}^* \right| \end{aligned}$$

where $w_i^* = \sigma_{w_1}(w_i)$. That is, $W^* = \sigma_{w_1}(W)$. By Corollary 5.4.4, there exists a permutation on $[n]$ such that $\rho_\theta(v_i) = w_i^*$, $1 \leq i \leq m$. Since $\sigma_{w_1} = \sigma_{w_1}^{-1}$, and letting $f = \sigma_{w_1} \rho_\theta$, we have $f(v_i) = w_i$, $1 \leq i \leq m$. This composition is illustrated in Figure 5.1.

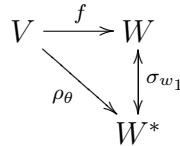


Figure 5.1: Diagram for Corollary 5.4.6

(\Leftarrow) Since $f = \sigma_{w_1}\rho_\theta$ and $f(v_i) = w_i$ for all i , $1 \leq i \leq m$, we have $\sigma_{w_1}\rho_\theta(v_i) = w_i$ or, equivalently,

$$\rho_\theta(v_i) = \sigma_{w_1}^{-1}(w_i) = \sigma_{w_1}(w_i) = w_1\Delta w_i \quad (5.4.1)$$

By applying Proposition 5.4.3 to V , we can see

$$\left| v_{l_1} \cap \dots \cap v_{l_k} \right| = \left| \rho_\theta(v_{l_1}) \cap \dots \cap \rho_\theta(v_{l_k}) \right|$$

for any k , $1 \leq k \leq m$ and for any $\{l_1, \dots, l_k\}$, $1 \leq l_1 < l_2 < \dots < l_k \leq m$, and so by (5.4.1)

$$\left| v_{l_1} \cap \dots \cap v_{l_k} \right| = \left| (w_1\Delta w_{l_1}) \cap \dots \cap (w_1\Delta w_{l_k}) \right|$$

Therefore, V and W satisfy Condition 5.4.5. □

Condition 5.4.7. For each k , $1 \leq k \leq m$, and for all sequences $1 \leq l_1 < l_2 < \dots < l_k \leq m$,

$$\left| (v_1\Delta v_{l_1}) \cap \dots \cap (v_1\Delta v_{l_k}) \right| = \left| (w_1\Delta w_{l_1}) \cap \dots \cap (w_1\Delta w_{l_k}) \right|$$

Corollary 5.4.8. For $V = \{v_1, v_2, \dots, v_m\}$ and $W = \{w_1, w_2, \dots, w_m\}$, V and W satisfy Condition 5.4.7 if and only if there exists an automorphism $f \in \text{Aut}(Q_n)$ such that $f(v_i) = w_i$, $f = \sigma_{w_1\Delta\theta(v_1)}\rho_\theta$, θ a permutation of $[n]$, $1 \leq i \leq m$.

Proof. (\Rightarrow) By Condition 5.4.7, we have

$$\begin{aligned} |(v_1 \Delta v_{i_1}) \cap \dots \cap (v_1 \Delta v_{i_k})| &= |(w_1 \Delta w_{i_1}) \cap \dots \cap (w_1 \Delta w_{i_k})| \\ |\sigma_{v_1}(v_{i_1}) \cap \dots \cap \sigma_{v_1}(v_{i_k})| &= |\sigma_{w_1}(w_{i_1}) \cap \dots \cap \sigma_{w_1}(w_{i_k})| \\ |v_{i_1}^* \cap \dots \cap v_{i_k}^*| &= |w_{i_1}^* \cap \dots \cap w_{i_k}^*| \end{aligned}$$

where $v_i^* = \sigma_{v_1}(v_i)$ and $w_i^* = \sigma_{w_1}(w_i)$. That is, $V^* = \sigma_{v_1}(V)$ and $W^* = \sigma_{w_1}(W)$. By Corollary 5.4.4, there exists a permutation on $[n]$ such that $\rho_\theta(v_i^*) = w_i^*$, $1 \leq i \leq m$. Since $\sigma_\alpha = \sigma_\alpha^{-1}$ for all α , and letting $f = \sigma_{w_1} \rho_\theta \sigma_{v_1} = \sigma_{w_1 \Delta \theta(v_1)} \rho_\theta$, we have $f(v_i) = w_i$, $1 \leq i \leq m$. This composition is illustrated in Figure 5.2.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \uparrow \sigma_{v_1} & & \uparrow \sigma_{w_1} \\ V^* & \xrightarrow{\rho_\theta} & W^* \end{array}$$

Figure 5.2: Diagram for Corollary 5.4.8

(\Leftarrow) Since $f = \sigma_{w_1 \Delta \theta(v_1)} \rho_\theta = \sigma_{w_1} \rho_\theta \sigma_{v_1}$ and $f(v_i) = w_i$ for all i , $1 \leq i \leq m$, we have $\sigma_{w_1} \rho_\theta \sigma_{v_1}(v_i) = w_i$ or, equivalently,

$$\rho_\theta \sigma_{v_1}(v_i) = \sigma_{w_1}^{-1}(w_i) = \sigma_{w_1}(w_i) = w_1 \Delta w_i$$

and, thus,

$$\rho_\theta(v_1 \Delta v_i) = w_1 \Delta w_i \tag{5.4.2}$$

Letting $v_i^* = v_1 \Delta v_i$ and applying Proposition 5.4.3 to V , we can see that, for any

k , $1 \leq k \leq m$, and for any $\{l_1, \dots, l_k\}$, $1 \leq l_1 < l_2 < \dots < l_k \leq m$,

$$\begin{aligned}
\left| (v_1 \Delta v_{l_1}) \cap \dots \cap (v_1 \Delta v_{l_k}) \right| &= \left| v_{l_1}^* \cap \dots \cap v_{l_k}^* \right| \\
&= \left| \rho_\theta(v_{l_1}^*) \cap \dots \cap \rho_\theta(v_{l_k}^*) \right| \\
&= \left| \rho_\theta(v_1 \Delta v_{l_1}) \cap \dots \cap \rho_\theta(v_1 \Delta v_{l_k}) \right| \\
&= \left| (w_1 \Delta w_{l_1}) \cap \dots \cap (w_1 \Delta w_{l_k}) \right| \quad (\text{by (5.4.2)})
\end{aligned}$$

Therefore, V and W satisfy Condition 5.4.7. □

Thus, Q_n is intersection m -transitive.

Notation. For $A \subset \mathbf{V}_m$, $A = \{a_1, \dots, a_k\}$, let $\bar{A} = \mathbf{V}_m \setminus A$ and define

$$s(A) = \{p : p \in a_1 \cap \dots \cap a_k, p \notin b, b \in \bar{A}\}$$

and let $n(A) = |s(A)|$. That is, $n(A)$ counts the number of elements that are common to every member of A but do not appear in any member of \bar{A} .

Example 5.4.9. Let $V = \{13, 245, 1234, 23457\}$. If $A_1 = \{13, 1234\}$, then $s(A_1) = \{1\}$ and $n(A_1) = 1$. If $A_2 = \{245, 1234, 23457\}$, then $s(A_2) = \{24\}$ and $n(A_2) = 2$. If $A_3 = \{13\}$, then $s(A_3) = \emptyset$ and $n(A_3) = 0$.

Lemma 5.4.10. For a subset A of \mathbf{V}_m ,

$$\prod_{1 \leq k \leq m} \prod_{\substack{A \subset \mathbf{V}_m \\ |A|=k}} n(A)!$$

counts the number of possible automorphisms, f , that can be constructed from Corollary 5.4.4.

Proof. $n(A)$ counts the number of elements common to each set in A that do not appear in any other member of \mathbf{V}_m . Since $f = \rho_\theta$, we have that θ can map any one of these elements to any corresponding element in \mathbf{W}_m . (By Lemma 5.4.2 we know $n(A) = n(B)$, B the corresponding k -subset in \mathbf{W}_m . For, if not, then θ would not be a permutation.) Therefore, $n(A)$ partitions $[n]$ into disjoint subsets. Restricting our view to $s(A)$, we have that there are $n(A)!$ possibilities for constructing θ . Thus, by considering all possible subsets of \mathbf{V}_m , we obtain the desired result. \square

Corollary 5.4.11. *Lemma 5.4.10 also counts the number of possible automorphisms, f , in Corollaries 5.4.6 and 5.4.8.*

Proof. In both Corollary 5.4.6 and Corollary 5.4.8, $f = \sigma_\alpha \rho_\theta$ for some α . In each case, there exists only one possibility for σ_α , so we need only count the number of possible permutations θ in ρ_θ . This is precisely what is given in Lemma 5.4.10. \square

Theorem 5.4.12. *Distance transitivity is equivalent to intersection 2-transitivity in Q_n .*

Proof. By Proposition 5.3.3, we just need to show that distance transitivity is necessary for intersection 2-transitivity.

Since Condition 5.4.1 and Condition 5.4.5 can be generalized to Condition 5.4.7, we only need to consider this latter situation and the three conditions that follow from it:

$$\begin{aligned} |v_1 \Delta v_1| &= |w_1 \Delta w_1| \\ |v_1 \Delta v_2| &= |w_1 \Delta w_2| \\ |(v_1 \Delta v_1) \cap (v_1 \Delta v_2)| &= |(w_1 \Delta w_1) \cap (w_1 \Delta w_2)| \end{aligned}$$

or equivalently,

$$\begin{aligned} |\emptyset| &= |\emptyset| \\ |v_2^*| &= |w_2^*| \\ |\emptyset \cap v_2^*| &= |\emptyset \cap w_2^*| \end{aligned}$$

where $v_2^* = v_1 \Delta v_2$ and $w_2^* = w_1 \Delta w_2$.

Let \mathbf{V}_2 and \mathbf{W}_2 be two 2-sets such that $d(v_1, v_2) = d(w_1, w_2)$. Then,

$$d(\emptyset, v_2^*) = d(v_1 \Delta v_1, v_1 \Delta v_2) = d(v_1, v_2) = d(w_1, w_2) = d(w_1 \Delta w_1, w_1 \Delta w_2) = d(\emptyset, w_2^*)$$

Thus, $d(\emptyset, v_2^*) = d(\emptyset, w_2^*)$ and we have $|v_2^*| = |w_2^*|$. Also, $|\emptyset \cap A| = |\emptyset| = 0$ for any set $A \subset [n]$. Therefore, \mathbf{V}_2 and \mathbf{W}_2 satisfy Condition 5.4.7 for $m = 2$. Since Q_n is distance transitive, there exists an $f \in \text{Aut}(Q_n)$ such that $f(v_i) = w_i$, $i = 1, 2$. Using the automorphism associated with each $\mathbf{V}_2, \mathbf{W}_2$ pair, we have Q_n is intersection 2-transitive. Hence, in Q_n , distance transitivity is equivalent to intersection 2-transitivity. \square

5.5 Δ m -Transitivity in G

First, let us see how intersection and symmetric difference are interrelated.

Proposition 5.5.1. *For any set $\{x_1, x_2, \dots, x_m\}$,*

$$|x_1 \Delta x_2 \Delta \dots \Delta x_m| = \sum_{\substack{i=1 \\ 1 \leq l_1 < l_2 < \dots < l_i \leq m}}^m (-2)^{i-1} |x_{l_1} \cap \dots \cap x_{l_i}|$$

Proof. Let us proceed by induction on m . For $m = 1$ we have $|x_1| = (-2)^0 |x_1|$. Now,

assume this equality holds true for $m - 1$. Then,

$$\begin{aligned}
|x_1 \Delta \dots \Delta x_{m-1} \Delta x_m| &= |x_1 \Delta \dots \Delta x_{m-1}| + |x_m| - 2|(x_1 \Delta \dots \Delta x_{m-1}) \cap x_m| \\
&= |x_1 \Delta \dots \Delta x_{m-1}| + |x_m| \\
&\quad - 2|(x_1 \cap x_m) \Delta (x_2 \cap x_m) \Delta \dots \Delta (x_{m-1} \cap x_m)| \\
&= \sum_{\substack{i=1 \\ 1 \leq l_1 < \dots < l_i \leq m-1}}^{m-1} (-2)^{i-1} |x_{l_1} \cap \dots \cap x_{l_i}| \\
&\quad + |x_m| - 2 \sum_{\substack{i=1 \\ 1 \leq l_1 < \dots < l_i \leq m-1}}^{m-1} (-2)^{i-1} |(x_{l_1} \cap x_m) \cap \dots \cap (x_{l_i} \cap x_m)| \\
&= \sum_{\substack{i=1 \\ 1 \leq l_1 < \dots < l_i \leq m-1}}^{m-1} (-2)^{i-1} |x_{l_1} \cap \dots \cap x_{l_i}| \\
&\quad + |x_m| - 2 \sum_{\substack{i=1 \\ 1 \leq l_1 < \dots < l_i \leq m-1}}^{m-1} (-2)^{i-1} |x_{l_1} \cap \dots \cap x_{l_i} \cap x_m| \\
&= \sum_{\substack{i=1 \\ 1 \leq l_1 < \dots < l_i \leq m-1}}^{m-1} (-2)^{i-1} |x_{l_1} \cap \dots \cap x_{l_i}| \\
&\quad - 2 \sum_{\substack{i=0 \\ 1 \leq l_1 < \dots < l_i \leq m-1}}^{m-1} (-2)^{i-1} |x_{l_1} \cap \dots \cap x_{l_i} \cap x_m| \\
&= \sum_{\substack{i=1 \\ 1 \leq l_1 < \dots < l_i \leq m}}^m (-2)^{i-1} |x_{l_1} \cap \dots \cap x_{l_i}|
\end{aligned}$$

□

Condition 5.5.2. *There exists two subsets of $[n]$, A and B , such that, for each k ,*

$$\left| (A \Delta v_{l_1}) \Delta \dots \Delta (A \Delta v_{l_k}) \right| = \left| (B \Delta w_{l_1}) \Delta \dots \Delta (B \Delta w_{l_k}) \right|$$

$$1 \leq l_1 < l_2 < \dots < l_k \leq m, 1 \leq k \leq m.$$

Definition 5.5.3. If, for every V_m and W_m that satisfy Condition 5.5.2, there exists an automorphism $f \in \text{Aut}(G)$ such that $f(v_i) = w_i$ for all i , $1 \leq i \leq m$, then G is Δ m -transitive.

Theorem 5.5.4. For any subgraph G of Q_n , intersection m -transitivity is equivalent to Δ m -transitivity.

Proof. Let V_m and W_m be two m -sets that satisfy Condition 5.3.1. Then, for each k , $1 \leq k \leq m$ and for all sequences $1 \leq l_1 < \dots < l_k \leq m$,

$$|v_{l_1}^* \cap \dots \cap v_{l_k}^*| = |w_{l_1}^* \cap \dots \cap w_{l_k}^*|$$

where $v_i^* = A\Delta v_i$ and $w_i^* = B\Delta w_i$.

By Proposition 5.5.1, we have that, for each k , $1 \leq k \leq m$ and for all sequences $1 \leq p_1 < \dots < p_k \leq m$,

$$\begin{aligned} \left| v_{p_1}^* \Delta \dots \Delta v_{p_k}^* \right| &= \sum_{\substack{i=1 \\ i \leq p_{l_1} < \dots < p_{l_i} \leq m}}^m (-2)^{i-1} \left| v_{p_{l_1}}^* \cap \dots \cap v_{p_{l_i}}^* \right| \\ &= \sum_{\substack{i=1 \\ i \leq p_{l_1} < \dots < p_{l_i} \leq m}}^m (-2)^{i-1} \left| w_{p_{l_1}}^* \cap \dots \cap w_{p_{l_i}}^* \right| \\ &= \left| w_{p_1}^* \Delta \dots \Delta w_{p_k}^* \right| \end{aligned}$$

Therefore, V_m and Q_m satisfy Condition 5.5.2. Since G is intersection m -transitive, there is some $f \in \text{Aut}(G)$ such that $f(v_i) = w_i$ for all $1 \leq i \leq m$. Using the corresponding automorphism for each set $\{V_m, W_m\}$, we can see that G is Δ m -transitive.

Now, let V_m and W_m be two m -sets that satisfy Condition 5.5.2. Then, for $k = 1$, we have $|v_{l_1}^*| = |w_{l_1}^*|$ for all l_1 , $1 \leq l_1 \leq m$, which is equivalent to Condition 5.3.1

with $k = 1$.

Rewriting Proposition 5.5.1 we have, for any t ,

$$(-2)^{t-1}|x_1 \cap \dots \cap x_t| = |x_1 \Delta \dots \Delta x_t| - \sum_{\substack{i=1 \\ i \leq l_1 < \dots < l_i \leq t}}^{t-1} (-2)^{i-1} |x_{l_1} \cap \dots \cap x_{l_i}|$$

So, for $k = 2$, we have, for any $1 \leq p_1 < p_2 \leq m$,

$$\begin{aligned} -2 \left| v_{p_1}^* \cap v_{p_2}^* \right| &= \left| v_{p_1}^* \Delta v_{p_2}^* \right| - \left| v_{p_1}^* \right| - \left| v_{p_2}^* \right| \\ &= \left| w_{p_1}^* \Delta w_{p_2}^* \right| - \left| w_{p_1}^* \right| - \left| w_{p_2}^* \right| \\ &= -2 \left| w_{p_1}^* \cap w_{p_2}^* \right| \end{aligned}$$

That is, $\left| v_{p_1}^* \cap v_{p_2}^* \right| = \left| w_{p_1}^* \cap w_{p_2}^* \right|$. Using this equality and incrementing k from 3 to m , we can see that we have $\left| v_{p_1}^* \cap \dots \cap v_{p_k}^* \right| = \left| w_{p_1}^* \cap \dots \cap w_{p_k}^* \right|$ for all k , $1 \leq k \leq m$, and for all sequences $1 \leq p_1 < \dots < p_k \leq m$. Hence, V_m and W_m satisfy Condition 5.3.1. Since G is Δ m -transitive, we can use these associated automorphisms for each pair V_m, W_m and conclude that G is intersection m -transitive. \square

Therefore, Q_n is Δ m -transitive.

5.6 Middle 2 Levels of Q_{2k+1}

The middle 2 levels of Q_{2k+1} are L_k and L_{k+1} .

Definition 5.6.1. Let $Q_n(X)$ be the induced subgraph of Q_n with vertex set X .

Recall the following definition from Section 2.1

First, let us show that distance is preserved when restricting Q_n to two adjacent levels. Let $n = 2k + 1$.

Proposition 5.6.2. For all $x, y \in V(Q_n(L_k, L_{k+1}))$, $d_{Q_n(L_k, L_{k+1})}(x, y) = d_{Q_n}(x, y)$.

Proof. Clearly, $d_{Q_n(L_k, L_{k+1})}(x, y) \geq d_{Q_n}(x, y)$.

Consider the path P with initial vertex x and degree sequence s_1, s_2, \dots, s_l . Then

$$P = \{x, x\Delta s_1, x\Delta s_1\Delta s_2, \dots, x\Delta s_1\Delta \dots \Delta s_l\}$$

Let $y \in V(Q_n(L_k, L_{k+1}))$ and $d_{Q_n}(x, y) = l$. For $x \in L_k$, add the following three restrictions to P : $s_j \in x\Delta y$, $s_i \neq s_j$ for $1 \leq i, j \leq l$, $i \neq j$; $s_{2j+1} \in y$, $0 \leq j \leq \lfloor \frac{l-1}{2} \rfloor$; $s_{2k} \in x$, $1 \leq j \leq \lceil \frac{l-1}{2} \rceil$. Now,

$$s_1\Delta s_2\Delta \dots \Delta s_l = x\Delta y$$

which gives

$$x\Delta(s_1\Delta s_2\Delta \dots \Delta s_l) = x\Delta(x\Delta y) = y$$

So P is a path of length l between x and y . By the restrictions on P , $s_1 \notin x$, which implies $x\Delta s_1 \in L_{k+1}$. Now, $s_2 \in L_k$, so $(x\Delta s_1)\Delta s_2 \in L_k$. Continuing this, we see that $x\Delta s_1\Delta \dots \Delta s_t$ is in L_k if t is even and in L_{k+1} if t is odd. That is, P is a path of length l from x to y that alternates between L_k and L_{k+1} . Thus, for $x \in L_k$, $d_{Q_n(L_k, L_{k+1})}(x, y) \leq d_{Q_n}(x, y)$.

For $x \in L_{k+1}$ we may proceed in the same fashion with the following 3 restrictions on P : $s_j \in x\Delta y$, $s_i \neq s_j$ for $1 \leq i, j \leq l$, $i \neq j$; $s_{2j+1} \in x$, $0 \leq j \leq \lfloor \frac{l-1}{2} \rfloor$; $s_{2j} \in y$, $1 \leq j \leq \lceil \frac{l-1}{2} \rceil$. Again, $x\Delta s_1\Delta \dots \Delta s_l = y$, so P is a path of length l between x and y . By the restrictions on P , we have $s_1 \in x$. That is, $x\Delta s_1 \in L_k$. Similarly, $s_2 \notin x$, which gives $(x\Delta s_1)\Delta s_2 \in L_{k+1}$. Continuing in this manner, we can see that

$x\Delta s_1\Delta \dots \Delta s_t$ is a member of L_{k+1} if t is even and a member of L_k if t is odd.

Therefore, for $x \in L_{k+1}$, $d_{Q_n(L_k, L_{k+1})}(x, y) \leq d_{Q_n}(x, y)$.

Thus, for all $x, y \in V(Q_n(L_k, L_{k+1}))$, $d_{Q_n(L_k, L_{k+1})}(x, y) \leq d_{Q_n}(x, y)$. \square

Now, we need to determine the automorphisms in $Q_{2k+1}(L_k, L_{k+1})$.

Notation. Let $\text{Stab}_k(x)$ denote $\text{Stab}(x) \cap \text{Aut}(Q_n(L_k, L_{k+1}))$, where $\text{Stab}(x)$ is the stabilizer subgroup of $\text{Aut}(Q_n)$.

Theorem 5.6.3. *The automorphism group of the middle two levels of Q_n is*

$$\text{Aut}(Q_n(L_k, L_{k+1})) = \langle \sigma_{[n]}, \rho_\theta | \theta \in \mathcal{S}_n \rangle$$

Proof. Consider the group $\mathcal{G} = \langle \sigma_A, \rho_\theta | A = [n], \theta \in \mathcal{S}_n \rangle$. Recall that $\rho_\theta \sigma_A = \sigma_{\theta(A)} \rho_\theta$.

Now, $A = [n]$ or \emptyset , so

$$\rho_\theta \sigma_A = \sigma_{\theta(A)} \rho_\theta = \sigma_A \rho_\theta \tag{5.6.1}$$

Also,

$$(\sigma_A \rho_\theta)^{-1} = (\rho_\theta)^{-1} (\sigma_A)^{-1} = \rho_{\theta^{-1}} \sigma_A = \rho_{\theta'} \sigma_A = \sigma_{\theta'(A)} \rho_{\theta'} = \sigma_A \rho_{\theta'} \tag{5.6.2}$$

where $\theta' = \theta^{-1}$ and noting that the order of σ_A is 2.

Thus, any composition of the elements of \mathcal{G} can be expressed as some $\sigma_A \rho_\theta$ with $A = [n]$ or \emptyset and $\theta \in \mathcal{S}_n$.

Let $x \in L_k$. Then $\sigma_A \rho_\theta(x) = \sigma_A(\rho_\theta(x)) = \sigma_A(y) \in L_k$, with $y = \rho_\theta(x)$. Now, for $A = \emptyset$, $\sigma_\emptyset(y) = y \in L_k$, and if $A = [n]$ we have $\sigma_{[n]}(y) = [n]\Delta y \in L_{k+1}$. Similarly, we can show that for any $x \in L_{k+1}$, $\sigma_A \rho_\theta(x) \in V(Q_n(L_k, L_{k+1}))$. Therefore, for all $g \in \mathcal{G}$, g maps $V(Q_n(L_k, L_{k+1}))$ to itself, and similarly for $E(Q_n(L_k, L_{k+1}))$. Since

$g \in \text{Aut}(Q_n)$, $g \in \text{Aut}(Q_n(L_k, L_{k+1}))$. So, by (5.6.1) and (5.6.2), we can see that \mathcal{G} is a group under composition with identity $\sigma_\emptyset \rho_{(1)}$.

Now, consider two vertices x and y in $V(Q_n(L_k, L_{k+1}))$. If $\text{wt}(x) = \text{wt}(y)$, then we can find a permutation $\theta \in \mathcal{S}_n$ such that $\rho_\theta(x) = y$. If $\text{wt}(x) \neq \text{wt}(y)$, we have $\text{wt}(y\Delta[n]) = \text{wt}(x)$, and by the previous statement, we can find a permutation θ so that $\rho_\theta(x) = y\Delta[n]$. Thus, by (5.6.1), $\sigma_{[n]}\rho_\theta(x) = y$. Therefore, \mathcal{G} acts transitively on $V(Q_n(L_k, L_{k+1}))$.

We just need to consider $|\text{Aut}(Q_n(L_k, L_{k+1}))| = |\text{orbit}(x)||\text{Stab}_k(x)|$. Since $|V(Q_n(L_k, L_{k+1}))| = \binom{n}{k} + \binom{n}{k+1} = 2\binom{n}{k}$ and each vertex can be sent to any other vertex under an automorphism, $|\text{orbit}(x)| = 2\binom{n}{k}$. Now, let us determine the size of the stabilizer of x . For this, we need the following lemma.

Lemma 5.6.4. *For any $f \in \text{Stab}_k(x)$, $f = \rho_\theta$, where $\theta \in \mathcal{S}_n$.*

Proof. Let $f \in \text{Stab}_k(x)$ and let $\mathcal{L}_i = \{y \in Q_n(L_k, L_{k+1}) | d(x, y) = i\}$, $0 \leq i \leq n$. Now, for each $x \in V(Q_n)$, there is a unique x^c such that $d_{Q_n}(x, x^c) = n$. So, by Proposition 5.6.2, for $x \in V(Q_n(L_k, L_{k+1}))$, x^c is also the unique element of $Q_n(L_k, L_{k+1})$ such that $d_{Q_n(L_k, L_{k+1})}(x, x^c) = n$. Therefore, $f(x^c) = x^c$ and, f fixes $\mathcal{L}_n = \{x^c\}$. Without loss of generality, we may assume $x \in L_k$. For a set $z \subset V(Q_n(L_k, L_{k+1}))$, define $f(z) = f(\{z_1, z_2, \dots, z_j\}) = \{f(z_1), f(z_2), \dots, f(z_j)\}$. Then, $\mathcal{L}_0 = \{x\}$, $\mathcal{L}_n = \{x^c\}$, and, again, $f(x^c) = x^c$.

Thus, f fixes \mathcal{L}_0 and \mathcal{L}_n and permutes the members of \mathcal{L}_1 and \mathcal{L}_{n-1} among themselves. Since $\mathcal{L}_1 = \{x \cup \{i\} | i \in x^c\}$, we have that f gives rise to a permutation φ_{x^c} on the elements of x^c . Note that when $\rho_{\varphi_{x^c}}$ is applied to \mathcal{L}_n , \mathcal{L}_n is fixed, and when applied to \mathcal{L}_{n-1} , this level has its members permuted.

We can extend φ_{x^c} to $\theta_0 \in \mathcal{S}_n$ by defining a permutation, φ_x , on the elements of x and setting $\theta_0 = \varphi_x \circ \varphi_{x^c}$. Note that $\rho_\theta \in \text{Aut}(Q_n(L_k, L_{k+1}))$. Again, ρ_{θ_0} leaves

\mathcal{L}_0 fixed and permutes the members of \mathcal{L}_1 , while its action on \mathcal{L}_n and \mathcal{L}_{n-1} remains unchanged. Therefore, ρ_{θ_0} agrees with f on levels $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_{n-1}$ and \mathcal{L}_n .

Define $f_1 = \rho_{\theta_0}^{-1}f$. We have $f_1 \in \text{Aut}(Q_n(L_k, L_{k+1}))$ and f_1 is the identity on $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_{n-1}$ and \mathcal{L}_n . Also, as the composition of automorphisms of $Q_n(L_k, L_{k+1})$ is an automorphism and, therefore, distance preserving, f_1 preserves distances. Hence f_1 permutes the neighbors of y for any $y \in \mathcal{L}_1 \cup \mathcal{L}_{n-1}$. Let $y \in \mathcal{L}_{n-1}$. Then $N(y) \cap \mathcal{L}_{n-2} = \{y \cup \{j\} | j \in x\}$. Since f_1 permutes the neighbors of y , for each y , f_1 gives rise to a permutation ψ_x of the elements of x . We need the following proposition.

Proposition 5.6.5. *Let $x \in L_k$. If $\theta \in \mathcal{S}_n$ is a permutation such that ρ_θ is the identity on \mathcal{L}_0 and \mathcal{L}_1 , then θ must be the identity on every element in x^c .*

Proof. Let ρ_θ be the identity on \mathcal{L}_0 and \mathcal{L}_1 . Then $\rho_\theta(x) = x$ and, for all $y \in \mathcal{L}_1$, $\rho_\theta(y) = y$, where $y = x \cup \{j\}$, $j \in x^c$. Thus, for each y ,

$$x \cup \{j\} = y = \rho_\theta(y) = \rho_\theta(x \cup \{j\}) = \rho_\theta(x) \cup \{\theta(j)\} = x \cup \{\theta(j)\}$$

and we have $\theta(j) = j$, $i \in x^c$. Since $\mathcal{L}_1 = \{x \cup \{j\} | j \in x^c\}$, we have, for all j , θ fixes j . That is, θ acts as the identity on every element of x^c . \square

Since f_1 fixes \mathcal{L}_0 and \mathcal{L}_1 , by Proposition 5.6.5, any permutation $\theta_1 \in \mathcal{S}_n$ that agrees with f_1 on \mathcal{L}_0 and \mathcal{L}_1 must be the identity on any element in x^c . Thus, for each y , and for $\theta_1 \in \mathcal{S}_n$ where $\theta_1 = \psi_x \circ (1) = \psi_x$, clearly we have that ρ_{θ_1} acts as the identity on $\mathcal{L}_0, \mathcal{L}_1$, and \mathcal{L}_n . Also, since $\rho_\theta(y) = y$, for any $y \in \mathcal{L}_1$, we have $\rho_\theta(y^c) = y^c$ for every $y^c \in \mathcal{L}_{n-1}$. Hence, ρ_{θ_1} fixes \mathcal{L}_{n-1} . We need to show that for any $y \in \mathcal{L}_1 \cap \mathcal{L}_{n-1}$, ρ_{θ_1} permutes the neighbors in \mathcal{L}_2 and \mathcal{L}_{n-2} consistently.

We have, for $y \in \mathcal{L}_{n-1}$ and $z \in \mathcal{L}_{n-2}$, z a neighbor of y ,

$$d(x^c, z) = 2 = d(f_1(x^c), f_1(z)) = d(x^c, f_1(z))$$

and

$$d(y, z) = 1 = d(f_1(y), f_1(z)) = d(y, f_1(z))$$

So, $f_1(z) \in \mathcal{L}_{n-2}$ and $f_1(z)$ is a neighbor of y . For any permutation θ in \mathcal{S}_n , for ρ_θ to agree with f_1 , we must have $f_1(z) = f_1(y \cup \{j\}) = f_1(y) \cup \{\theta_i(j)\} = y \cup \{\theta_i(j)\}$, where $y = x^c \setminus \{i\}$, $j \in x$, $\theta_i(j) \in x$, for some $\theta_i \in \mathcal{S}_n$.

Let $z' = y \cup \{j'\}$, another neighbor of y . Then,

$$2 = d(z, z') = d(f_1(z), f_1(z')) = d(y \cup \{\theta_i(j)\}, y \cup \{\theta_{i'}(j')\})$$

which gives $\theta_i(j) \neq \theta_{i'}(j')$ for all $j, j' \in x$, $j \neq j'$.

Now, consider $\bar{z} = y' \cup \{j\}$, $y' \neq y$ (That is, $y' = x^c \setminus \{i'\}$, $i' \neq i$). Then,

$$2 = d(z, \bar{z}) = d(f_1(z), f_1(\bar{z})) = d(y \cup \{\theta_i(j)\}, y' \cup \{\theta_{i'}(j)\})$$

Since $d(y, y') = 2$, we must have $\theta_i(j) = \theta_{i'}(j)$ for all $i, i' \in x^c$, for all $j \in x$.

Thus, we have shown that any θ that permutes any lower neighborhood of y acts consistently on the other members of \mathcal{L}_{n-2} . That is, such a θ would permute these neighborhoods so that distance is preserved as in f_1 . Now, restricting our view to the members of \mathcal{L}_{n-2} , we have $N(y) = \{y \cup \{j\} | j \in x\}$ and

$$\begin{aligned}
\rho_{\theta_1}(N(y)) &= \rho_{\theta_1}(\{y \cup \{j\} | j \in x\}) \\
&= \{\rho_{\theta_1}(y \cup \{j\}) | j \in x\} \\
&= \{y \cup \{\theta_1(j)\} | j \in x\} \\
&= \{y \cup \{\theta_1(j)\} | \theta(j) \in x\} \\
&= N(y)
\end{aligned}$$

Similarly, restricting our view to \mathcal{L}_2 , for $\bar{y} \in \mathcal{L}_1$, we can see $\rho_{\theta_1}(N(\bar{y})) = N(\bar{y})$.

Thus, θ_1 agrees with f_1 on \mathcal{L}_2 and \mathcal{L}_{n-2} , as we needed to show.

Define $f_2 = \rho_{\theta_1}^{-1}f_1$. Then f_2 acts as the identity on \mathcal{L}_i , $i = 0, 1, 2, n-2, n-1, n$.

We need another proposition.

Proposition 5.6.6. *For $x \in L_k$, if ρ_θ acts as the identity on every member of \mathcal{L}_{n-1} and \mathcal{L}_{n-2} , then θ must be the identity on every element of x .*

Proof. Let ρ_θ be the identity on \mathcal{L}_{n-1} and \mathcal{L}_{n-2} . Then, $\rho_\theta(y) = y$ for all $y \in \mathcal{L}_{n-1}$ and $\rho_\theta(z) = z$ for any $z \in \mathcal{L}_{n-2}$. Now, for each z , $z = y \cup \{i\}$, $i \in x$, and

$$y \cup \{i\} = z = \rho_\theta(z) = \rho_\theta(y \cup \{i\}) = \rho_\theta(y) \cup \{\theta(i)\} = y \cup \{\theta(i)\}$$

Therefore, we have $\theta(i) = i$ for all $i \in x$ and so θ acts as the identity on each element of this set. □

By Propositions 5.6.5 and 5.6.6, any $\theta_2 \in \mathcal{S}_n$ that agrees with f_2 on these levels must be the identity on all elements of $x \cup x^c = [n]$. Thus, $\theta_2 = id$. Define $f_3 = \rho_{\theta_2}^{-1}f_2$. So, f_3 acts as the identity on \mathcal{L}_i , $i = 0, 1, 2, 3, n-3, n-2, n-1, n$, and, again, by Propositions 5.6.5 and 5.6.6, we have $\theta_3 = id$. Continue defining $f_{t+1} = \rho_{\theta_t}^{-1}f_t$ for

$4 \leq t < k$, noting that, in each case, $\theta_t = id$. Now, $f_k = id$ on levels $\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_k$, and $\mathcal{L}_{n-k}, \mathcal{L}_{n-k+1}, \dots, \mathcal{L}_n$. That is, f_k is the identity on all levels \mathcal{L}_i , $0 \leq i \leq n$. Thus,

$$\begin{aligned}
id &= f_k \\
&= \rho_{\theta_{k-1}}^{-1} f_{k-1} \\
&= \rho_{\theta_{k-1}}^{-1} \rho_{\theta_{k-2}}^{-1} \cdots \rho_{\theta_1}^{-1} \rho_{\theta_0}^{-1} f \\
&= (\rho_{\theta_0} \rho_{\theta_1} \cdots \rho_{\theta_{k-2}} \rho_{\theta_{k-1}})^{-1} f
\end{aligned}$$

So, $f = \rho_{\theta_0} \rho_{\theta_1} \cdots \rho_{\theta_{k-2}} \rho_{\theta_{k-1}} = \rho_\theta$, where $\theta = \theta_0 \theta_1 \dots \theta_{k-1} \in \mathcal{S}_n$. Therefore, for any $f \in \text{Stab}_k(x)$, f can be written as ρ_θ , for some $\theta \in \mathcal{S}_n$. \square

Let $x \in L_k$. Then, by Lemma 5.6.4, we can apply a permutation such that we permute the k elements in x and, separately, permute the $n - k$ elements not in x . From this, we can see $|\text{Stab}_k(x)| = k!(n - k)!$. If $x \in L_{k+1}$, we can apply the same process to arrive at the same result. Thus,

$$\begin{aligned}
|\text{Aut}(Q_n(L_k, L_{k+1}))| &= |\text{orbit}(x)| |\text{Stab}_k(x)| \\
&= 2 \binom{n}{k} (k!(n - k)!) \\
&= 2 \left(\frac{n!}{k!(n - k)!} \right) k!(n - k)! \\
&= 2n!
\end{aligned}$$

Since $|\mathcal{G}| = |A| |\mathcal{S}_n| = 2n!$, we have \mathcal{G} is the full automorphism group of $Q_n(L_k, L_{k+1})$. \square

Corollary 5.6.7. *Let φ_A be a permutation of $A \subset [n]$ and let $f = \rho_\theta \in \text{Stab}_k(x)$,*

with $\theta \in \mathcal{S}_n$. Then $\theta = \varphi_x \circ \varphi_{x^c}$.

Proof. By the proof of Proposition 5.6.4, we can see that

$$\begin{aligned} f &= \rho_{\theta_0} \rho_{\theta_1} \cdots \rho_{\theta_{k-2}} \rho_{\theta_{k-1}} \\ &= \rho_{\varphi_x \circ \varphi_{x^c}} \rho_{\psi_x} \rho_{(1)} \cdots \rho_{(1)} \\ &= \rho_{\varphi_x \circ \varphi_{x^c} \circ \psi_x} \\ &= \rho_{\varphi'_x \circ \varphi_{x^c}} \end{aligned}$$

where $\varphi'_x = \varphi_x \circ \psi_x$. Therefore, for any $f \in \text{Stab}_k(x)$, f can be written as some ρ_θ , where $\theta = \varphi'_x \circ \varphi_{x^c}$. \square

In order to look at the distance m -transitivity of the middle two levels of Q_n , we must first determine how distance is affected by this level restriction. In doing so, let us redefine \mathbf{V}_m and \mathbf{W}_m , so that they fit this restriction.

Let $v_i, w_i \in \mathcal{P}([n])$, $\text{wt}(v_i), \text{wt}(w_i) = k$ or $k+1$, and define $\mathbf{V}_m = \{v_1, \dots, v_m\}$ and $\mathbf{W}_m = \{w_1, \dots, w_m\}$, $1 \leq m \leq 2\binom{n}{k}$.

5.7 Distance m -Transitivity in $Q_{2k+1}(L_k, L_{k+1})$

For two sets \mathbf{V}_m and \mathbf{W}_m with $d(v_i, v_j) = d(w_i, w_j)$ for all i, j , $1 \leq i, j \leq m$, there exists an automorphism $f \in \text{Aut}(Q_n(L_k, L_{k+1}))$ such that $f(v_i) = w_i$, for all i , $1 \leq i \leq m$ only when $m = 1, 2$. For $i = 1$ this is clear, since this implies we just need find an f that maps one element, x , to an element y , where $\text{wt}(x) = \text{wt}(y)$. This is easily achieved by applying a permutation θ such that $\theta(x) = y$, giving $f = \rho_\theta \in \text{Aut}(Q_n(L_k, L_{k+1}))$. For $i = 2$, see Proposition 5.7.1.

Example 5.7.2 illustrates that distance m -transitivity in $Q_{2k+1}(L_k, L_{k+1})$ does not necessarily hold for $m \geq 3$.

Proposition 5.7.1. *For two 2-sets \mathbf{V}_2 and \mathbf{W}_2 with $d(v_1, v_2) = d(w_1, w_2)$, there is an automorphism $f \in \text{Aut}(Q_n(L_k, L_{k+1}))$ such that $f(v_i) = w_i$, $i = 1, 2$. That is, $Q_n(L_k, L_{k+1})$ is distance transitive.*

Proof. Let $\text{wt}(v_1) = \text{wt}(w_1)$. Then v_1 and w_1 are members of the same level and we must have $\text{wt}(v_2) = \text{wt}(w_2)$ in order to satisfy the distance condition. Now, letting r represent the number of elements common to both v_1 and v_2 ,

$$d(w_1, w_2) = d(v_1, v_2) = |v_1 \Delta v_2| = \text{wt}(v_1) + \text{wt}(v_2) - 2r = \text{wt}(w_1) + \text{wt}(w_2) - 2r$$

Thus, the number of elements common to both v_1 and v_2 is equal to the number of elements common to both w_1 and w_2 . Construct a permutation θ by mapping a common element of v_1 and v_2 to an element common to both w_1 and w_2 . The remaining unmapped elements appear at most once between v_1 and v_2 (and similarly in w_1 and w_2), so we may extend θ by mapping a unique element of v_i to a unique element of w_i , $i = 1, 2$. Thus, we have constructed a permutation θ such that, with $f = \rho_\theta$, $f(v_i) = w_i$, $i = 1, 2$.

Let $\text{wt}(v_1) \neq \text{wt}(w_1)$. Then v_1 and w_1 are members of different levels. Apply $\sigma_{[n]}$ to \mathbf{W}_2 to obtain \mathbf{W}_2^* . That is, $\mathbf{W}_2^* = \{w_1^*, w_2^*\} = \{\sigma_{[n]}(w_1), \sigma_{[n]}(w_2)\}$. Now, $\text{wt}(v_1) = \text{wt}(w_1^*)$ and we proceed as above to find a permutation θ on $[n]$ such that $\rho_\theta(v_i) = w_i^*$, $i = 1, 2$. Since $\sigma_{[n]} = \sigma_{[n]}^{-1}$ and letting $f = \sigma_{[n]}\rho_\theta$, we have $f(v_i) = w_i$, $i = 1, 2$. □

Example 5.7.2. For $n \geq 9$, let $V = \{12, 13, 14\} + \{\{8, 9, \dots, k+5\}\}$ and $W = \{56, 57, 67\} + \{\{8, 9, \dots, k+5\}\}$. Then, $|V| = |W| = 3$, $\text{wt}(v_i) = \text{wt}(w_i) = k$,

$1 \leq i \leq 3$, and $d(v_i, v_j) = d(w_i, w_j) = 2$ for all i, j . However, there is no f such that $f(v_i) = (w_i)$, $1 \leq i \leq 3$. Since f must map elements of weight k to elements of weight k , we have that $A = \emptyset$ for any $f = \sigma_A \rho_\theta$. Thus, $f = \rho_\theta$. In order for θ to be a valid permutation, we must have that, for all $x \in \{8, 9, \dots, k+5\}$, $\theta(x) = y$, $y \in \{8, 9, \dots, k+5\}$. (For, if not, then f would send x to 5, 6 or 7, and while $x \in v_i$ for all i , none of these elements appear in every member of W .) Thus, f must map $\{1, 2\}$ to $\{5, 6\}$ via a permutation. By Example 5.2.2, no such f exists. Therefore, the middle 2 levels of Q_n are not distance 3-transitive for $n \geq 9$.

For $m > 3$, extend this example by adding the set $\{x, y\} + \{\{8, 9, \dots, k+5\}\}$, $k+6 \leq x, y \leq n$, $x \neq y$, to both V and W .

5.8 Intersection m -Transitivity in $Q_{2k+1}(L_k, L_{k+1})$

Recall that \mathbf{V}_m and \mathbf{W}_m were redefined to fit into these middle 2 levels. Let $n = 2k + 1$ and let $V = \{v_1, v_2, \dots, v_m\}$ and $W = \{w_1, w_2, \dots, w_m\}$, $v_i, w_i \in L_k, L_{k+1}$ for $1 \leq i \leq m$.

Lemma 5.8.1. *V and W satisfy Condition 5.4.1 if and only if there exists an automorphism $f \in \text{Aut}(Q_n(L_k, L_{k+1}))$ such that $f(v_i) = w_i$, $f = \rho_\theta$, θ a permutation on $[n]$, $1 \leq i \leq m$.*

Proof. (\Leftarrow) Since $f = \rho_\theta$ and setting $X = V$, this is just an extension of Proposition 5.4.3.

(\Rightarrow) (Induction on k) If $k = 0$, then $L_0 = \{\emptyset\}$ and $L_1 = \{1\}$. Then there exists only three possibilities: $V = \{\emptyset\}$, $V = \{1\}$, and $V = \{\emptyset, 1\}$. In all three of these cases, we must have $W = V$, so $\theta = (1)$ and f is just the identity map.

Let $p = \max_{v_{i_1}, \dots, v_{i_k} \in V} \left\{ k \mid v_{i_1} \cap \dots \cap v_{i_k} \neq \emptyset \right\}$ and let x_1 be the element in this

v_{i_1}, \dots, v_{i_p} . Then there exists some $y_1 \in w_{i_1} \cap \dots \cap w_{i_p}$ and we can define $\theta(x_1) = y_1$. Set $v'_l = v_l \setminus \{x_1\}$, $w'_l = w_l \setminus \{y_1\}$ and $V' = \{v'_l | 1 \leq l \leq m\}$.

Now,

$$\left| v_{i_1} \cap \dots \cap v_{i_p} \right| - 1 = \left| w_{i_1} \cap \dots \cap w_{i_p} \right| - 1$$

which gives:

$$\left| v'_{i_1} \cap \dots \cap v'_{i_p} \right| = \left| w'_{i_1} \cap \dots \cap w'_{i_p} \right|$$

Applying the same process, let $q = \max_{v'_{i_1}, \dots, v'_{i_k} \in V'} \{k \mid v'_{i_1} \cap \dots \cap v'_{i_k} \neq \emptyset\}$ and let x_2 be the element in this $v'_{i_1}, \dots, v'_{i_q}$. Then there is some $y_2 \in w'_{i_1} \cap \dots \cap w'_{i_q}$ and we can define $\theta(x_2) = y_2$. Set $v''_l = v'_l \setminus \{x_2\}$, $w''_l = w'_l \setminus \{y_2\}$.

As above, we have

$$\left| v'_{i_1} \cap \dots \cap v'_{i_p} \right| - 1 = \left| w'_{i_1} \cap \dots \cap w'_{i_p} \right| - 1$$

which yields:

$$\left| v''_{i_1} \cap \dots \cap v''_{i_p} \right| = \left| w''_{i_1} \cap \dots \cap w''_{i_p} \right|$$

Thus, Condition 5.4.1 holds for V in $[n] \setminus \{x_1, x_2\}$ and W in $[n] \setminus \{y_1, y_2\}$. Since both $[n] \setminus \{x_1, x_2\}$ and $[n] \setminus \{y_1, y_2\}$ are equivalent to $[n-2] = [2(k-1) + 1]$, we have, by induction, there exists $\theta' : [n-2] \rightarrow [n-2]$ and a $f' = \rho_{\theta'}$. Extending θ' by θ , we have a map $\theta : [n] \rightarrow [n]$ and, hence, $f = \rho_{\theta}$, with $f(v_i) = w_i$, $1 \leq i \leq m$. \square

Condition 5.8.2. For each k , $1 \leq k \leq m$, and for all sequences $1 \leq l_1 < l_2 < \dots < l_k \leq m$,

$$\left| v_{l_1} \cap \dots \cap v_{l_k} \right| = \left| ([n] \Delta w_{l_1}) \cap \dots \cap ([n] \Delta w_{l_k}) \right|$$

Lemma 5.8.3. *V and W satisfy Condition 5.8.2 if and only if there exists an automorphism $f \in \text{Aut}(Q_n(L_k, L_{k+1}))$ such that $f(v_i) = w_i$, $f = \sigma_{[n]}\rho_\theta$, θ a permutation of $[n]$, $1 \leq i \leq m$.*

Proof. (\Rightarrow) By Condition 5.8.2, we have

$$\begin{aligned} |v_{i_1} \cap \dots \cap v_{i_k}| &= |([n]\Delta w_{i_1}) \cap \dots \cap ([n]\Delta w_{i_k})| \\ &= |\sigma_{[n]}(w_{i_1}) \cap \dots \cap \sigma_{[n]}(w_{i_k})| \\ &= |w_{i_1}^* \cap \dots \cap w_{i_k}^*| \end{aligned}$$

where $w_i^* = \sigma_{[n]}(w_i)$. That is, $W^* = \sigma_{[n]}(W)$. By Lemma 5.8.1, there exists a permutation on $[n]$ such that $\rho_\theta(v_i) = w_i^*$, $1 \leq i \leq m$. Since $\sigma_{[n]} = \sigma_{[n]}^{-1}$, and letting $f = \sigma_{[n]}\rho_\theta$, we have $f(v_i) = w_i$, $1 \leq i \leq m$.

(\Leftarrow) Since $f = \sigma_{[n]}\rho_\theta$ and $f(v_i) = w_i$ for all i , $1 \leq i \leq m$, we have $\sigma_{[n]}\rho_\theta(v_i) = w_i$ or, equivalently,

$$\rho_\theta(v_i) = \sigma_{[n]}^{-1}(w_i) = \sigma_{[n]}(w_i) = [n]\Delta w_i \quad (5.8.1)$$

By applying Proposition 5.4.3 to V , we can see

$$|v_{i_1} \cap \dots \cap v_{i_k}| = |\rho_\theta(v_{i_1}) \cap \dots \cap \rho_\theta(v_{i_k})|$$

for any k , $1 \leq k \leq m$ and for any $\{l_1, \dots, l_k\}$, $1 \leq l_1 < l_2 < \dots < l_k \leq m$, and so by (5.8.1)

$$|v_{i_1} \cap \dots \cap v_{i_k}| = |([n]\Delta w_{i_1}) \cap \dots \cap ([n]\Delta w_{i_k})|$$

Therefore, V and W satisfy Condition 5.8.2. □

Hence, $Q_n(L_k, L_{k+1})$ is intersection m -transitive, and thus, by Theorem 5.5.4, we

have $Q_n(L_k, L_{k+1})$ is Δ m -transitive.

Proposition 5.8.4. *Lemma 5.4.10 counts the number of possible automorphisms, f , that can be constructed from Lemma 5.8.1.*

Proof. Let $V = \mathbf{V}_m$ and $W = \mathbf{W}_m$. Since the proof of Lemma 5.4.10 is independent of the elements in \mathbf{V}_m and \mathbf{W}_m and $f = \rho_\theta$, the result still holds for $Q_n(L_k, L_{k+1})$. \square

Corollary 5.8.5. *Lemma 5.4.10 also counts the number of possible automorphisms, f , in Lemma 5.8.3.*

Proof. This is clear upon setting $\alpha = [n]$ in the proof of Corollary 5.4.11. \square

Chapter 6

Q_n^*

6.1 Introduction

The diameter of the n dimensional hypercube, Q_n is n . Given that distance is preserved in the middle level graph, the diameter of $Q_n(L_k, L_{k+1})$ also achieves this value. Let us define a new graph so that the diameter is significantly reduced in size.

Definition 6.1.1. Let Q_n^* be the graph with vertex set $V(Q_n^*) = V(Q_n)$ and edge set $E(Q_n^*) = E(Q_n) \cup \{\langle x, x^c \rangle \mid x \in V(Q_n^*)\}$.

Definition 6.1.2. An edge of the form $\langle x, x^c \rangle$ is called a complementary edge.

Notation. Let $d(u, v) = d_{Q_n}(u, v)$ and $d^*(u, v) = d_{Q_n^*}(u, v)$.

Clearly, $d^*(u, v) \leq d(u, v)$.

Let us consider some of the basic properties of Q_n^* .

6.2 Properties of Q_n^*

Lemma 6.2.1. $\text{Aut}(Q_n) \subset \text{Aut}(Q_n^*)$.

Proof. Since $V(Q_n) = V(Q_n^*)$, we only need to consider the action of an automorphism on the edge set. Let $x, y \in V(Q_n)$.

Let $\varphi \in \text{Aut}(Q_n)$ and assume that vertex x is adjacent to y in Q_n^* . If $\langle x, y \rangle \in E(Q_n)$, then $\langle \varphi(x), \varphi(y) \rangle \in E(Q_n)$ and we have $\langle \varphi(x), \varphi(y) \rangle \in E(Q_n^*)$. If $\langle x, y \rangle \notin E(Q_n)$, then $y = x^c = x\Delta[n]$. Thus, we have

$$\begin{aligned} \langle \varphi(x), \varphi(y) \rangle &= \langle \varphi(x), \varphi(x\Delta[n]) \rangle \\ &= \langle \varphi(x), \varphi(x)\Delta\varphi([n]) \rangle \\ &= \langle \varphi(x), \varphi(x)\Delta[n] \rangle \\ &= \langle \varphi(x), (\varphi(x))^c \rangle \in E(Q_n^*) \end{aligned}$$

Therefore, $\text{Aut}(Q_n) \subset \text{Aut}(Q_n^*)$. □

Proposition 6.2.2. *For any two vertices u and v in Q_n^* , a shortest path between u and v uses at most one complementary edge.*

Proof. Assume not. Then for some $u, v \in V(Q_n^*)$, the shortest path between u and v traverses at least two complementary edges. Let $\langle w, w^c \rangle$ and $\langle x, x^c \rangle$ be the first two such edges in this path. Without loss of generality, we may assume that, of these four vertices, w is visited first along the uv -path and $d(w, x) \leq d(w, x^c)$. Thus, our path is of the form

$$u \rightarrow \dots \rightarrow w \rightarrow w^c \rightarrow \dots \rightarrow x^c \rightarrow x \rightarrow \dots \rightarrow v$$

For $d(w, x) = m$, we may write $x = w\Delta\{i_1, i_2, \dots, i_m\}$, $i_j \in [n]$, $1 \leq j \leq m$, and $x^c = (w\Delta\{i_1, \dots, i_m\})\Delta[n]$. So, the path between w and x through w^c and x^c uses

only edges from Q_n except for the two complementary edges and, thus, is of length $1 + d(w^c, x^c) + 1 = 1 + m + 1 = m + 2 > m > d(w, x)$. Therefore, we have arrived at a contradiction for the use of two complementary edges. Repeating this argument for each pair of $\langle w, w^c \rangle, \langle x, x^c \rangle$, we can see that a shortest path between u and v uses at most one complementary edge. \square

Lemma 6.2.3. *The diameter of Q_n^* is $\lceil \frac{n}{2} \rceil$.*

Proof. Let $u, v \in V(Q_n^*)$. Without loss of generality, we may assume $u = \emptyset$. Let $\text{wt}(v) = i$. If $i \leq \lfloor \frac{n}{2} \rfloor$, then $d(u, v) \leq \lfloor \frac{n}{2} \rfloor$ and $d^*(u, v) \leq \lfloor \frac{n}{2} \rfloor$.

On the other hand, when $i > \lfloor \frac{n}{2} \rfloor$, we have $d(u, v) > \lfloor \frac{n}{2} \rfloor$, or $d(u, v) \geq \lceil \frac{n}{2} \rceil$. By Proposition 6.2.2, we have that any shortest path between \emptyset and v uses at most one complementary edge. If we do not use such an edge between \emptyset and v , the length of the path is equal to the distance in Q_n , namely $d(u, v) = i \geq \lceil \frac{n}{2} \rceil$. However, if we use an edge $\langle w, w^c \rangle$, $\text{wt}(w) = j$, $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$, we have the path

$$\emptyset = u \rightarrow \dots \rightarrow w \rightarrow w^c \rightarrow \dots \rightarrow v$$

Thus,

$$\begin{aligned} d^*(u, v) &\leq d^*(u, w) + d^*(w, w^c) + d^*(w^c, v) \\ &\leq d(u, w) + 1 + d(w^c, v) \\ &= j + 1 + ((n - j) - i) \\ &= n - i + 1 \\ &< n - \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil + 1 \end{aligned} \tag{6.2.1}$$

Therefore, a shortest path must include a complementary edge and we have

$$\text{diam}(Q_n^*) \leq \lceil \frac{n}{2} \rceil.$$

By the above argument, we have that the distance between $u = \emptyset$ and v is greatest when $\lceil \frac{n}{2} \rceil \leq \text{wt}(v) \leq n$ and a shortest path will include an edge of the form $\langle w, w^c \rangle$. Therefore, we wish to maximize $n - i + 1$, where $\text{wt}(v) = i$. This occurs exactly when $\text{wt}(v) = \lceil \frac{n}{2} \rceil$, yielding $\text{diam}(Q_n^*) \geq \lceil \frac{n}{2} \rceil$.

$$\text{Hence, } \text{diam}(Q_n^*) = \lceil \frac{n}{2} \rceil. \quad \square$$

Proposition 6.2.4. *Let $u, v \in V(Q_n^*)$ such that $d(u, v) \geq \lceil \frac{n}{2} \rceil$. The number of shortest paths between u and v in Q_n^* , each containing a distinct complementary edge, is $2^{n-d(u,v)}$.*

Proof. Without loss of generality, we may assume $u = \emptyset$ and, thus, $\lceil \frac{n}{2} \rceil \leq \text{wt}(v) \leq n$. Let $X = [n] \setminus v$. Form a path from \emptyset to v as follows:

1. Construct a path from \emptyset to $x \subset X$ using edges native to Q_n
2. Traverse the edge $\langle x, x^c \rangle$
3. Complete the path from x^c to v , using no complementary edges

The length of this path is $1 + |[n] \setminus v| = n + 1 - \text{wt}(v)$. Since $\lceil \frac{n}{2} \rceil \leq \text{wt}(v) \leq n$, the length of this path is at most $n + 1 - \lceil \frac{n}{2} \rceil = \lceil \frac{n}{2} \rceil = \text{diam}(Q_n^*)$. By construction and Proposition 6.2.2, this does, in fact, give a shortest path in Q_n^* . Also, since steps 1 and 3 are computed in the shortest possible manner, this method determines every path of the necessary type.

Now, for each unique complementary edge chosen in step 2, edge in there are exactly $2^{n-\text{wt}(v)}$ (*i.e.*, the total number of subsets of $[n] \setminus v$) choices for x in step 1, and 1 choice for x^c in step 3. Thus, there exists $2^{n-\text{wt}(v)} = 2^{n-d(u,v)}$ shortest paths, each using a different complementary edge. □

Lemma 6.2.5. *At least $2^{\lfloor n/2 \rfloor}$ complementary edges must be removed from Q_n^* in order for the diameter to increase.*

Proof. By the proof of Lemma 6.2.3, the removal of a complementary edge will only affect the diameter of Q_n^* if the weight of v is between $\lfloor \frac{n}{2} \rfloor$ and n , inclusive, when considering a path between \emptyset and v . By Proposition 6.2.4, there are $2^{n-\text{wt}(v)}$ possible complementary edges to use when forming a shortest path between \emptyset and v .

Let $\text{wt}(v_1) = i$, $\text{wt}(v_2) = j$, $\lfloor \frac{n}{2} \rfloor \leq j < i \leq n$ and let $0 \leq \text{wt}(w) \leq \lfloor \frac{n}{2} \rfloor$. By (6.2.1), we have

$$\begin{aligned}
|\text{shortest } \emptyset, v_2\text{-path}| &= d^*(\emptyset, v_2) \\
&\leq d^*(\emptyset, w) + d^*(w, w^c) + d^*(w^c, v_2) \\
&\leq d^*(\emptyset, w) + d^*(w, w^c) + d^*(w^c, v_1) + d^*(v_1, v_2) \\
&\leq |\emptyset, v_1\text{-path using 1 complementary edge}| + |v_1, v_2\text{-path}|
\end{aligned}$$

Thus, the length of a shortest \emptyset, v_2 -path is not longer than a \emptyset, v_1 -path through a complementary edge and a v_1, v_2 -path, combined.

Hence, for any two vertices v_1 and v_2 in Q_n^* such that $\lfloor \frac{n}{2} \rfloor \leq \text{wt}(v_1), \text{wt}(v_2) \leq n$, $\text{wt}(v_1) > \text{wt}(v_2)$ and $v_2 \subset v_1$, if a complementary edge can be used in a shortest path between \emptyset and v_1 , then it can also be used when forming a shortest path between \emptyset and v_2 . Therefore, we can assume $\text{wt}(v) = \lfloor \frac{n}{2} \rfloor$. Thus, at least $2^{n-\lfloor n/2 \rfloor} = 2^{\lfloor n/2 \rfloor}$ complementary edges must be removed from Q_n^* in order for the diameter to increase. □

Lemma 6.2.6. *For $k \geq 1$, any odd length cycle in Q_{2k}^* contains an odd number of complementary edges.*

Proof. Since traversing an edge from Q_n changes the parity of a vertex, every cycle must contain an even number of ordinary edges. On the other hand, in Q_{2k}^* , moving along a complementary edge preserves the parity of a vertex. Thus, any odd cycle in Q_{2k}^* must contain an odd number of complementary edges. \square

Proposition 6.2.7. *The length of the shortest odd cycle in Q_{2k}^* is $2k + 1$, $k \geq 1$.*

Proof. The cycle

$$\emptyset \rightarrow 1 \rightarrow 12 \rightarrow \dots \rightarrow [n] \rightarrow \emptyset$$

uses one complementary edge and is of length $2k + 1$. By construction, we can see that no shorter odd cycle that includes \emptyset and one complementary edge can be formed. Therefore, any shortest odd length cycle containing one complementary edge has length $2k + 1$.

Let C be an odd cycle of shortest length. By Lemma 6.2.6, C must contain an odd number of complementary edges. If C contains one complementary edge, then, by above, the length of C is $2k + 1$. If C contains at least 3 complementary edges, then we can write C as

$$v_1 \rightarrow v_1^c \rightarrow w_{1,1} \rightarrow w_{1,2} \rightarrow \dots w_{1,l_1} \rightarrow v_2 \rightarrow v_2^c \rightarrow w_{2,1} \rightarrow \dots \rightarrow w_{2,l_2} \rightarrow v_3 \rightarrow \dots \rightarrow$$

$$v_m \rightarrow v_m^c \rightarrow w_{m,1} \rightarrow \dots \rightarrow w_{m,l_m} \rightarrow v_{m+1} = v_1$$

and we have 2 cases to consider:

Case (1): $d(v_j, v_{j+1}^c) > k$ for all j , $1 \leq j \leq m$, or

Case (2): For some t , $d(v_t, v_{t+1}^c) \leq k$.

In Case (1), since $d(v_j, v_{j+1}^c) > k$, $1 \leq j \leq m$, then $d(v_j^c, v_{j+1}) > k$. Taking any series of three complementary edges and two path segments, we see the length of this

cycle piece is greater than $1 + k + 1 + k + 1 = 2k + 3$, which is longer than the shortest odd cycle with one complementary edge. Thus, we have obtained a contradiction to our minimal length assumption.

For Case (2), we have some t so that $d(v_t, v_{t+1}^c) \leq k$, and the piece of cycle under consideration looks like:

$$v_t \rightarrow v_t^c \rightarrow w_{t,1} \rightarrow w_{t,2} \rightarrow \dots \rightarrow w_{t,l_t} \rightarrow v_{t+1} \rightarrow v_{t+1}^c$$

The length of this cycle segment is $1 + d(v_t^c, v_{t+1}) + 1 = 2 + d(v_t, v_{t+1}^c) > d(v_t, v_{t+1}^c)$. Thus, we could replace this piece of the cycle by a path with no complementary edges and shorten the cycle, again contradicting the minimality of the length of C (Note that this replacement still yields a cycle of odd length with an odd number of complementary edges.)

Hence, C , is an odd cycle of shortest length equal to $2k + 1$. □

Proposition 6.2.8. *For $n = 2k + 1$, Q_n^* is 2-colorable.*

Proof. Consider Q_n with bipartition (X, Y) , where X and Y are the partitions $X = \{v \mid \text{wt}(v) \text{ is even}\}$ and $Y = \{v \mid \text{wt}(v) \text{ is odd}\}$, $v \in V(Q_n)$. Now, $E(Q_n^*) = E(Q_n) \cup \{\langle v, v^c \rangle\}$. If $\text{wt}(v)$ is even, say $2i$, then $\text{wt}(v^c) = n - 2i = 2k + 1 - 2i = 2(k - i) + 1$, which is odd. Therefore, (X, Y) is still a valid partition of the vertex set of Q_n^* , since any complementary edges join vertices whose weights are of different parities and we have Q_n^* is 2-colorable. □

Proposition 6.2.9. *For $n = 2k$, Q_n^* is 4-colorable.*

Proof. For $i = 0, 1$, let $Q_n^i = \{x \in Q_n \mid x_n = i\}$. Then each Q_n^i is isomorphic to the bipartite graph Q_{n-1} . So color Q_n^0 with colors 1 and 2, and color Q_n^1 with colors 3

and 4. This yields a 4-coloring of Q_n^* . □

Proposition 6.2.10. Q_4^* is not 3-colorable.

Proof. First, let us show that, $\alpha(Q_4^*)$, the maximum size of an independent set, S , in Q_4^* is 5. Let $S = \{1, 2, 3, 4, 1234\}$. Then $|S| = 5$ and it is clear that S is independent. Thus, $\alpha(Q_4^*) \geq 5$.

Now, assume S is an independent set of size at least 6. Without loss of generality, we may assume S contains \emptyset , and, so we must have the other vertices in S are of weight 2 or 3. Now, there is a perfect matching between the vertices of weight 2 using only complementary edges: $\{\langle 12, 34 \rangle, \langle 13, 24 \rangle, \langle 23, 14 \rangle\}$. Thus, S can contain at most 3 vertices of weight 2, and, therefore, must also contain at least 2 vertices of weight 3. Again, without loss of generality, we may assume that S contains vertices 123 and 124. Now, there are only three vertices of weights 2 and 3 that are not adjacent to 123 or 124, namely, 34, 134, and 234. Since $|S| \geq 6$, we must have all three of these vertices in S . But then S contains the edge $\langle 34, 134 \rangle$, and S is not independent. Contradiction.

Hence, the maximum size of an independent set in Q_4^* is less than 6. In fact, the maximum size of an independent set is 5.

Now, $|S| = 5 < \frac{1}{3}|V(Q_n^*)|$. Therefore, partitioning $V(Q_n^*)$ into independent sets, we would have at least 4 parts. Thus, Q_4^* is not 3-colorable. □

Recall $\kappa(G)$, the connectivity of G , is the minimum size of a vertex set V such that $G - V$ is disconnected or has only one vertex.

Proposition 6.2.11. For $n \geq 2$, $\kappa(Q_n^*) = n + 1$.

Proof. Since $Q_2^* \cong K_4$, we have $\kappa(Q_2^*) = \kappa(K_4) = 3$. Now, consider Q_3^* with bipartition $X = \{v \mid \text{wt}(v) \text{ is even}\}$ and $Y = \{v \mid \text{wt}(v) \text{ is odd}\}$. Since, for all $v \in V(Q_3^*)$,

$\deg(v) = 4$, we have that $Q_3^* \cong K_{4,4}$. Therefore, $\kappa(Q_3^*) = \kappa(K_{4,4}) = 4$, as desired.

Let $n \geq 4$. For any $n \geq 1$, the neighborhood of a vertex in Q_n^* forms a separating set. Thus, since $\deg(x) = n + 1$ for all $x \in V(Q_n^*)$, we have $\kappa(Q_n^*) \leq n + 1$.

Now, we must show that $\kappa(Q_n^*) \geq n + 1$ to arrive at our desired result. Partition the vertex set into 2 parts: $X = \{v | n \notin v\}$ and $Y = \{v | n \in v\}$. Then every antipodal edge occurs between a vertex of X and a vertex of Y , and we have that both X and Y can be viewed as Q_{n-1} . Letting S be a separating set of Q_n^* , we have two cases to consider:

Case (1): $X - S$ and $Y - S$ are both connected

Case (2): at least one of $X - S$ and $Y - S$ is disconnected.

In Case (1), if both $X - S$ and $Y - S$ are connected, then S must contain at least one endpoint of every edge between X and Y . That is, $|S| \geq 2^{n-1}$. But $2^{n-1} > n + 1$ for $n \geq 4$ contradicting $\kappa(Q_n^*) \leq n + 1$.

Case (2): Without loss of generality, assume $X - S$ is disconnected. Thus, since $\kappa(Q_{n-1}) = n - 1$, S has at least $n - 1$ vertices in X . Now, every vertex, x , in X is adjacent to two vertices in Y , namely $x \cup \{n\}$ and x^c . If S contains zero or one vertex of Y , then $Y - S$ is still connected and every vertex in $X - S$ has at least one neighbor in $Y - S$. Therefore, $Q_n^* - S$ is still connected when S contains less than two vertices of Y . Hence, $\kappa(Q_n^*) \geq (n - 1) + 2 = n + 1$. \square

6.3 m -Transitivity in Q_n^*

Proposition 6.3.1. Q_n^* is distance m -transitive for $m < 4$.

Proof. Since $V(Q_n^*) = V(Q_n)$, $\text{Aut}(Q_n) \subset \text{Aut}(Q_n^*)$, and Q_n is distance m -transitive for $m < 4$, it follows that Q_n^* is distance m -transitive, $m < 4$. \square

By Example 5.2.2 we can see that Q_n^* is not distance m -transitive for $m \geq 4$.

Let \mathbf{V}_m and \mathbf{W}_m be defined as in Section 5.1. That is, let $v_i, w_i \in \mathcal{P}([n])$ and define $\mathbf{V}_m = \{v_0, \dots, v_m\}$ and $\mathbf{W}_m = \{w_0, \dots, w_m\}$, $0 \leq m \leq 2^n$.

Proposition 6.3.2. Q_n^* is intersection m -transitive.

Proof. Again, since $V(Q_n^*) = V(Q_n)$, $\text{Aut}(Q_n) \subset \text{Aut}(Q_n^*)$, and Q_n is intersection m -transitive, it follows that Q_n^* is intersection m -transitive. \square

Proposition 6.3.3. Q_n^* is Δ m -transitive.

Proof. As above, $V(Q_n^*) = V(Q_n)$ and $\text{Aut}(Q_n) \subset \text{Aut}(Q_n^*)$. Thus, since Q_n is Δ m -transitive, so is Q_n^* . \square

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