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Some geometric properties of certain toric varieties and Schubert varieties

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SOME GEOMETRIC PROPERTIES
OF
CERTAIN TORIC VARIETIES AND SCHUBERT VARIETIES

A dissertation presented

by

Justin Allen Brown

to

The Department of Mathematics

In partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in the field of

Mathematics

Northeastern University

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April, 2009

SOME GEOMETRIC PROPERTIES
OF
CERTAIN TORIC VARIETIES AND SCHUBERT VARIETIES

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Justin Allen Brown

ABSTRACT OF DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate School of Arts and Sciences of
Northeastern University, April, 2009

Abstract

This thesis has three distinct chapters: Bruhat-Hibi toric varieties, Gorenstein Schubert varieties in a minuscule G/P, and Wahl's conjecture for a minuscule G/P. We begin with a study of toric varieties associated to Bruhat lattices for a minuscule G/P. Our main result is a combinatorial characterization of the singular loci of these toric varieties. In the next chapter, we are concerned with Schubert varieties in a minuscule G/P, for which we give a combinatorial characterization for them to be arithmetically Gorenstein. In the last chapter, we prove Wahl's conjecture for a minuscule G/P.

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Introduction

The results of this thesis may be put into the following three headings: Bruhat-Hibi toric varieties, Gorenstein Schubert varieties in a minuscule G/P, and Wahl's conjecture for a minuscule G/P. Accordingly, the results are organized in three chapters, one for each of the three headings. We now give a brief description of the results in each of the three chapters.

1. Bruhat-Hibi toric varieties

This thesis project began with a conjecture of Lakshmibai and Gonciulea in [**13**] concerning the singular loci of toric degenerations of cones over Grassmannians. Toric varieties arising in this setting can be realized as the vanishing set of binomials given by the join and meet relations on a distributive lattice. Specifically, let K denote the base field which we assume to be algebraically closed of arbitrary characteristic. Given a distributive lattice $\mathcal{L}$, let $X(\mathcal{L})$ denote the affine variety in $\mathbb{A}^{\#\mathcal{L}}$ whose vanishing ideal is generated by the binomials $X_\tau X_\varphi - X_{\tau\vee\varphi}X_{\tau\wedge\varphi}$ in the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$ (here, $\tau \vee \varphi$ (resp. $\tau \wedge \varphi$) denotes the *join* - the smallest element of $\mathcal{L}$ greater than both τ, φ (resp. the *meet* - the largest element of $\mathcal{L}$ smaller than both τ, φ)). These varieties were extensively studied by Hibi in [**14**] where Hibi proves that $X(\mathcal{L})$ is a normal variety. On the other hand, Eisenbud-Sturmfels show in [**9**] that a binomial prime ideal is toric (here, "toric ideal" is in the sense of [**33**]). Thus one obtains that $X(\mathcal{L})$ is a normal toric variety. We shall refer to such a $X(\mathcal{L})$ as a *Hibi toric variety.*

For $\mathcal{L}$ being the Bruhat poset of Schubert varieties in a minuscule G/P, it is shown in [**12**] that $X(\mathcal{L})$ flatly deforms to $\widehat{G/P}$ (the cone over G/P), i.e., there exists a flat family over $\mathbb{A}^1$ with $\widehat{G/P}$ as the generic fiber and $X(\mathcal{L})$ as the special fiber. As mentioned above, the authors in [**13**] make a conjecture on the singular locus of $X(\mathcal{L})$ for $\mathcal{L}$ a Bruhat poset of Schubert varieties in the Grassmannian. (The statement of the conjecture is precisely the "Main Result" below.) The sufficiency part of the conjecture was proven in [**13**], using the Jacobian criterion for smoothness. The necessary part of the conjecture has been proven in [**1**], using certain desingularization techniques while relating these toric varieties to mirror symmetry.

Neither of [**1, 13**] brings out the relationship between the singularities of $X(\mathcal{L})$ and the combinatorics of the polyhedral cone associated to $X(\mathcal{L})$. Lakshmibai suggested we relate the singularities of $X(\mathcal{L})$ and the combinatorics of the associated polyhedral cone. The study of such a relationship led us to consider more general distributive lattices which we call "lattices with JIGL," namely, distributive lattices such that the partially ordered subset of join irreducible elements (see Definition 2.5) forms a grid lattice (a finite sublattice of $\mathbb{Z} \times \mathbb{Z}$). In §4, we show that all minuscule lattices are in fact lattices with JIGL (by a minuscule lattice, we mean the Bruhat poset of Schubert varieties in a minuscule G/P). For such a lattice $\mathcal{L}$, we call $X(\mathcal{L})$ a *Bruhat-Hibi toric variety*.

The goal of the first chapter of this thesis is to study Hibi toric varieties for which the associated distributive lattice is a lattice with JIGL. The main result gives the irreducible components of the singular loci of such Hibi toric varieties. This is the culmination of §5 and §6. We give the statement of the main result here (see Theorem 6.15):

Main Result. Let $\mathcal{L}$ be a distributive lattice with JIGL. Then

$$\operatorname{Sing} X(\mathcal{L}) = \bigcup_{(\alpha,\beta)} Z_{\alpha,\beta}$$

where (α, β) is an (unordered) pair of incomparable join-meet irreducible elements in $\mathcal{L}$, and $Z_{\alpha,\beta} = \{P \in X(\mathcal{L}) \subset \mathbb{A}^{\#\mathcal{L}} \mid P(\theta) = 0, \forall \theta \in [\alpha \wedge \beta, \alpha \vee \beta]\}$. (Here, $P(\theta)$ represents the θ-coordinate of $P \in \mathbb{A}^{\#\mathcal{L}}$ for $\theta \in \mathcal{L}$.)

As a consequence, we obtain a description of the singular locus of $X(\mathcal{L})$ in terms of the faces of the polyhedral cone associated to $X(\mathcal{L})$; further, we obtain that $\operatorname{Sing} X(\mathcal{L})$ is pure of codimension 3, with generic singularities being of cone type.

In §7, we study the divisors and line bundles of $X(\mathcal{L})$ where $\mathcal{L}$ is a lattice with JIGL. In §8, we show that "Young lattices" are also lattices with JIGL, therefore the main result above extends to even more Hibi toric varieties of interest (Hibi toric varieties associated to a Young lattice also appear in [**12**] as toric degenerations of the cone over partial flag varieties).

The reader might wonder if the main result above could extend to $X(\mathcal{L})$, where $\mathcal{L}$ is any lattice other than a lattice with JIGL. To answer this question, we give a counter example in §9. The example uses a lattice for which the poset of join irreducibles is a sublattice of $\mathbb{Z}^3$. We show that the singular locus is not characterized as in the main result above.

Returning to the original case of toric degenerations of the cone over the Grassmannian, we conclude this chapter by giving multiplicity formulae at specific points of $X(\mathcal{L})$ when $\mathcal{L}$ is the well known distributive lattice $I_{d,n}$ (see §10).

2. Arithmetically Gorenstein Schubert varieties

In the next chapter, we are concerned with the problem of characterizing the arithmetically Gorenstein Schubert varieties in a minuscule G/P (G a semisimple

algebraic group and P a minuscule parabolic subgroup). Let W be the Weyl group and T a maximal torus. For $w \in W$, we denote by $X(w)$ the Schubert variety in G/P associated to the T-fixed point wP, i.e., $X(w)$ is the Zariski closure of BwP in G/P, with the canonical reduced scheme structure.

While all Schubert varieties are Cohen-Macaulay (cf.[**29**]), not all of the Schubert varieties are smooth; one has (thanks to the works of several mathematicians during 1980's and 1990's) a complete classification of smooth Schubert varieties (see [**2**] for a detailed account on this). The Gorenstein property is a geometric property in between Cohen-Macaulayness and smoothness properties.

The problem of classifying the Gorenstein Schubert varieties is an open problem. Recently, Woo and Yong (cf. [**36**]) have given a characterization of the (geometrically) Gorenstein Schubert varieties in the flag variety $SL(n)/B$. As a consequence, one obtains a combinatorial characterization of the Gorenstein Schubert varieties in the Grassmannian. The characterization can be described as follows. Let w be an element in the Weyl group of G/P, then w is associated to a Young diagram. Then $X(w)$ is Gorenstein if and only if the outer corners of the Young diagram associated to w lie on the same anti-diagonal.

In Chapter 2, we begin by proving a stronger result than that of [**36**] (see also [**34**]), namely, while in [**34, 36**], the authors give a characterization of Gorenstein Schubert varieties, we give a characterization of the Gorenstein property even for the cones over Schubert varieties. As a consequence, it turns out that a Schubert variety in the Grassmannian is arithmetically Gorenstein (for the Plücker embedding) if and only if it is geometrically Gorenstein.

Next, we extend the combinatorial characterization of [**36**] to arithmetically Gorenstein Schubert varieties in the orthogonal Grassmannian, i.e., G/P for $G = SO(m)$ and P a maximal parabolic subgroup associated to the right end root if m is odd, or

one of the right end roots if m is even. (The indexing of the Dynkin diagram is as in [**5**].)

The proofs of these results rely on the fact that the coordinate ring of a Schubert variety is a "Hodge algebra" in the sense of [**7**], $R(w)$ having a set of algebra generators indexed by $H(w)$, the Bruhat poset of Schubert subvarieties of $X(w)$. For $X(w)$ a Schubert variety in a minuscule G/P, we have that $H(w)$ is a distributive lattice. Now using a result of Stanley (cf.[**31**]), we have that $R(w)$ is Gorenstein if and only if the poset of join-irreducibles of $H(w)$ is a ranked poset (i.e., all maximal chains in $H(w)$ have the same length).

In the remaining minuscule cases, if G is $SP(2n)$ and $P = P_1$ (the maximal parabolic corresponding to the left end root in the Dynkin diagram), then $G/P \cong \mathbb{P}^{2n-1}$; all of the Schubert varieties in G/P are smooth (and hence Gorenstein). If G is $SO(2n)$, and $P = P_1$ (the maximal parabolic corresponding to the left end root), then one easily checks that the poset of join-irreducibles of $H(w)$ is a ranked poset for all Schubert varieties $X(w)$ in G/P. Hence all of the Schubert varieties are Gorenstein. If G is E_6 or E_7 and P is a minuscule parabolic (using the above criterion for Gorenstein property), we have listed the arithmetically Gorenstein Schubert varieties (for the canonical projective embedding $X(w) \hookrightarrow \mathrm{Proj}\,(H^0\,(G/P, L))$, L being the ample generator of $\mathrm{Pic}(G/P) \cong \mathbb{Z}$).

3. Wahl's conjecture for a minuscule G/P

In [**35**], Wahl conjectured that the "Gaussian map" is surjective for the variety G/P, G a complex semisimple group and P a parabolic subgroup. To be specific, let X be a non-singular projective variety over $\mathbb{C}$. For ample line bundles L and M over X, consider the natural restriction map (called the *Gaussian*)

$$H^0(X \times X, \mathcal{I}_\Delta \otimes p_1^* L \otimes p_2^* M) \to H^0(X, \Omega_X^1 \otimes L \otimes M)$$

where $\mathcal{I}_\Delta$ denotes the ideal sheaf of the diagonal Δ in $X \times X$, p_1 and p_2 the two projections of $X \times X$ on X, and Ω^1_X the sheaf of differential 1-forms of X. Kumar proved Wahl's conjecture in [**17**], using representation theoretic techniques.

In [**19**], the authors considered Wahl's conjecture in positive characteristics, and observed that Wahl's conjecture will follow if there exists a Frobenius splitting of $X \times X$ which compatibly splits the diagonal and which has the maximum possible order of vanishing along the diagonal; this stronger statement was formulated as a conjecture in [**19**] which we shall refer to as the *LMP-conjecture* in the sequel:

> **LMP-conjecture:** For any G/P, there exists a splitting of $G/P \times G/P$ that compatibly splits the diagonal copy of G/P with maximal multiplicity.

Subsequently, in [**25**], Mehta-Parameswaran proved the LMP-conjecture for the Grassmannian. Recently, Lakshmibai-Raghavan-Sankaran (cf.[**22**]) extended the result of [**25**] to symplectic and orthogonal Grassmannians.

Our goal in this chapter is to show that the LMP conjecture (and hence Wahl's conjecture) holds in all characteristics for a minuscule G/P (of course, if G is the special orthogonal group $SO(m)$, then one should not allow characteristic 2). The main philosophy in [**25, 22**] consists of reducing the LMP-conjecture for a G/P to the problem of finding a section $\varphi \in H^0(G/B, K^{-1}_{G/B})$ ($K_{G/B}$ being the canonical bundle on G/B) which has maximum possible order of vanishing along P/B. Here, we further reduce this problem to computing the order of vanishing (along P/B) of the highest weight vector f_d in $H^0(G/B, L(\omega_d))$, for every fundamental weight ω_d of G.

CHAPTER 1

Bruhat-Hibi toric varieties

The main result of this chapter is the description of the singular locus of Bruhat-Hibi toric varieties. We do so by isolating a specific property that all minuscule lattices share. Thus, we actually determine the singular locus of a wider class of Hibi toric varieties, specifically all varieties $X(\mathcal{L})$ such that $\mathcal{L}$ is a lattice with join irreducibles forming a grid lattice.

1. Affine toric varieties

In this section, we recall some basic definitions concerning affine toric varieties. Let $T = (K^*)^m$ be an m-dimensional torus.

Definition 1.1. (cf. [**11**], [**10**]) An *equivariant affine embedding* of a torus T is an affine variety $X \subseteq \mathbb{A}^l$ containing T as a dense open subset and equipped with a T-action $T \times X \to X$ extending the action $T \times T \to T$ given by multiplication. If in addition X is normal, then X is called an *affine toric variety*.

1.2. The cone associated to a toric variety. Let X be an affine toric variety, and T the embedded torus. Let M be the character group of T, and N the $\mathbb{Z}$-dual of M. Recall (cf. [**11**], [**10**]) that there exists a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}} (= N \otimes_{\mathbb{Z}} \mathbb{R})$ such that

$$K[X] = K[S_\sigma],$$

where S_σ is the subsemigroup $\sigma^\vee \cap M$, $\sigma^\vee$ being the cone in $M_\mathbb{R}$ dual to σ, namely, $\sigma^\vee = \{f \in M_\mathbb{R} \mid f(v) \geq 0, v \in \sigma\}$. Note that S_σ is a finitely generated subsemigroup in M.

We shall denote X also by X_σ. We may suppose, without loss of generality, that σ spans $N_\mathbb{R}$ so that the dimension of σ equals $\dim N_\mathbb{R} = \dim T$. (Here, by dimension of σ, one means the vector space dimension of the span of σ.)

A *face* τ of σ is the intersection $\tau = \sigma \cap u^\perp = \{v \in \sigma \mid u\,(v) = 0\}$ for any $u \in \sigma^\vee$. When τ is a face of a cone σ, we denote $\tau \leq \sigma$.

1.3. The distinguished point P_τ. Each face τ determines a (closed) point P_τ in X_σ, namely, it is the point corresponding to the maximal ideal in $K[S_\sigma]$ given by the kernel of $e_\tau : K[S_\sigma] \to K$, where for $u \in S_\sigma$, we have

$$e_\tau(u) = \begin{cases} 1, & \text{if } u \in \tau^\perp, \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau^\perp$ denotes $\{u \in M_\mathbb{R} \,|\, u(v) = 0, \forall v \in \tau\}$.

1.4. Orbit decomposition. Let O_τ denote the T-orbit in X_σ through P_τ. We have the following orbit decomposition in X_σ:

$$X_\sigma = \bigcup_{\theta \leq \sigma} O_\theta, \qquad \overline{O_\tau} = \bigcup_{\theta \geq \tau} O_\theta,$$

and $\dim\, \tau + \dim\, O_\tau = \dim\, X_\sigma$. See [**11**], [**10**] for details.

2. Distributive lattices

Let $(\mathcal{L}, \leq)$ be a poset, i.e, a finite partially ordered set. We shall suppose that $\mathcal{L}$ is *bounded*, i.e., it has a unique maximal, and a unique minimal element, denoted $\widehat{1}$ and $\widehat{0}$ respectively. For $\mu, \lambda \in \mathcal{L}, \mu \leq \lambda$, we shall denote

$$[\mu, \lambda] := \{\tau \in \mathcal{L}, \mu \leq \tau \leq \lambda\}$$

We shall refer to $[\mu, \lambda]$ as the *interval from* μ *to* λ.

Definition 2.1. The ordered pair (λ, μ) is called a *cover* (and we also say that λ *covers* μ or μ is *covered* by λ) if $[\mu, \lambda] = \{\mu, \lambda\}$.

Definition 2.2. A *lattice* is a partially ordered set $(\mathcal{L}, \leq)$ such that for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$ and $x \wedge y$, called the *join*, respectively the *meet* of x and y, defined by:

$$x \vee y \geq x, \ x \vee y \geq y, \text{ and if } z \geq x \text{ and } z \geq y, \text{ then } z \geq x \vee y,$$

$$x \wedge y \leq x, \ x \wedge y \leq y, \text{ and if } z \leq x \text{ and } z \leq y, \text{ then } z \leq x \wedge y.$$

It is easy to check that the operations $\vee$ and $\wedge$ are commutative and associative.

Definition 2.3. A lattice is called *distributive* if the following identities hold:

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z). \end{aligned}$$

Definition 2.4. Given a lattice $\mathcal{L}$, a subset $\mathcal{L}' \subset \mathcal{L}$ is called a *sublattice* of $\mathcal{L}$ if $x, y \in \mathcal{L}'$ implies $x \wedge y \in \mathcal{L}'$, $x \vee y \in \mathcal{L}'$; $\mathcal{L}'$ is called an *embedded sublattice of* $\mathcal{L}$ if

$$\tau, \phi \in \mathcal{L}, \quad \tau \vee \phi, \tau \wedge \phi \in \mathcal{L}' \quad \Rightarrow \quad \tau, \phi \in \mathcal{L}'.$$

Definition 2.5. An element z of a lattice $\mathcal{L}$ is called *join-irreducible* (respectively *meet-irreducible*) if $z = x \vee y$ (respectively $z = x \wedge y$) implies $z = x$ or $z = y$. The set of join-irreducible (respectively meet-irreducible) elements of $\mathcal{L}$ is denoted by $J(\mathcal{L})$ (respectively $M(\mathcal{L})$), or just by J (respectively M) if no confusion is possible.

Definition 2.6. An element in $J(\mathcal{L}) \cap M(\mathcal{L})$ is called *irreducible*.

The following Lemma is easily checked.

LEMMA 2.7. *With the notations as above, we have*

(a) $J = \{\tau \in \mathcal{L} \mid$ *there exists at most one cover of the form* $(\tau, \lambda)\}$.

(b) $M = \{\tau \in \mathcal{L} \mid$ *there exists at most one cover of the form* $(\lambda, \tau)\}$.

DEFINITION 2.8. A subset I of a poset P is called an *ideal* of P if for all x, $y \in P$,

$$x \in I \text{ and } y \leq x \text{ imply } y \in I.$$

THEOREM 2.9 (cf. [**3**]). *Let $\mathcal{L}$ be a distributive lattice with $\hat{0}$, and P the poset of its nonzero join-irreducible elements. Then $\mathcal{L}$ is isomorphic to the lattice of ideals of P, by means of the lattice isomorphism*

$$\alpha \mapsto I_\alpha := \{\tau \in P \mid \tau \leq \alpha\}, \qquad \alpha \in \mathcal{L}.$$

We will often use the theorem above to view $\mathcal{L}$ as the lattice of ideals of $J(\mathcal{L})$. Thus the notation I_α as defined above will be used frequently.

LEMMA 2.10. *Let (τ, λ) be a cover in $\mathcal{L}$. Then I_τ equals $I_\lambda \dot{\cup} \{\beta\}$ for some $\beta \in J(\mathcal{L})$.*

PROOF. Let (τ, λ) be a cover in $\mathcal{L}$, then clearly $I_\tau \supsetneq I_\lambda$. Let $H = I_\tau \setminus I_\lambda$, we have that H is non-empty.

Assume, if possible, that β_1, β_2 are two distinct elements in $H \subset J(\mathcal{L})$. Denote $H_1 = I_{\beta_1} \cup I_\lambda$, $H_2 = I_{\beta_2} \cup I_\lambda$. Clearly H_1 and H_2 are ideals in $J(\mathcal{L})$, thus by Theorem 2.9 they correspond to elements in $\mathcal{L}$, call them ϕ_1 and ϕ_2, respectively. We thus have

$$\phi_1, \phi_2 \leq \tau, \quad \phi_1, \phi_2 \gneq \lambda.$$

Note that we have $\phi_1 \neq \phi_2$, and $\phi_1 \neq \lambda \neq \phi_2$. Therefore the interval $[\lambda, \tau]$ has cardinality of at least three, implying that τ does not cover λ, a contradiction.

Therefore we have that the cardinality of H is equal to one, and $I_\tau = I_\lambda \dot{\cup} \{\beta\}$ for some element β. $\square$

3. The variety $X(\mathcal{L})$

Consider the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$; let $\mathfrak{a}(\mathcal{L})$ be the ideal generated by $\{X_\alpha X_\beta - X_{\alpha\vee\beta}X_{\alpha\wedge\beta}, \alpha, \beta \in \mathcal{L}\}$. Then one knows (cf.[**14**]) that $K[X_\alpha, \alpha \in \mathcal{L}]/\mathfrak{a}(\mathcal{L})$ is a normal domain; in particular, we have that $\mathfrak{a}(\mathcal{L})$ is a prime ideal. Let $X(\mathcal{L})$ be the affine variety of the zeroes in $K^{\#\mathcal{L}}$ of $\mathfrak{a}(\mathcal{L})$. Then $X(\mathcal{L})$ is an affine normal variety defined by binomials. On the other hand, by [**9**], we have that a binomial prime ideal is toric (here, "toric ideal" is in the sense of [**33**, Chapter 4]). Hence $X(\mathcal{L})$ is a toric variety for the action by a suitable torus T, and thus $\dim X(\mathcal{L}) = \dim T$.

In the sequel, we shall denote $R(\mathcal{L}) := K[X_\alpha, \alpha \in \mathcal{L}]/\mathfrak{a}(\mathcal{L})$. Further, for $\alpha \in \mathcal{L}$, we shall denote the image of X_α in $R(\mathcal{L})$ by x_α.

Definition 3.1. The variety $X(\mathcal{L})$ will be called a *Hibi toric variety.*

Remark 3.2. An extensive study of $X(\mathcal{L})$ appears first in [**14**].

Theorem 3.3 (cf. [**20**]). *The dimension of $X(\mathcal{L})$ is equal to $\#J(\mathcal{L})$. Further, $\dim X(\mathcal{L})$ equals the cardinality of the set of elements in a maximal chain in (the graded poset) $\mathcal{L}$.*

Proposition 3.4. *(cf.* [**13**, Proposition 5.16]*) $X(\mathcal{L}')$ is a subvariety of $X(\mathcal{L})$ if and only if $\mathcal{L}'$ is an embedded sublattice of $\mathcal{L}$.*

3.5. Multiplicity of $X(\mathcal{L})$ at the origin. In this subsection, we shall determine the multiplicity of $X(\mathcal{L})$ at the origin. We first recall some definitions.

Let B be a $\mathbb{Z}_+$-graded, finitely generated K-algebra, $B = \oplus B_m$. Let $\phi_m(B)$ denote the Hilbert function:

$$\phi_m(B) = \dim_K B_m.$$

Let $P_B(x)$ denote the Hilbert polynomial of B; recall that

(1) $P_B(x) \in \mathbb{Q}[x]$,

(2) $\deg P_B(x) = \dim \mathrm{Proj} B = s$, say,

(3) the leading coefficient of $P_B(x)$ is of the form $\frac{e_B}{s!}$.

Definition 3.6. The number e_B is called the *degree* of the graded ring B, or also the degree of $\mathrm{Proj}\, B$.

Theorem 3.7. *The degree of $K[X(\mathcal{L})]$ is equal to the number of maximal chains in $\mathcal{L}$.*

Proof. Let $\mathfrak{a}(\mathcal{L})$ be as above. We begin by putting a monomial order on $K[X_\alpha,\, \alpha \in \mathcal{L}]$. Consider the reverse partial order on $\mathcal{L}$, and extend it to a total order, denoted $\leq_{tot}$, on the variables $\{X_\alpha,\, \alpha \in \mathcal{L}\}$. We now take the monomial order defined as follows. For $\alpha_1 \leq_{tot} \ldots \leq_{tot} \alpha_r$, $\beta_1 \leq_{tot} \ldots \leq_{tot} \beta_s$, we say $X_{\alpha_1} \cdots X_{\alpha_r} \prec X_{\beta_1} \cdots X_{\beta_s}$ if and only if either $r < s$ or $r = s$ and there exists a $t < r$ such that $\alpha_1 = \beta_1, \ldots, \alpha_t = \beta_t$, $\alpha_{t+1} <_{tot} \beta_{t+1}$. From [**13**], we have that $\{X_\alpha X_\beta - X_{\alpha\wedge\beta} X_{\alpha\vee\beta} \mid \alpha, \beta \in \mathcal{L} \text{ non-comparable}\}$ is a Gröbner basis for $\mathfrak{a}(\mathcal{L})$ for this monomial order. Hence, letting I be the ideal generated by initial terms of elements of $\mathfrak{a}(\mathcal{L})$, we have that $\{X_\alpha X_\beta \mid \alpha, \beta \text{ non-comparable}\}$ is a generating set for I. Let us denote $K[X(\mathcal{L})]$ by S, and $K[X_\alpha,\, \alpha \in \mathcal{L}]/I$ by R. By [**8**, §15.8], we have a flat degeneration of $\mathrm{Spec}(S)$ to $\mathrm{Spec}(R)$. Hence, the degree of S equals the degree of R.

Let $J = \{j_1, \ldots, j_s\}$ be a subset of $\mathcal{L}$ such that $X_{j_1} \cdots X_{j_s} \notin I$. Note that J is thus a chain of length $s-1$ in $\mathcal{L}$. We have

$$R = K \oplus \bigoplus_{J=\{j_1,\ldots,j_s\}} (X_{j_1} \cdots X_{j_s})\, K[X_{j_1}, \ldots, X_{j_s}],$$

where J runs over all chains of any length in $\mathcal{L}$. Therefore, we have

$$\phi_m(R) = \dim R_m = \sum_{J=\{j_1,\ldots,j_s\}} \binom{s + (m-s) - 1}{m-s} = \sum_{J=\{j_1,\ldots,j_s\}} \binom{m-1}{s-1}.$$

Note that for m sufficiently large, the leading term appears in the summation above only for J of maximal cardinality s. The result follows from this. $\square$

Next we recall $mult_P X$, the multiplicity of an algebraic variety X at a point $P \in X$: Let $\mathcal{O}_{X,P} = (A, \mathfrak{m})$. Let C_P be the tangent cone at P, namely $C_P = \operatorname{Spec} A(P)$, where $A(P) = gr(A, \mathfrak{m})$. Then the multiplicity of X at P is defined to be:

$$mult_P X = \deg \operatorname{Proj} A(P) \, (= \deg A(P)).$$

Hence (using the notation from §3.5) we obtain, $e_B = mult_0 \operatorname{Spec}(B)$, the multiplicity of $\operatorname{Spec}(B)$ at the origin.

As a direct consequence of Theorem 3.7, we have:

THEOREM 3.8. *The multiplicity of $X(\mathcal{L})$ at the origin is equal to the number of maximal chains in $\mathcal{L}$.*

3.9. Cone and dual cone of $X(\mathcal{L})$. As above, denote the poset of join-irreducibles in $\mathcal{L}$ by J. Let $\mathcal{I}(J)$ denote the poset of ideals of J. For $A \in \mathcal{I}(J)$, let $\mathbf{m}_A$ denote the monomial

$$\mathbf{m}_A := \prod_{\tau \in A} y_\tau$$

in the polynomial algebra $K[y_\tau, \tau \in J(\mathcal{L})]$. If α is the element of $\mathcal{L}$ such that $I_\alpha = A$ (cf. Theorem 2.9), then we shall denote $\mathbf{m}_A$ also by $\mathbf{m}_\alpha$. Consider the surjective algebra map

$$F : K[X_\alpha, \alpha \in \mathcal{L}] \to K[\mathbf{m}_A, A \in \mathcal{I}(J)], \; X_\alpha \mapsto \mathbf{m}_\alpha.$$

THEOREM 3.10 (cf. [**14**], [**20**]). *We have an isomorphism $K[X(\mathcal{L})] \cong K[\mathbf{m}_A, A \in \mathcal{I}(J)]$.*

Let us denote the torus acting on the toric variety $X(\mathcal{L})$ by T; by Theorem 3.3, we have, $\dim T = \#J(\mathcal{L}) = d$, say. Identifying T with $(K^*)^d$, let $\{f_z, z \in J(\mathcal{L})\}$

denote the standard $\mathbb{Z}$-basis for $X(T)$, namely, for $t = (t_z)_{z \in J(\mathcal{L})}$, $f_z(t) = t_z$. Denote $M := X(T)$; let N be the $\mathbb{Z}$-dual of M, and $\{e_y, y \in J(\mathcal{L})\}$ be the basis of N dual to $\{f_z, z \in J(\mathcal{L})\}$. For $A \in \mathcal{I}(J)$, set

$$f_A := \sum_{z \in A} f_z.$$

Let $\sigma \subset N_{\mathbb{R}}$ be the cone such that $X(\mathcal{L}) = X_\sigma$.

As an immediate consequence of Theorem 3.10, we have

Proposition 3.11. *The semigroup S_σ is generated by $f_A, A \in \mathcal{I}(J)$.*

Let $M(J(\mathcal{L}))$ be the set of maximal elements in the poset $J(\mathcal{L})$. Let $Z(J(\mathcal{L}))$ denote the set of all covers in the poset $J(\mathcal{L})$.

Proposition 3.12 (cf. [**20**], Proposition 4.7). *The cone σ is generated by*

$$\{e_z \mid z \in M(J(\mathcal{L}))\} \cup \{e_{y'} - e_y \mid (y, y') \in Z(J(\mathcal{L}))\}.$$

3.13. The sublattice D_τ. We shall concern ourselves just with the closed points in $X(\mathcal{L})$. So in the sequel, by a point in $X(\mathcal{L})$, we shall mean a closed point. Let τ be a face of σ, and P_τ the distinguished point (see §1.3).

For a point $P \in X(\mathcal{L})$ (identified with a point in $\mathbb{A}^{\#\mathcal{L}}$), let us denote by $P(\alpha)$, the αth coordinate of P. Let

$$D_\tau = \{\alpha \in \mathcal{L} \mid P_\tau(\alpha) \neq 0\}.$$

We have the following lemma.

Lemma 3.14 (cf. [**20**]). *D_τ is an embedded sublattice. Conversely, let D be an embedded sublattice in $\mathcal{L}$. Then D determines a unique face τ' of σ such that $D_{\tau'} = D$.*

Thus in view of the lemma above, we have a bijection

$$\{\text{faces of } \sigma\} \overset{bij}{\leftrightarrow} \{\text{embedded sublattices of } \mathcal{L}\}.$$

PROPOSITION 3.15 (cf. [**20**]). *Let τ be a face of σ. Then we have $\overline{O_\tau} = X(D_\tau)$.*

4. Grid lattices and minuscule lattices

In this section, we restrict our attention to a specific class of distributive lattices. Give $\mathbb{N} \times \mathbb{N}$ the lattice structure

$$(\alpha_1, \alpha_2) \wedge (\beta_1, \beta_2) = (\delta_1, \delta_2), \ (\alpha_1, \alpha_2) \vee (\beta_1, \beta_2) = (\gamma_1, \gamma_2),$$

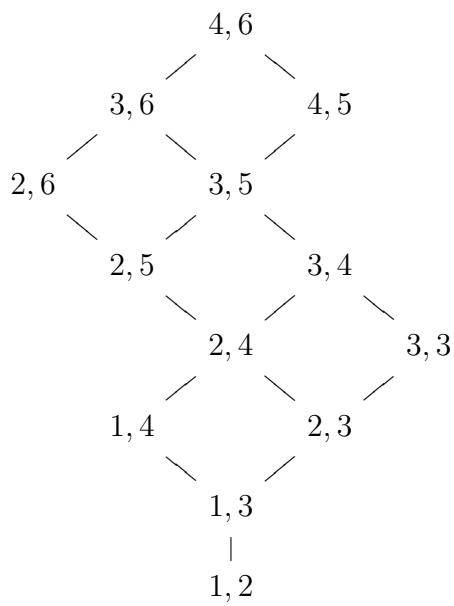
where $\delta_i = \min\{\alpha_i, \beta_i\}$, $\gamma_i = \max\{\alpha_i, \beta_i\}$.

DEFINITION 4.1. Let J be a finite, distributive sublattice of $\mathbb{N} \times \mathbb{N}$, such that if α covers β in J, then α covers β in $\mathbb{N} \times \mathbb{N}$ as well. Then we say J is a *grid lattice*.

REMARK 4.2. For J a grid lattice, we have the following:

(1) J is a distributive lattice.
(2) For any $\mu \in J$, there exist at most two distinct covers of the form (α, μ) in J, i.e., there are at most two elements in J covering μ.
(3) For any $\lambda \in J$, λ covers at most two distinct elements in J.
(4) If α, β are two covers of μ in J, then $\alpha \vee \beta$ covers both α, β; thus the interval $[\mu, \alpha \vee \beta]$ is a rank 2 subposet of J.

EXAMPLE 4.3. The following is an example of a grid lattice.



DEFINITION 4.4. We now turn our attention to distributive lattices $\mathcal{L}$ such that $J(\mathcal{L})$ is a grid lattice. We will refer to such a lattice as a *lattice with JIGL*. (JIGL is short for Join Irreducibles (forming a) Grid Lattice.)

As stated at the beginning of this chapter, our main goal is to study Bruhat-Hibi toric varieties (defined in Definition 4.9). We now show the connection between Bruhat-Hibi toric varieties and lattices with JIGL.

4.5. Minuscule weights and lattices. Let G be a semisimple, simply connected algebraic group. Let T be a maximal torus in G. Let $X(T)$ be the character group of T, and B a Borel subgroup containing T. Let R be the root system of G relative to T; let R^+ (resp. $S = \{\alpha_1, \cdots, \alpha_l\}$) be the set of positive (resp. simple) roots in R relative to B (here, l is the rank of G). Let $\{\omega_i, 1 \leq i \leq l\}$ be the fundamental weights. Let W be the Weyl group of G, and $(,)$ a W-invariant inner product on $X(T) \otimes \mathbb{R}$. For generalities on semisimple algebraic groups, we refer the reader to [**4**].

Let P be a maximal parabolic subgroup of G with ω as the associated fundamental weight. Let W_P be the Weyl group of P (note that W_P is the subgroup of W generated by $\{s_\alpha \mid \alpha \in S_P \subset S\}$). Let $W^P = W/W_P$. We have that the Schubert varieties of G/P are indexed by W^P, and thus W^P can be given the partial order induced by the inclusion of Schubert varieties.

Definition 4.6. A fundamental weight ω is called *minuscule* if $\langle \omega, \beta \rangle \, (= \frac{2(\omega,\beta)}{(\beta,\beta)}) \leq 1$ for all $\beta \in R^+$; the maximal parabolic subgroup associated to ω is called a *minuscule parabolic subgroup*.

Remark 4.7 (cf [**15**]). Let P be a maximal parabolic subgroup; if P is minuscule then W/W_P is a distributive lattice.

Definition 4.8. For P a minuscule parabolic subgroup, we call $\mathcal{L} = W/W_P$ a *minuscule lattice*.

Definition 4.9. We call $X(\mathcal{L})$ a *Bruhat-Hibi toric variety* (B-H toric variety for short) if $\mathcal{L}$ is a minuscule lattice.

For convenience, we list all of the minuscule fundamental weights here. Following the indexing of the simple roots as in [**5**], we have the complete list of minuscule weights for each type of semisimple algebraic group:

$$\begin{aligned}
\text{Type } \mathbf{A_n} &: \text{ Every fundamental weight is minuscule} \\
\text{Type } \mathbf{B_n} &: \omega_n \\
\text{Type } \mathbf{C_n} &: \omega_1 \\
\text{Type } \mathbf{D_n} &: \omega_1, \omega_{n-1}, \omega_n \\
\text{Type } \mathbf{E_6} &: \omega_1, \omega_6 \\
\text{Type } \mathbf{E_7} &: \omega_7.
\end{aligned}$$

There are no minuscule weights in types $\mathbf{E_8}, \mathbf{F_4}$, or $\mathbf{G_2}$.

Before proving that each minuscule lattice is a lattice with JIGL, we must introduce some additional lattice notation. For a poset P, let $\mathcal{I}(P)$ represent the lattice of ideals of P. Thus for a distributive lattice $\mathcal{L}$, $\mathcal{L} = \mathcal{I}(J(\mathcal{L}))$ (cf. Theorem 2.9). (Notice that the empty set is considered the minimal ideal, and in Theorem 2.9 we do not include the minimal element in P. Therefore, in this section, $\mathcal{I}(J)$ will have a minimal element that is not an element of J.)

For $k \in \mathbb{N}$, let $\underline{k}$ be the the totally ordered set with k elements. The symbols $\oplus$ and $\times$ denote the disjoint union and (Cartesian) product of posets.

Let $\mathbf{X_n}(\omega_i)$ denote the minuscule lattice W/W_P where P is a parabolic subgroup associated to ω_i in the root system of type $\mathbf{X_n}$.

Theorem 4.10 (cf. [**28**], Propositions 3.2 and 4.1[1]). *The minuscule lattices have the following combinatorial descriptions:*

$$\begin{aligned}
&\mathbf{A_{n-1}}(\omega_j) \cong \mathcal{I}\left(\mathcal{I}\left(\underline{j-1} \oplus \underline{n-j-1}\right)\right)\\
&\mathbf{C_n}(\omega_1) \cong \underline{2n}\\
&\mathbf{B_n}(\omega_n) \cong \mathbf{D_{n+1}}(\omega_{n+1}) \cong \mathbf{D_{n+1}}(\omega_n) \cong \mathcal{I}(\mathcal{I}(\mathcal{I}(\underline{1} \oplus \underline{n-2})))\\
&\mathbf{D_n}(\omega_1) \cong \mathcal{I}^{n-1}(\underline{1} \oplus \underline{1})\\
&\mathbf{E_6}(\omega_1) \cong \mathbf{E_6}(\omega_6) \cong \mathcal{I}^4(\underline{1} \oplus \underline{2})\\
&\mathbf{E_7}(\omega_7) \cong \mathcal{I}^5(\underline{1} \oplus \underline{2}).
\end{aligned}$$

This theorem is very convenient in working with the faces of B-H toric varieties, because the join irreducible lattice of each of these minuscule lattices is very easy to see, simply by eliminating one $\mathcal{I}(\cdot)$ operation. Our goal is to show that the join irreducibles of each minuscule lattice is in fact a grid lattice.

4.11. Minuscule lattices $\mathbf{A_{n-1}}(\omega_j)$.

[1]Our notation differs significantly than that used in [**28**]; namely, where we use $\mathcal{I}$, Proctor uses J; whereas we use J to signify the set of join irreducibles.

REMARK 4.12 (cf. [**28**], Proposition 4.2). The join irreducibles of the minuscule lattice $\mathbf{A_{n-1}}(\omega_j)$ are isomorphic to the lattice

$$\underline{j} \times \underline{n-j}.$$

Therefore, every element of $J(\mathbf{A_{n-1}}(\omega_j))$ can be written as the pair (a, b), for $1 \leq a \leq j$, $1 \leq b \leq n-j$. This leads us to the following result,

COROLLARY 4.13. *The minuscule lattice* $\mathbf{A_{n-1}}(\omega_j)$ *is a lattice with JIGL.*

4.14. Minuscule lattices $\mathbf{C_n}(\omega_1)$. This minuscule lattice is totally ordered, and the associated B-H toric variety is simply the affine space of dimension $2n$.

4.15. Minuscule lattices $\mathbf{B_{n-1}}(\omega_{n-1}) \cong \mathbf{D_n}(\omega_{n-1}) \cong \mathbf{D_n}(\omega_n)$. From Theorem 4.10, we have

$$J(\mathbf{D_n}(\omega_n)) \cong \mathcal{I}^2(\underline{1} \oplus \underline{n-3}) \cong \mathbf{A_{n-1}}(\omega_2).$$

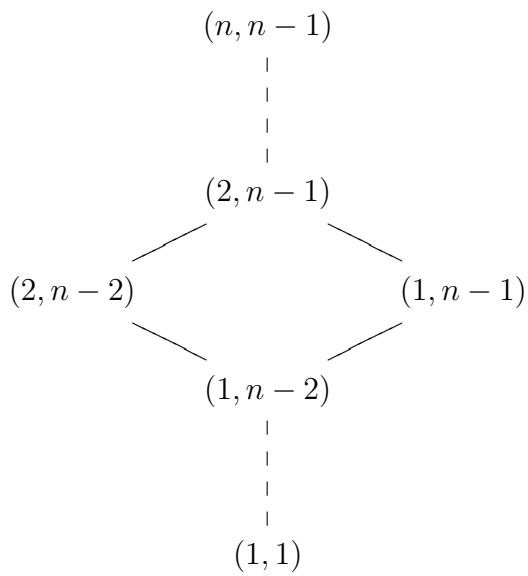
It is a well known result that $\mathbf{A_{n-1}}(\omega_2)$ represents the lattice of Schubert varieties in the Grassmannian of 2-planes in K^n, and the Schubert varieties are indexed by $I_{2,n} = \{(i_1, i_2) \mid 1 \leq i_1 < i_2 \leq n\}$. Therefore,

$$J(\mathbf{D_n}(\omega_n)) \cong I_{2,n}.$$

The lattice $I_{2,n}$ is therefore distributive, (being another minuscule lattice), and clearly a grid lattice. This leads to the following result,

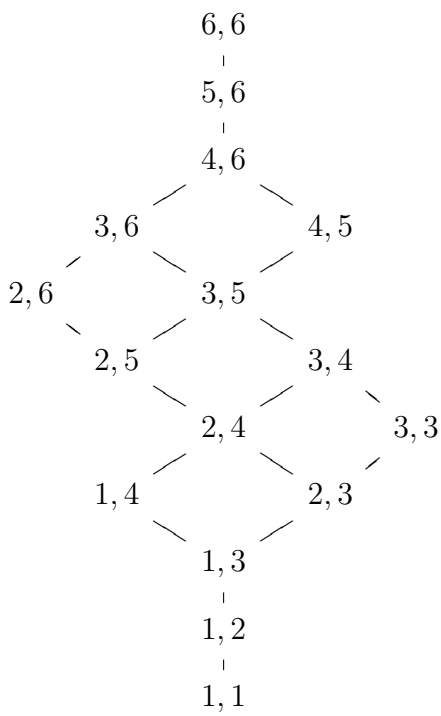
COROLLARY 4.16. *The minuscule lattices* $\mathbf{B_{n-1}}(\omega_{n-1})$, $\mathbf{D_n}(\omega_{n-1})$, *and* $\mathbf{D_n}(\omega_n)$ *are lattices with JIGL.*

4.17. Minuscule lattices $\mathbf{D_n}(\omega_1)$. From Theorem 4.10, we have $J(\mathbf{D_n}(\omega_1)) \cong \mathcal{I}^{n-2}(\underline{1} \oplus \underline{1})$. This lattice of join irreducibles is isomorphic to the following sublattice of $\mathbb{N} \times \mathbb{N}$:

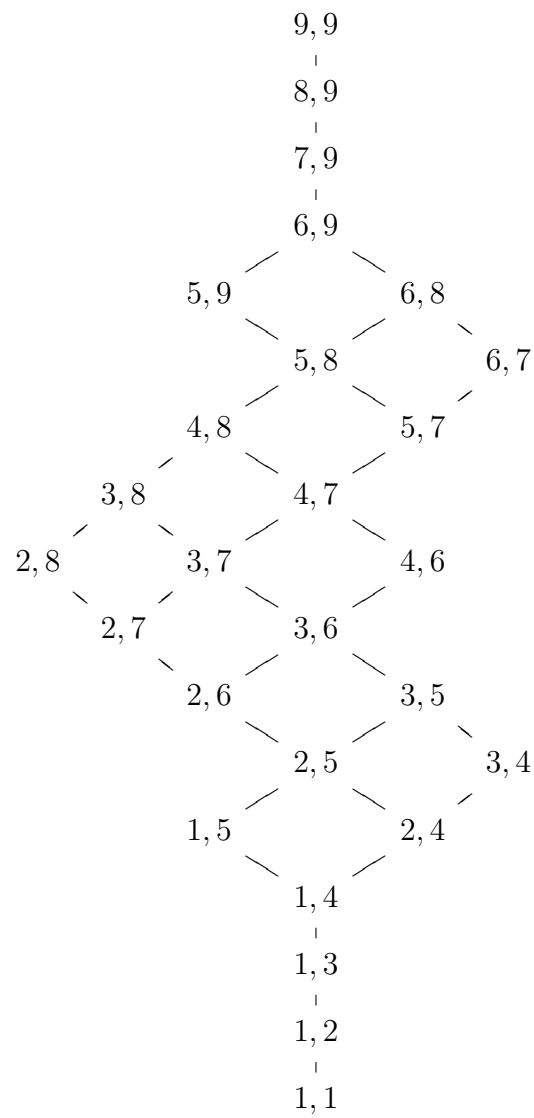


Clearly this is a grid lattice.

4.18. Minuscule lattices $\mathbf{E_6}(\omega_1) \cong \mathbf{E_6}(\omega_6)$, and $\mathbf{E_7}(\omega_7)$. Let $\mathbf{E_6} = \mathbf{E_6}(\omega_1) = \mathbf{E_6}(\omega_6)$ and $\mathbf{E_7} = \mathbf{E_7}(\omega_7)$. Since there are only two exceptional cases, it is best to explicitly give the grid lattice structure to the join irreducibles. Thus, we have the two join irreducible lattices below, with each lattice point given coordinates in $\mathbb{N} \times \mathbb{N}$. Coincidentally, $J(\mathbf{E_6}) = \mathbf{D_5}(\omega_5)$ and $J(\mathbf{E_7}) = \mathbf{E_6}$.



$J(\mathbf{E_6})$



$J(\mathbf{E_7})$

This completes the individual discussion for each type of minuscule lattice, leading us to the following result.

Corollary 4.19. *If $\mathcal{L}$ is a minuscule lattice, then $J(\mathcal{L})$ is a grid lattice.*

5. Results on a lattice with JIGL

For this entire section, we let $\mathcal{L}$ be a lattice with JIGL. Let J denote $J(\mathcal{L})$. It will often be useful to view elements of $\mathcal{L}$ as ideals in J. Recall that for $x, y \in \mathcal{L}$, $x \geq y$ if and only if $I_x \supseteq I_y$ as ideals in J.

LEMMA 5.1. *Given $\gamma_1, \gamma_2 \in J$, $(\gamma_1 \wedge \gamma_2)_{\mathcal{L}}$ belongs to J and is in fact equal to $(\gamma_1 \wedge \gamma_2)_J$.*

PROOF. Let $\theta = (\gamma_1 \wedge \gamma_2)_J$ and $\phi = (\gamma_1 \wedge \gamma_2)_{\mathcal{L}}$. Clearly $\theta \in I_{\gamma_1} \cap I_{\gamma_2} = I_\phi$. Therefore $I_\theta \subset I_\phi$. Let now $\eta \in I_\phi (\subset J)$. Then $\eta \leq \phi$, and thus η is less than or equal to both γ_1 and γ_2 in $\mathcal{L}$, and therefore in J. Hence $\eta \leq \theta$, and thus $I_\phi \subset I_\theta$. The result follows. □

LEMMA 5.2. *Let (α, β) be an incomparable pair of irreducibles (cf. Definition 2.6) in $\mathcal{L}$. Then*

(1) *α, β are meet irreducibles in J,*
(2) *$(\alpha \wedge \beta)_{\mathcal{L}} = (\alpha \wedge \beta)_J \in J$.*

PROOF. Part (2) follows from Lemma 5.1, (note that $\alpha, \beta \in J$). Now say $\alpha = (\gamma_1 \wedge \gamma_2)_J$ for an incomparable pair (γ_1, γ_2) in J. Lemma 5.1 implies that $\alpha = (\gamma_1 \wedge \gamma_2)_{\mathcal{L}}$, a contradiction since α is meet irreducible in $\mathcal{L}$. Part (1) follows. □

Thus an incomparable pair (α, β) of irreducibles in $\mathcal{L}$ determines a (unique) non-meet irreducible in J (namely, $(\alpha \wedge \beta)_{\mathcal{L}} = (\alpha \wedge \beta)_J$). We shall now show (cf. Lemma 5.5 below) that conversely a non-meet irreducible element μ in J determines a unique incomparable pair (α, β) of irreducibles in $\mathcal{L}$. We first prove a couple of preliminary results:

LEMMA 5.3. *Let μ be a non-meet irreducible element in J. Then μ determines an incomparable pair (α, β) of elements (in J) both of which are meet irreducible in J.*

PROOF. Let $\mu = (\mu_1, \mu_2)$ (considered as an element of $\mathbb{N} \times \mathbb{N}$). Since μ is non-meet irreducible element in J, there exist $x = (x_1, x_2), y = (y_1, y_2)$ in J, $x, y > \mu$ such that $x_2 > \mu_2, y_1 > \mu_1$. Define $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ in J as

$$\alpha = \text{ the maximal element } x > \mu \text{ in } J \text{ such that } x_1 = \mu_1,$$

$$\beta = \text{ the maximal element } y > \mu \text{ in } J \text{ such that } y_2 = \mu_2.$$

Clearly α, β are both meet-irreducible in J (note that $(\mu_1 + 1, \alpha_2)$ (resp. $(\beta_1, \mu_2 + 1)$) is the unique element in J covering α (resp. β) in J). Also, it is clear that (α, β) is an incomparable pair. □

Let μ, α, β be as in the above Lemma. In particular, we have, $\mu_1 = \alpha_1 < \beta_1$, $\mu_2 = \beta_2 < \alpha_2$.

LEMMA 5.4. *With notation as in Lemma 5.3, we have,*

(1) $(\alpha \vee \beta)_J = (\beta_1, \alpha_2)$.
(2) *α is the maximal element of the set $\{x = (x_1, x_2) \in J \mid x_1 = \alpha_1\}$, and β is the unique maximal element of the set $\{x = (x_1, x_2) \in J \mid x_2 = \beta_2\}$.*

PROOF. Assertion (2) is immediate from the definition of α, β. Assertion (1) is also clear. □

LEMMA 5.5. *Let μ, α, β be as in Lemma 5.3. Then α and β are irreducibles in $\mathcal{L}$. Thus the non-meet irreducible element μ of J determines a unique incomparable pair of irreducibles in $\mathcal{L}$.*

PROOF. We will show the result for α (the proof for β being similar). Since $\alpha \in J$, α is join irreducible in $\mathcal{L}$. It remains to show that α is meet irreducible in $\mathcal{L}$. If possible, let us assume that there exists an incomparable pair (θ_1, θ_2) in $\mathcal{L}$ such

that $\theta_1 \wedge \theta_2 = \alpha$; without loss of generality, we may suppose that θ_1 and θ_2 both cover α. Then there exist (cf. Lemma 2.10) $\gamma, \delta \in J$ such that

$$I_{\theta_1} = I_\alpha \dot{\cup} \{\gamma\},\ I_{\theta_2} = I_\alpha \dot{\cup} \{\delta\}.$$

We have

$$I_\gamma \cap I_\delta \subset I_{\theta_1} \cap I_{\theta_2} = I_\alpha. \tag{$*$}$$

Also, γ, δ are either covers of α in J, or non-comparable to α. (They cannot be less than α because they are not in I_α.)

Case 1: Suppose γ and δ are covers of α in J. Then α is not meet irreducible in J, a contradiction (cf. Lemma 5.2,(1)).

Case 2: Suppose γ covers α in J, and δ is non-comparable to α. Let $\delta = (\delta_1, \delta_2)$, $\xi = (\xi_1, \xi_2) = (\alpha \vee \delta)_J$. Then the fact that $\xi > \alpha$ (since α, δ are incomparable) implies (in view of Lemma 5.4, (2)) that $\xi_1 > \mu_1$; hence $\delta_1 (= \xi_1) \geq \mu_1 + 1$, and $\delta_2 < \alpha_2$. Also, $\gamma = (\mu_1 + 1, \alpha_2)$ (cf. Lemma 5.4, (2)). Therefore $\gamma \wedge \delta = (\mu_1 + 1, \delta_2)$, but this element is non-comparable to α, and thus $I_\gamma \cap I_\delta \not\subset I_\alpha$, a contradiction to $(*)$. Hence we obtain that the possibility "γ covers α in J and δ is non-comparable to α" does not exist. A similar proof shows that the possibility "δ covers α in J and γ is non-comparable to α" does not exist.

Case 3: Suppose both $\gamma = (\gamma_1, \gamma_2)$ and $\delta = (\delta_1, \delta_2)$ are non-comparable to $\alpha = (\mu_1, \alpha_2)$. As in Case 2, we must have $\delta_2 < \alpha_2$, and thus $\delta_1 > \mu_1$. Similarly, $\gamma_2 < \alpha_2$, $\gamma_1 > \mu_1$. Thus the minimum of $\{\gamma_1, \delta_1\}$ is still greater than μ_1, therefore $I_\gamma \cap I_\delta \not\subset I_\alpha$, a contradiction to $(*)$.

Thus our assumption that α is non-meet irreducible in $\mathcal{L}$ is wrong, and it follows that α (and similarly β) is meet irreducible in $\mathcal{L}$. $\square$

We continue with the above notation; in particular, we denote $\mu = (\mu_1, \mu_2), \mu_1 = \alpha_1 < \beta_1$, $\mu_2 = \beta_2 < \alpha_2$.

LEMMA 5.6. *Let $x = (x_1, x_2) \in J$. If $x \notin I_\alpha \cup I_\beta$, then $x > \alpha \wedge \beta$.*

PROOF. By hypothesis, we have $x \not\leq \alpha$, $x \not\leq \beta$.

We first claim that $x_1 > \alpha_1$; for, if possible, let us assume $x_1 \leq \alpha_1$. Since $x \not\leq \alpha$, we must have $x_2 > \alpha_2$. Thus $x \vee \alpha = (\alpha_1, x_2)$ ($> \alpha$, since $\alpha \not\geq x$); but this is a contradiction, by the property of α (cf. Lemma 5.4,(2)). Hence our assumption is wrong, and we get $x_1 > \alpha_1$.

Similarly, we have, $x_2 > \beta_2$, and the result follows (note that by our notation (and definition of α, β), we have $\alpha \wedge \beta = (\alpha_1, \beta_2)$). □

DEFINITION 5.7. For an incomparable (unordered) pair (α, β) of irreducible elements in $\mathcal{L}$, define

$$\mathcal{L}_{\alpha,\beta} = \mathcal{L} \setminus [\alpha \wedge \beta, \alpha \vee \beta].$$

PROPOSITION 5.8. *$\mathcal{L}_{\alpha,\beta}$ is an embedded sublattice.*

PROOF. First, we show that $\mathcal{L}_{\alpha,\beta}$ is a sublattice. To do this, we identify $\mathcal{L}$ with the "lattice of ideals" of J. Thus, for $x \in \mathcal{L}_{\alpha,\beta}$, either $I_x \not\supset (I_\alpha \cap I_\beta)$ or $I_x \not\subset (I_\alpha \cup I_\beta)$, by definition of $\mathcal{L}_{\alpha,\beta}$. Note that $I_\alpha \cap I_\beta = I_{\alpha\wedge\beta}$, and $I_\alpha \cup I_\beta = I_{\alpha\vee\beta}$.

Case 1: Let $x, y \in \mathcal{L}_{\alpha,\beta}$ such that I_x, $I_y \not\supset I_{\alpha\wedge\beta}$. Then clearly $I_x \cap I_y \not\supset I_{\alpha\wedge\beta}$; and thus $x \wedge y \in \mathcal{L}_{\alpha,\beta}$. We also have (by the definition of ideals) that $\alpha \wedge \beta \notin I_x, I_y$ (note that $\alpha \wedge \beta \in J$ (cf. Lemma 5.2,(2))), therefore $\alpha \wedge \beta \notin I_x \cup I_y$, and therefore $x \vee y \in \mathcal{L}_{\alpha,\beta}$.

Case 2: Let $x, y \in \mathcal{L}_{\alpha,\beta}$ such that $I_x \not\supset I_{\alpha\wedge\beta}$ and $I_y \not\subset I_{\alpha\vee\beta}$. Then clearly $I_x \cap I_y \not\supset I_{\alpha\wedge\beta}$ and $I_x \cup I_y \not\subset I_\alpha \cup I_\beta$. Hence, $x \vee y, x \wedge y$ are in $\mathcal{L}_{\alpha,\beta}$.

Case 3: Let $x, y \in \mathcal{L}_{\alpha,\beta}$ such that I_x, $I_y \not\subset I_{\alpha\vee\beta}$. Clearly $I_x \cup I_y \not\subset I_\alpha \cup I_\beta$; hence, $x \vee y \in \mathcal{L}_{\alpha,\beta}$.

Claim: $I_x \cap I_y \not\subset I_\alpha \cup I_\beta$.

Note that Claim implies that $x \wedge y \in \mathcal{L}_{\alpha,\beta}$. If possible, let us assume that $I_x \cap I_y \subset I_\alpha \cup I_\beta$. Now the hypothesis that $I_x,\ I_y \not\subset I_\alpha \cup I_\beta$ implies that there exist $\theta, \delta \in J$ such that $\theta \in I_x, \theta \notin I_\alpha \cup I_\beta$, and $\delta \in I_y, \delta \notin I_\alpha \cup I_\beta$. Now $I_\theta \cap I_\delta \subset I_x \cap I_y \subset I_\alpha \cup I_\beta$ (note that by our assumption, $I_x \cap I_y \subset I_\alpha \cup I_\beta$). Hence we obtain that either $\theta \wedge \delta \leq \alpha$ or $\theta \wedge \delta \leq \beta$; let us suppose $\theta \wedge \delta \leq \alpha$ (proof is similar if $\theta \wedge \delta \leq \beta$). By Lemma 5.6, we have that both $\theta, \delta \geq \alpha \wedge \beta$, and hence $\theta \wedge \delta \geq \alpha \wedge \beta$. Thus

$$\alpha \geq \theta \wedge \delta \geq \alpha \wedge \beta = (\alpha_1, \beta_2) \tag{$**$}$$

Let $\xi(= (\xi_1, \xi_2)) = \theta \wedge \delta$. Then $(**)$ implies that $\xi_1 = \alpha_1$; hence at least one of $\{\theta_1, \delta_1\}$, say θ_1 equals α_1. This implies that $\theta_2 > \alpha_2$ (since $\theta \notin I_\alpha$). This contradicts Lemma 5.4(2). Hence our assumption is wrong and it follows that $I_x \cap I_y \not\subset I_\alpha \cup I_\beta$.

This completes the proof in Case 3. Thus we have shown that $\mathcal{L}_{\alpha,\beta}$ is a sublattice.

Next, we will show that $\mathcal{L}_{\alpha,\beta}$ is an embedded sublattice. Let $x, y \in \mathcal{L}$ be such that $x \vee y, x \wedge y$ are in $\mathcal{L}_{\alpha,\beta}$. We need to show that $x, y \in \mathcal{L}_{\alpha,\beta}$. This is clear if either $x \wedge y \not\leq \alpha \vee \beta$ or $x \vee y \not\geq \alpha \wedge \beta$ (in the former case, $x, y \not\leq \alpha \vee \beta$, and in the latter case, $x, y \not\geq \alpha \wedge \beta$). Let us then suppose that $x \wedge y \leq \alpha \vee \beta$ and $x \vee y \geq \alpha \wedge \beta$; this implies that $x \wedge y \not\geq \alpha \wedge \beta$ and $x \vee y \not\leq \alpha \vee \beta$ (since, $x \vee y, x \wedge y$ are in $\mathcal{L}_{\alpha,\beta}$), i.e., $I_x \cap I_y \not\supset I_\alpha \cap I_\beta$ and $I_x \cup I_y \not\subset I_\alpha \cup I_\beta$. We will now show that $x, y \in \mathcal{L}_{\alpha,\beta}$.

Since $\alpha \wedge \beta \notin I_x \cap I_y$, we have that one of the elements $\{x, y\}$ must not be greater than or equal to $\alpha \wedge \beta$, say $x \not\geq \alpha \wedge \beta$. This implies that $x \in \mathcal{L}_{\alpha,\beta}$. It remains to show that $y \in \mathcal{L}_{\alpha,\beta}$. If $y \not\geq \alpha \wedge \beta$, then we would obtain that $y \in \mathcal{L}_{\alpha,\beta}$. Let us then assume that $y \geq \alpha \wedge \beta$; i.e. $I_y \supset I_\alpha \cap I_\beta$. Note that for any $\delta \in I_x$, we have $\delta \leq x$ and thus $\delta \not\geq \alpha \wedge \beta$. By Lemma 5.6, $\delta \in I_\alpha \cup I_\beta$, and therefore $I_x \subset I_\alpha \cup I_\beta$. Since by hypothesis $I_x \cup I_y \not\subset I_\alpha \cup I_\beta$, we must have $I_y \not\subset I_\alpha \cup I_\beta$. Therefore, $y \in \mathcal{L}_{\alpha,\beta}$.

This completes the proof of the assertion that $\mathcal{L}_{\alpha,\beta}$ is an embedded sublattice, and therefore the proof of the Proposition. $\square$

6. Singular locus of $X(\mathcal{L})$

In this section, we determine the singular locus of $X(\mathcal{L})$, $\mathcal{L}$ being a lattice with JIGL. Let σ be the cone associated to $X(\mathcal{L})$. We follow the notation of §1 and §2.

Definition 6.1. A face τ of σ is a *singular* (resp. *non-singular*) face if P_τ is a singular (resp. non-singular) point of X_σ.

Definition 6.2. Let us denote by W the set of generators for σ as described in Proposition 3.12. Let τ be a face of σ, and let D_τ be as in § 3.13. Define

$$W(\tau) = \{v \in W \mid f_{I_\alpha}(v) = 0, \forall\, \alpha \in D_\tau\}.$$

Then $W(\tau)$ gives a set of generators for τ.

6.3. Determination of $W(\tau)$. Let (α, β) be an incomparable (unordered) pair of irreducible elements of $\mathcal{L}$. By Proposition 5.8, $\mathcal{L}_{\alpha,\beta}$ is an embedded sublattice of $\mathcal{L}$ ($\mathcal{L}_{\alpha,\beta}$ being as in Definition 5.7). Let $\tau_{\alpha,\beta}$ be the face of σ corresponding to $\mathcal{L}_{\alpha,\beta}$ (cf. Lemma 3.14; note that $D_{\tau_{\alpha,\beta}} = \mathcal{L}_{\alpha,\beta}$). Let us denote $\tau = \tau_{\alpha,\beta}$. Following the notation of §4, let $\mu(= (\mu_1, \mu_2)) = \alpha \wedge \beta, \alpha_1 = \mu_1, \beta_2 = \mu_2$. Since μ is not meet irreducible in J, there are two elements A and B in J covering μ, namely, $A = (\alpha_1, \beta_2 + 1), B = (\alpha_1 + 1, \beta_2)$. Also, we have that $A \vee B$ (in the lattice J) covers both A and B, (cf. Remark 4.2). Let $C = (A \vee B)_J$; then $C = (\alpha_1 + 1, \beta_2 + 1)$.

It will aid our proof below to notice a few facts about the generating set $W(\tau)$ of τ. First of all, $e_{\hat{1}}$ is not a generator for any $\tau_{\alpha,\beta}$; because $\hat{1} \in \mathcal{L}_{\alpha,\beta}$ for all pairs (α, β), and $e_{\hat{1}}$ is non-zero on $f_{I_{\hat{1}}}$.

Secondly, for any cover $(y, y'), y > y'$ in $J(\mathcal{L})$, $e_{y'} - e_y$ is not a generator of τ if $y' \in \mathcal{L}_{\alpha,\beta}$, because $f_{I_{y'}}(e_{y'} - e_y) \neq 0$. Thus, in determining the elements of $W(\tau)$, we need only be concerned with elements $e_{y'} - e_y$ of W such that $y' \in J \cap [\alpha \wedge \beta, \alpha \vee \beta]$.

Lemma 6.4. $J \cap [\alpha \wedge \beta, \alpha \vee \beta] = \{x \in J \mid x \in [\mu, \alpha] \cup [\mu, \beta]\}$.

PROOF. The inclusion $\supseteq$ is clear. To show the inclusion $\subseteq$, let $x \in J \cap [\alpha \wedge \beta, \alpha \vee \beta]$. If possible, assume $x \notin [\mu, \alpha] \cup [\mu, \beta]$; the assumption implies that $x \notin I_\alpha \cup I_\beta (= I_{\alpha \vee \beta})$. Hence we obtain that $x \not\leq \alpha \vee \beta$, a contradiction to the hypothesis that $x \in [\alpha \wedge \beta, \alpha \vee \beta]$. □

LEMMA 6.5. *The set* $\{x \in J \mid x \notin I_\alpha \cup I_\beta\}$ *has a unique minimal element; moreover that element is* C.

PROOF. For any x in this set, we have $x > \alpha \wedge \beta$ (cf. Lemma 5.6). Hence by Lemma 5.4,(2), and the hypothesis that $x \notin I_\alpha \cup I_\beta$, we obtain that $x_1 > \alpha_1$, $x_2 > \beta_2$. Therefore,

$$\{x \in J \mid x \notin I_\alpha \cup I_\beta\} = \{x \in J \mid x_1 > \alpha_1,\, x_2 > \beta_2\}.$$

This set clearly has a minimal element, namely $C = (\alpha_1 + 1, \beta_2 + 1)$. □

THEOREM 6.6. *Following the notation from above, we have*

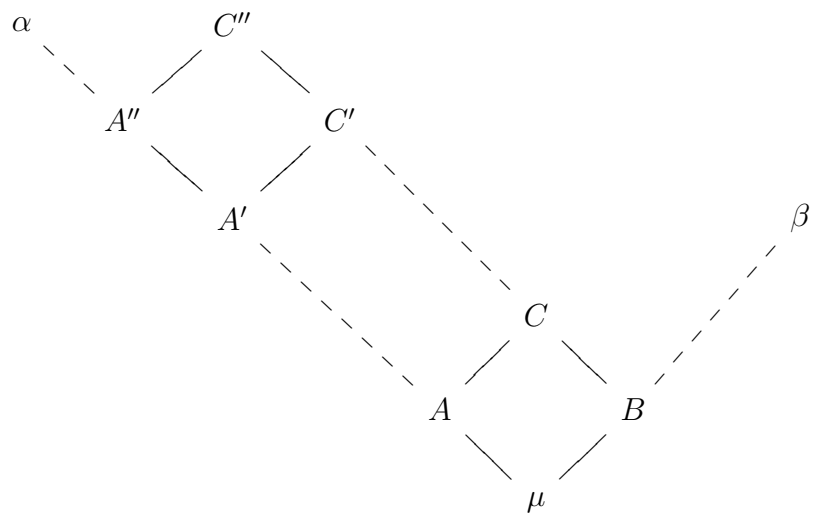
$$W(\tau) = \{e_\mu - e_A,\, e_\mu - e_B,\, e_A - e_C,\, e_B - e_C\}.$$

PROOF. **Claim 1:** $W(\tau) \supset \{e_\mu - e_A,\, e_\mu - e_B,\, e_A - e_C,\, e_B - e_C\}$.

We must show that for any $x \in \mathcal{L}_{\alpha,\beta}$, f_{I_x} is zero on these four elements of W. If possible, let us assume that there exists a $x \in \mathcal{L}_{\alpha,\beta}$ such that f_{I_x} is non-zero on some of the above four elements. Then clearly $x \geq \mu (= \alpha \wedge \beta)$. Hence $x \not\leq \alpha \vee \beta$ (since $x \notin [\alpha \wedge \beta, \alpha \vee \beta]$), i.e., $I_x \not\subset I_\alpha \cup I_\beta$. Therefore I_x contains some join irreducible γ such that $\gamma \not\leq \alpha, \beta$; hence, $I_\gamma \not\subset I_\alpha \cup I_\beta$. This implies (cf. Lemma 6.5) that $\gamma \geq C$. Hence we obtain that $C \in I_x$. Therefore, $x \geq C$, and f_{I_x} is zero on all of the four elements of Claim 1, a contradiction to our assumption. Hence our assumption is wrong and Claim 1 follows.

Claim 2: $W(\tau) = \{e_\mu - e_A,\, e_\mu - e_B,\, e_A - e_C,\, e_B - e_C\}$.

In view of § 6.3, it is enough to show that for all $\theta \in J \cap [\alpha \wedge \beta, \alpha \vee \beta]$, the element $e_\theta - e_\delta \in W$ which is different from the four elements of Claim 1 is not in $W(\tau)$. In view of Lemma 6.4, it suffices to examine all covers in J of all elements in $([\mu, \alpha] \cup [\mu, \beta])_J$. This diagram represents the part of the grid lattice J we are concerned with:



In the diagram above, consider $A' = (\alpha_1, \beta_2 + n)$, $A'' = (\alpha_1, \beta_2 + n + 1)$, and $C' = (\alpha_1 + 1, \beta_2 + n)$. Note that all elements of $J \cap [\mu, \alpha]$ can be written in the form of A'. Thus, we need to check elements $e_{A'} - e_{A''}$ and $e_{A'} - e_{C'}$ in W.

First, we observe that $C' \in \mathcal{L}_{\alpha,\beta}$, and $f_{I_{C'}}$ is non-zero on $e_{A'} - e_{A''}$. Hence $e_{A'} - e_{A''} \notin W(\tau)$.

Next, let $x = (A' \vee C)_{\mathcal{L}}$, (note that x is not in J, and thus does not appear on the diagram above). Then $I_x = I_{A'} \cup I_C$; and we have $x \in \mathcal{L}_{\alpha,\beta}$ (since, $C \notin I_\alpha \cup I_\beta$ and $x > C$, we have, $x \not\leq \alpha \vee \beta$). Moreover, f_{I_x} is non-zero on $e_{A'} - e_{C'}$. Hence $e_{A'} - e_{C'} \notin W(\tau)$.

This completes the proof for the interval $[\mu, \alpha]$, and a similar discussion yields the same result for the interval $[\mu, \beta]$.

Thus Claim 2 (and hence the Theorem) follows. $\square$

As an immediate consequence of Theorem 6.6, we have the following

THEOREM 6.7. *Let (α, β) be an incomparable pair of irreducibles in $\mathcal{L}$. We have an identification of the (open) affine piece in $X(\mathcal{L})$ corresponding to the face $\tau_{\alpha,\beta}$ with the product $Z \times (K^*)^{\#J(\mathcal{L})-3}$, where Z is the cone over the quadric surface $x_1x_4 - x_2x_3 = 0$ in $\mathbb{P}^3$.*

LEMMA 6.8. *The dimension of the face $\tau_{\alpha,\beta}$ equals 3.*

PROOF. By Theorem 6.6, a set of generators for $\tau_{\alpha,\beta}$ is given by $\{e_\mu - e_A,\, e_\mu - e_B,\, e_A - e_C,\, e_B - e_C\}$. We see that a subset of three of these generators is linearly independent. Thus if the fourth generator can be put in terms of the first three, the result follows. Notice that

$$(e_\mu - e_A) - (e_\mu - e_B) + (e_A - e_C) = e_B - e_C.$$

□

The following two lemmas hold for a general toric variety.

LEMMA 6.9. *Let X_τ be an affine toric variety with τ as the associated cone. Then X_τ is a non-singular variety if and only if it is non-singular at the distinguished point P_τ.*

PROOF. Only the implication $\Leftarrow$ requires a proof. Let then P_τ be a smooth point. Let us assume (if possible) that $\operatorname{Sing} X_\tau \neq \varnothing$. We have the following facts:

(1) $\operatorname{Sing} X_\tau$ is a closed T-stable subset of X_τ.
(2) $P_\tau \in \overline{O_\theta}$, for every face θ of τ (see §1.4); in particular, $P_\tau \in \overline{O_\theta}$, for some face θ such that P_θ is a singular point, (such a θ exists, since by our assumption Sing X_τ is non-empty).

Therefore we obtain that $P_\tau \in \text{Sing } X_\tau$, a contradiction. Hence our assumption is wrong and the result follows. □

LEMMA 6.10. *Let τ be a face of σ. Then P_τ is a smooth point of X_σ if and only if P_τ is a smooth point of U_τ, i.e., if and only if τ is generated by a part of a basis of N (N is the $\mathbb{Z}$ dual of the character group of the torus).*

PROOF. U_τ is a principal open subset of X_σ. Hence X_σ is non-singular at P_τ if and only if U_τ is non-singular at P_τ. By Lemma 6.9, U_τ is non-singular at P_τ if and only if U_τ is a non-singular variety; but by §2.1 of [**11**], this is true if and only if τ is generated by a part of a basis of N. □

Returning to our study of $X(\mathcal{L})$, we combine the two lemmas above with Theorem 6.7 and we obtain the following:

THEOREM 6.11. *$P_\tau \in Sing\, X_\sigma$, for $\tau = \tau_{\alpha,\beta}$. Further, the singularity at P_τ is of the same type as that at the vertex of the cone over the quadric surface $x_1x_4 - x_2x_3 = 0$ in $\mathbb{P}^3$.*

Next, we will show that the faces containing some $\tau_{\alpha,\beta}$ are the only singular faces.

LEMMA 6.12. *Let $(y, y'), y > y'$ be a cover in J. Then either $e_{y'} - e_y \in W(\tau_{\alpha,\beta})$ for some incomparable pair (α, β) of irreducibles in $\mathcal{L}$, or y, y' are comparable to every other element of J.*

PROOF. **Case 1:** Let y' be non-meet irreducible in J.

In view of the hypothesis, we can find an incomparable pair (α, β) of irreducibles in $\mathcal{L}$ such that $y' = \alpha \wedge \beta$, as shown in Lemma 5.5 (with $\mu = y'$). Thus $e_{y'} - e_y = e_\mu - e_A$ or $e_{y'} - e_y = e_\mu - e_B$ as in Theorem 6.6.

Case 2: Let y' be meet irreducible, but not join irreducible (in J).

Let x_1 and x_2 be the two elements covered by y' in J (cf. Remark 4.2); thus $(x_1 \vee x_2)_J = y'$.

For convenience of notation, all join and meet operations in this proof will refer to the join and meet operations in the lattice J.

Claim (a): If both x_1 and x_2 are meet irreducible (in J), then y', y are comparable to every element of J.

If possible, let us assume that there exists a $z \in J$ such that z is non-comparable to y'. We first observe that z is non-comparable to both x_1 and x_2; for, say z, x_1 are comparable, then $z > x_1$ necessarily (since z, y' are non-comparable). This implies that $x_1 \leq z \wedge y' < y'$, and hence we obtain that $x_1 = z \wedge y' < y'$ (since (y', x_1) is a cover), a contradiction to the hypothesis that x_1 is meet irreducible. Thus we obtain that z is non-comparable to both x_1 and x_2. Now, we have, $z \vee x_i \geq z \vee y'$ (note that $x_i, i = 1, 2$ being meet irreducible in J, y' is the unique element covering $x_i, i = 1, 2$, and hence $z \vee x_i \geq y'$). Hence $(z \vee y' \geq) z \vee x_i \geq z \vee y'$, and we obtain

$$z \vee x_1 = z \vee y' = z \vee x_2.$$

On the other hand, the fact that $z \wedge y' < y'$ implies that $z \wedge y' \leq x_1$ or x_2. Let i be such that $z \wedge y' \leq x_i$. Then $z \wedge y' \leq z \wedge x_i \leq z \wedge y'$; therefore

$$z \wedge x_i = z \wedge y'.$$

Now

$$y' \wedge (x_i \vee z) = y' \wedge (y' \vee z) = y'; \quad (y' \wedge x_i) \vee (y' \wedge z) = x_i \vee (x_i \wedge z) = x_i.$$

Therefore J is not a distributive lattice (Definition 2.3), a contradiction. Hence our assumption is wrong and it follows that y' is comparable to every element of J, and since y is the unique cover of y', y is also comparable to every element of J. Claim (a) follows.

Continuing with the proof in Case 2, in view of Claim (a), we may suppose that x_1 is not meet irreducible (in J). Then by Lemma 5.3 (with $\mu = x_1$), there exists a unique incomparable pair (α, β) of meet irreducibles (in J) such that $x_1 = \alpha \wedge \beta$. In view of the fact that y' is a cover of x_1, we obtain that y' is equal to A or B (A, B being as in §6.3), say $y' = A$; this in turn implies that $y = C$ (C being as in §6.3; note that by hypothesis, y is the unique element covering y' in J). Therefore we obtain that $e_{x_1} - e_{y'} (= e_{x_1} - e_A), e_{y'} - e_y (= e_A - e_C)$ are in $W(\tau_{\alpha,\beta})$.

This completes the proof of the assertion in Case 2.

Case 3: Let y' be both meet irreducible and join irreducible in J.

If y' is comparable to every other element of J, then y is also comparable to every other element of J, since by hypothesis, y is the unique element covering y' in J; and the result follows.

Let then there exist a $z \in J$ such that z and y' are incomparable. This in particular implies that $y' \neq \hat{0}_J$; let $x \in J$ be covered by y' (in fact, by hypothesis, x is unique). Proceeding as in Case 2 (especially, the proof of Claim (a)), we obtain that x is non-meet irreducible. Hence taking $\mu = x$ in Lemma 5.3 and proceeding as in Case 2, we obtain that $e_{y'} - e_y$ is in $W(\tau_{\alpha,\beta})$ ((α, β) being the incomparable pair of irreducibles determined by μ).

This completes the proof of the Lemma. □

THEOREM 6.13. *Let τ be a face of σ such that D_τ is not contained in any $\mathcal{L}_{\alpha,\beta}$, for all incomparable pair (α, β) of irreducibles in $\mathcal{L}$; in other words τ does not contain any $\tau_{\alpha,\beta}$. Then τ is nonsingular.*

PROOF. As in Definition 6.2, let

$$W(\tau) = \{v \in W \mid f_{I_\alpha}(v) = 0, \forall\, \alpha \in D_\tau\}.$$

Then $W(\tau)$ gives a set of generators for τ. By Remark 6.10 and §2.1 of [**11**], for τ to be nonsingular, it must be generated by part of a basis for N (N being as in § 1.2). If $W(\tau)$ is linearly independent, then it would follow that τ is non-singular. (Generally this is not enough to prove that τ is nonsingular; but since all generators in W have coefficients equal to ± 1, any linearly independent subset of W will serve as part of a basis for N.)

If possible, let us assume that $W(\tau)$ is linearly dependent. Recall that the elements of W can be represented as all the line segments in the lattice J, with the exception of $e_{\hat{1}}$. Therefore, the linearly dependent generators $W(\tau)$ of τ must represent a "loop" of line segments in J. This loop will have at least one bottom corner, left corner, top corner, and right corner.

Let us fix an incomparable pair (α, β) of irreducibles in $\mathcal{L}$. By Theorem 6.6, we have that $W(\tau_{\alpha,\beta}) = \{e_\mu - e_A,\, e_\mu - e_B,\, e_A - e_C,\, e_B - e_C\}$ (notation being as in that Theorem). These four generators are represented by the four sides of a diamond in J. Thus, by hypothesis, the generators of τ represent a loop in J that does not traverse all four sides of the diamond representing all four generators of $\tau_{\alpha,\beta}$. We have the following identification for $\mathcal{L}_{\alpha,\beta}$:

$$\mathcal{L}_{\alpha,\beta} = \{x \in \mathcal{L} \mid f_{I_x} \equiv 0 \,\text{on}\, W(\tau_{\alpha,\beta})\}. \tag{$\dagger$}$$

The above identification for $\mathcal{L}_{\alpha,\beta}$ together with the hypothesis that $D_\tau \not\subseteq \mathcal{L}_{\alpha,\beta}$ implies the existence of a $\theta \in D_\tau \cap [\alpha \wedge \beta, \alpha \vee \beta]$; note that by ($\dagger$), we have

$$f_{I_\theta} \not\equiv 0 \,\text{on}\, W(\tau_{\alpha,\beta})$$

This implies in particular that $\theta \not\geq C$ (C being as the proof of Theorem 6.6); also, $\theta \geq \mu (= \alpha \wedge \beta)$, since $\theta \in [\alpha \wedge \beta, \alpha \vee \beta]$. Based on how θ compares to both A and B, we can eliminate certain elements of W from $W(\tau)$. There are four possibilities; we list all four, as well as the corresponding generators in $W(\tau_{\alpha,\beta})$ which are not in

$W(\tau)$, i.e., those generators v in $W(\tau_{\alpha,\beta})$ such that $f_{I_\theta}(v) \neq 0$:

$$\begin{aligned}
\theta \ngeq A,\, \theta \ngeq B \quad &\Rightarrow \quad e_\mu - e_A,\, e_\mu - e_B \notin W(\tau) \\
\theta \geq A,\, \theta \ngeq B \quad &\Rightarrow \quad e_A - e_C,\, e_\mu - e_B \notin W(\tau) \\
\theta \ngeq A,\, \theta \geq B \quad &\Rightarrow \quad e_\mu - e_A,\, e_B - e_C \notin W(\tau) \\
\theta \geq A,\, \theta \geq B \quad &\Rightarrow \quad e_A - e_C,\, e_B - e_C \notin W(\tau)
\end{aligned}$$

Therefore, we obtain

$$\text{neither } \{e_\mu - e_A, e_A - e_C\} \text{ nor } \{e_\mu - e_B, e_B - e_C\} \text{ is contained in } W(\tau) \qquad (*)$$

for any $\tau_{\alpha,\beta}$ ((α,β) being an incomparable pair of irreducibles in $\mathcal{L}$).

Let y', z' denote respectively, the left and right corners of our loop; let $(y, y'), (z, z')$ denote the corresponding covers (in J) which are contained in our loop. Now y', z' are non-comparable; hence, by Lemma 6.12 we obtain that (y, y') (resp. (z, z')) are contained in some $W(\tau_{\alpha,\beta})$ (resp. $W(\tau_{\alpha',\beta'})$). Hence we obtain (by Theorem 6.6, with notation as in that Theorem)

$$\{e_\mu - e_{y'}, e_{y'} - e_y\} = \{e_\mu - e_A, e_A - e_C\} \text{ or } \{e_\mu - e_B, e_B - e_C\}$$

But this contradicts $(*)$. Thus our loop in J that represented $W(\tau)$ cannot have both left and right corners; therefore $W(\tau)$ is not a loop at all, a contradiction. Hence, our assumption (that $W(\tau)$ is linearly dependent) is wrong, and the result follows. □

Combining the above Theorem with Theorem 6.11 and Lemma 6.8, we obtain our first main Theorem:

THEOREM 6.14. *Let $\mathcal{L}$ be a distributive lattice such that $J(\mathcal{L})$ is a grid lattice. Then*

(1) *Sing* $X(\mathcal{L}) = \bigcup_{(\alpha,\beta)} \overline{O}_{\tau_{\alpha,\beta}}$, *the union being taken over all incomparable pairs* (α, β) *of irreducibles in* $\mathcal{L}$.

(2) *Sing* $X(\mathcal{L})$ *is pure of codimension* 3 *in* $X(\mathcal{L})$*; further, the generic singularities are of cone type (more precisely, the singularity type is the same as that at the vertex of the cone over the quadric surface* $x_1x_4 - x_2x_3 = 0$ *in* $\mathbb{P}^3$ *).*

As a result of Corollary 4.19, we have the following, letting

$$\Phi = \{(\alpha, \beta) \mid \alpha, \beta \text{ non-comparable irreducibles in } \mathcal{L}\},$$

THEOREM 6.15. *For the B-H toric variety* $X(\mathcal{L})$,

$$Sing\, X(\mathcal{L}) = \bigcup_{(\alpha,\beta)\in\Phi} \overline{O}_{\tau_{\alpha,\beta}}.$$

In other words, $X(\mathcal{L})$ *is smooth at* P_τ *(*τ *being a face of* σ*) if and only if for each pair* $(\alpha, \beta) \in \Phi$*, there exists at least one* $\gamma \in [\alpha \wedge \beta, \alpha \vee \beta]$ *such that* $P_\tau(\gamma)$ *is non-zero.*

7. Divisors and line bundles

7.1. Weil divisors.

DEFINITION 7.2. For any variety X, a *Weil divisor* is an element of the free abelian group generated by irreducible closed subvarieties of codimension one in X. If X is a toric variety, a Weil divisor that is mapped to itself by the torus T is a T-Weil divisor.

Thus for X_σ where σ is a convex polyhedral cone, let $\tau_1, \ldots, \tau_d$ be the one-dimensional faces of σ. Then the orbit closures $\overline{O_{\tau_1}}, \ldots, \overline{O_{\tau_d}}$ are all of the T-stable irreducible subvarieties of codimension one. Thus, T-Weil divisors are of the form $\sum_{i=1}^{d} a_i \overline{O_{\tau_i}}$.

Now let $X(\mathcal{L})$ be a Hibi toric variety. Due to our (inclusion-reversing) correspondence between faces of the cone and embedded sublattices (see §3.13), we also have a correspondence between one-dimensional faces and maximal embedded sublattices. Here we give a method for finding these maximal embedded sublattices.

Let the elements of $J(\mathcal{L})$ generate $N = \mathbb{Z}^{\#J(\mathcal{L})}$. Let α_2 cover α_1 in $J(\mathcal{L})$, so we have a one-dimensional face generated by $e_{\alpha_1} - e_{\alpha_2}$ (using notation as in Proposition 3.12). We define the following subset of $\mathcal{L}$:

$$\mathcal{L}_{\alpha_1}^{\alpha_2} = \mathcal{L} \setminus \{a \in \mathcal{L} \mid a \geq \alpha_1 \text{ and } a \ngeq \alpha_2\}.$$

Clearly, $\mathcal{L}_{\alpha_1}^{\alpha_2}$ is the maximal subset of $\mathcal{L}$ with the property that for $a \in \mathcal{L}_{\alpha_1}^{\alpha_2}$, f_{I_a} vanishes on $e_{\alpha_1} - e_{\alpha_2}$.

Lemma 7.3. *$\mathcal{L}_{\alpha_1}^{\alpha_2}$ is an embedded sublattice.*

Proof. First we show that $\mathcal{L}_{\alpha_1}^{\alpha_2}$ is in fact a sublattice. Let $b, c \in \mathcal{L}_{\alpha_1}^{\alpha_2}$. If both $b, c \geq \alpha_1$, then we have $b, c \geq \alpha_2$, and thus $b \vee c$, $b \wedge c \geq \alpha_2$. Therefore $b \vee c$, $b \wedge c \in \mathcal{L}_{\alpha_1}^{\alpha_2}$. Now say $b, c \ngeq \alpha_1$. Viewing $\mathcal{L}$ as the lattice of ideals of $J(\mathcal{L})$, this implies $\alpha_1 \notin I_a \cup I_b$. Thus $a \vee b \ngeq \alpha_1$. Clearly $a \wedge b \ngeq \alpha_1$, therefore $a \wedge b$, $a \vee b \in \mathcal{L}_{\alpha_1}^{\alpha_2}$. Lastly, say $a \geq \alpha_2$, $b \ngeq \alpha_1$, but then clearly $a \vee b \geq \alpha_2$, and $a \wedge b \ngeq \alpha_1$. Thus we have shown that $\mathcal{L}_{\alpha_1}^{\alpha_2}$ is a sublattice of $\mathcal{L}$.

Next we show that $\mathcal{L}_{\alpha_1}^{\alpha_2}$ is an embedded sublattice. Let $a, b \in \mathcal{L}$ such that $a \vee b$, $a \wedge b \in \mathcal{L}_{\alpha_1}^{\alpha_2}$. We'll show that for $a \vee b \geq \alpha_2$, $a \wedge b \ngeq \alpha_1$ we must have $a, b \in \mathcal{L}_{\alpha_1}^{\alpha_2}$. (The other cases follow obviously.) Since $\alpha_2 \in I_a \cup I_b$, we must have one of $a, b \geq \alpha_2$, say a. Conversely, we have $\alpha_1 \notin I_a \cap I_b$. Clearly $\alpha_1 \in I_a$, we must have $b \ngeq \alpha_1$. Therefore $a, b \in \mathcal{L}_{\alpha_1}^{\alpha_2}$. □

We have that $X(\mathcal{L}_{\alpha_1}^{\alpha_2})$ is an irreducible closed subvariety of codimension one in $X(\mathcal{L})$ (from Proposition 3.15).

In addition, for any $\alpha \in J(\mathcal{L})$ such that α is a maximal element, then we have a one-dimensional face generated by e_α. The associated embedded sublattice is $\mathcal{L}^\alpha = \{a \in \mathcal{L} \mid a \not\geq \alpha\}$. (The proof that this is in fact an embedded sublattice follows very easily from the fact that α is both join-irreducible, and maximal.)

For a toric variety $X(\mathcal{L})$ where $\mathcal{L}$ is a lattice with JIGL, we have a unique maximal element of $J(\mathcal{L})$, say $\hat{1}$. Therefore, T-Weil divisors are of the form

$$D = a_{\hat{1}} X(\mathcal{L}^{\hat{1}}) + \sum_{(\alpha_j, \alpha_i) \in Z(J(\mathcal{L}))} a_i^j X(\mathcal{L}_{\alpha_i}^{\alpha_j}).$$

($Z(J(\mathcal{L}))$ is the set of all covers in $J(\mathcal{L})$, as in Proposition 3.12.)

7.4. Cartier divisors. From [**11**, §3.3], a Cartier divisor for a variety X is given by the data of a covering of X by affine open sets U_α, and nonzero rational functions f_α called local equations, such that the ratios f_α / f_β are nowhere zero regular functions on $U_\alpha \cap U_\beta$. A Cartier divisor D determines a Weil divisor, denoted $[D]$, by

$$[D] = \sum_{\operatorname{codim}(V,X)=1} \operatorname{ord}_V(D) \cdot V,$$

where $\operatorname{ord}_V(D)$ is the order of vanishing of an equation for D in the local ring along the subvariety V. For a normal variety X, the map $D \mapsto [D]$ embeds the group of Cartier divisors in the group of Weil divisors. A nonzero rational function f determines a principal divisor $\operatorname{div}(f)$ whose local equation in each open set is f.

Cartier divisors which are T-stable are called T-Cartier divisors. Let σ be a convex polyhedral cone, and $u \in \check{\sigma} \cap M$ (note that we are using notation introduced in §1.2). Let $x \in X_\sigma$, then x can be realized as a semigroup homomorphism from $\check{\sigma} \cap M$ to $\mathbb{C}$. The function χ^u is defined such that $\chi^u(x) = x(u)$.

LEMMA 7.5 ([**11**]). *A general T-Cartier divisor on X_σ has the form $div(\chi^u)$ for some $u \in M$. Moreover, the associated Weil divisor is given by*

$$[div(\chi^u)] = \sum_i \langle u, v_i \rangle D_i$$

where v_i is the first lattice point along the one-dimensional face τ_i, $\langle\, , \,\rangle$ is the canonical pairing on $N \times M$, and $D_i = \overline{O_{\tau_i}}$.

Now let $X(\mathcal{L})$ be a Hibi toric variety, where $\mathcal{L}$ is a lattice with JIGL. It is a straight forward application of the lemma above to see that T-Cartier divisors on $X(\mathcal{L})$ are of the form

$$\langle u, e_{\hat{1}} \rangle X(\mathcal{L}^{\hat{1}}) + \sum_{(\alpha_j,\alpha_i)\in Z(J(\mathcal{L}))} \left\langle u, e_{\alpha_i} - e_{\alpha_j} \right\rangle X(\mathcal{L}^{\alpha_j}_{\alpha_i}).$$

For $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ where n is the cardinality of $J(\mathcal{L})$, the sum above is equal to

$$u_n X(\mathcal{L}^{\hat{1}}) + \sum_{(\alpha_j,\alpha_i)\in Z(J(\mathcal{L}))} (u_i - u_j)\, X(\mathcal{L}^{\alpha_j}_{\alpha_i}).$$

EXAMPLE 7.6. Let us consider a basic grid lattice:

$$J(\mathcal{L}) \quad = \quad \begin{array}{ccccc} & & \alpha_4 & & \\ & \diagup & & \diagdown & \\ \alpha_2 & & & & \alpha_3 \\ & \diagdown & & \diagup & \\ & & \alpha_1 & & \end{array}$$

Let $D_i^j = X(\mathcal{L}^{\alpha_j}_{\alpha_i})$. Thus, a T-Weil divisor on $X(\mathcal{L})$ is of the form $f = xD_1^2 + yD_1^3 + zD_2^4 + wD_3^4$, for integers x, y, z, w (for now, we ignore the term corresponding to $\mathcal{L}^{\hat{1}}$, since it has no relations with the other subvarieties). If f is a T-Cartier divisor, then

there exists a $u = (u_1, u_2, u_3, u_4) \in \mathbb{Z}^4$ such that

$$\begin{aligned} u_1 - u_2 &= x \\ u_1 - u_3 &= y \\ u_2 - u_4 &= z \\ u_3 - u_4 &= w \end{aligned}$$

We have that, for f to be a Cartier divisor, $x - y + z - w = 0$.

Now, for $J(\mathcal{L})$ any grid lattice, a Cartier divisor has a relation like the one above for every rank 2 diamond in the grid lattice. All other relations are a result of these.

7.7. Line bundles. For a Cartier divisor D, the ideal sheaf $\mathcal{O}(-D)$ is the subsheaf of the sheaf of rational functions generated by f_α on U_α; the inverse sheaf $\mathcal{O}(D)$ is the subsheaf of $\mathcal{O}_X$ generated by $1/f_\alpha$ on U_α. Regarded as a line bundle, its transition functions from U_α to U_β are f_α/f_β.

Let $\mathrm{Pic}(X)$ be the group of all line bundles, up to isomorphism. As a result of [**11**, §3.4], we have that $\mathrm{rank}(\mathrm{Pic}(X)) \leq d - n$, where d is the number of one-dimensional faces in σ and n is the dimension of the variety. Thus, for $X(\mathcal{L})$ a Hibi toric variety where $J(\mathcal{L})$ is a grid lattice, we have $\mathrm{rank}(\mathrm{Pic}(X(\mathcal{L}))) \leq \#\{\text{covers in } J(\mathcal{L})\} + 1 - \#J(\mathcal{L})$. (Note that the "+1" comes from the one maximal element in $J(\mathcal{L})$.)

Proposition 7.8. *Let $\mathcal{L}$ be a lattice with JIGL, then rank(Pic($X(\mathcal{L})$)) is bounded by the number of minimal diamonds in $J(\mathcal{L})$.*

Proof. First, it is easily checked that the proposition holds for the two most basic grid lattices: first, if $J(\mathcal{L})$ is a totally ordered chain of $n + 1$ elements, then there are n covers. Hence, from the statement above, we have $\mathrm{rank}(\mathrm{Pic}(X(\mathcal{L}))) \leq$

$n+1-(n+1)=0$. Since there are no diamonds in a totally ordered chain, the proposition holds.

The second grid lattice we will check: let $J(\mathcal{L})$ be four elements in a minimal diamond, as in Example 7.6. Now we have $\text{rank}(\text{Pic}(X(\mathcal{L}))) \leq 4+1-4=1$. Once again, the proposition holds.

Finally, we let $J(\mathcal{L})$ be any grid lattice. Say that $J(\mathcal{L}) = J_1 \cup J_2$, where J_1 and J_2 are grid lattices, such that J_1 and J_2 intersect on some totally ordered interval of $n+1$ elements. Thus the intersection also includes n covers. (Note that it is impossible for the union of J_1 and J_2 to create new minimal diamonds in $J(\mathcal{L})$ that did not already exist in J_1 or J_2.) Let v_i denote the number of lattice points in J_i, and e_i denote the number of cover in J_i.

Therefore, we have a bound on $\text{rank}(\text{Pic}(X(\mathcal{L})))$; namely

$$\begin{aligned} rank(Pic(X(\mathcal{L}))) &\leq (e_1+e_2-n)+1-(v_1+v_2-(n+1)) \\ &= e_1+e_2+2-v_1-v_2 \\ &= (e_1+1-v_1)+(e_2+1-v_2) \end{aligned}$$

We can see that the bound for $\text{rank}(\text{Pic}(X(\mathcal{L})))$ is actually the sum of the bounds on the lattices corresponding to J_1 and J_2.

For any grid lattice $J(\mathcal{L})$ we have $J(\mathcal{L}) = \bigcup J_i$, where each J_i is either a minimal diamond or a totally ordered chain, and where the intersection of any pair $J_i \cap J_j$ is either empty or a chain. The proposition follows. $\square$

A T-Cartier divisor $D = \sum a_i D_i$ determines a rational convex polyhedron in $M_{\mathbb{R}}$ defined by

$$P_D = \{u \in M_{\mathbb{R}} \mid \langle u, v_i \rangle \geq -a_i \text{ for all } i\}$$

where v_i is the first lattice point on the one-dimensional face τ_i (τ_i corresponding to D_i).

LEMMA 7.9 ([**11**]). *The global sections of the line bundle $\mathcal{O}(D)$ are*

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbb{C} \cdot \chi^u.$$

Now let $X(\mathcal{L})$ be a Hibi toric variety, where $\mathcal{L}$ is a lattice with JIGL. Let D be a T-Cartier divisor, $D = \text{div}(\chi^u) = u_n X(\mathcal{L}^{\hat{1}}) + \sum_{(\alpha_j, \alpha_i)} (u_i - u_j)\, X(\mathcal{L}^{\alpha_j}_{\alpha_i})$ for some $u = (u_1, \ldots u_n)$. Therefore

$$P_D = \{v \in M_{\mathbb{R}} \mid v_n \geq -u_n,\ v_i - v_j \geq u_j - u_i \text{ for all covers } (\alpha_j, \alpha_i) \text{ in } J(\mathcal{L})\}.$$

Let us return to scenario of Example 7.6. The system of inequalities for $v \in P_D$ is:

$$\begin{aligned} v_4 &\geq -u_4 \\ v_1 - v_2 &\geq u_2 - u_1 \\ v_1 - v_3 &\geq u_3 - u_1 \\ v_2 - v_4 &\geq u_4 - u_2 \\ v_3 - v_4 &\geq u_4 - u_3 \end{aligned}$$

Thus, $v_i \geq -u_i$ for $1 \leq i \leq 4$. It is easy to see that this result can be expanded to any grid lattice. Therefore, for a T-Cartier divisor $D = \text{div}(\chi^u)$, $u \in M$, we have

$$P_D = \{v \in M_{\mathbb{R}} \mid v_i \geq -u_i \,\forall\, i\}.$$

8. Young lattices

In this section, we show that a Young lattice $\mathcal{L}$ is a lattice with JIGL. (We give the definition of a Young lattice below.) In [**12**, §10], it was shown that the multicone

over a partial flag variety SL_n/Q flatly degenerates to the toric variety $X(\mathcal{L})$ for $\mathcal{L}$ a Young lattice. Thus, we will be showing that Theorems 6.14, 6.15 hold for this toric degeneration of SL_n/Q, for Q a parabolic subgroup.

For Q some maximal parabolic subgroup, we have $Q = \bigcap_{i=1}^{r} P_{d_i}$ where P_{d_i} is the maximal parabolic subgroup corresponding to fundamental weight ω_{d_i}, and $1 \leq d_1 < \ldots < d_r \leq n-1$. We define the distributive lattice

$$H_Q = \dot{\bigcup}_{i=1}^{r} I_{d_i,n}.$$

Here, $I_{d,n} = \{\underline{i} = (i_1, \ldots, i_d) \mid 1 \leq i_1 < i_2 < \ldots < i_d \leq n\}$, with a partial order on $I_{d,n}$ given by $\underline{i} \leq \underline{j}$ if and only if $i_1 \leq j_1, \ldots, i_d \leq j_d$. For an element $\tau \in H_Q \cap I_{d_t,n}$, we say τ is of *type* d_t. The partial order on two elements of H_Q of type d_t is given by the partial order on $I_{d_t,n}$. For $\tau = (\tau_1, \ldots, \tau_{d_t})$, $\theta = (\theta_1, \ldots, \theta_{d_s})$ of type d_t, d_s respectively, we define $\tau \leq \theta$ if and only if

$$d_t \geq d_s, \text{ and } \tau_1 \leq \theta_1, \ldots, \tau_{d_s} \leq \theta_{d_s}.$$

H_Q is referred to as a *Young lattice*.

THEOREM 8.1 (cf. [**12**]). *The multicone over SL_n/Q flatly degenerates to $X(H_Q)$.*

A note about notation, we will use $[i, j]$ to denote the segment $i, i+1, \ldots, j$, and if there is a possibility that $j < i$, the segment is understood to be empty.

In the remainder of the section, we will show that H_Q is a lattice with JIGL. We begin by identifying $J(H_Q)$. Note that if both θ, $\delta \in H_Q$ are of type d, then $\theta \vee \delta$ is of type d. Thus, if an element in H_Q of type d is not join irreducible in $I_{d,n}$, it will not be join irreducible in H_Q. Thus, when identifying join irreducible elements in H_Q, we need only consider join irreducible elements in $I_{d_i,n}$ for each i, $1 \leq i \leq r$. From [**13**, Lemma 8.2], we have that $\underline{i} \in I_{d,n}$ is join irreducible if and only if $\underline{i}$ consists entirely

of one segment, or if $\underline{i}$ is equal to two disjoint segments, (μ, ν) such that μ starts with 1.

Let $\theta \in H_Q$ be of type d_i and $\delta \in H_Q$ be of type d_j, such that $d_i < d_j$. Then we have $\theta \vee \delta = (\max\{\theta_1, \delta_1\}, \ldots, \max\{\theta_{d_i}, \delta_{d_i}\})$. Note that $\theta \vee \delta$ is of type d_i.

Let $\tau \in H_Q$ be of type d_t, $\tau = ([1, s], [p, p + d_t - s - 1])$, where $0 \leq s \leq d_t$, $p > n - d_{t+1} + s + 1$. If $t = r$ (i.e., d_{t+1} does not exist), then there is no condition on p other than $p > s$.

Lemma 8.2. *The element τ as defined above is in $J(H_Q)$.*

Proof. Assume, if possible, that there exists θ, $\delta \in H_Q$ distinct from τ such that $\theta \vee \delta = \tau$. From the discussion above, we can see that τ is join irreducible as an element of $I_{d_t,n}$, thus we must have θ and δ of different types in H_Q, and one must be of the same type as τ. Therefore, without loss of generality, let θ be of type d_t and δ of type d_q where $d_t < d_q$; thus $\theta = (\theta_1, \ldots, \theta_{d_t})$, $\delta = (\delta_1, \ldots, \delta_{d_q})$. (Note that, by this argument, we can already see that the lemma is true when τ is type d_r, so we continue assuming $t < r$.)

Since $\tau = \theta \vee \delta$, we have θ_i, $\delta_i \leq \tau_i$ for $1 \leq i \leq d_t$. Therefore $(\theta_1, \ldots, \theta_s) = (1, \ldots, s) = (\delta_1, \ldots, \delta_s)$. We now examine $\{\delta_{s+1}, \ldots, \delta_q\}$. By necessity, we have $\delta_{s+1} \leq n - d_q + s + 1$ (for $\delta_{d_q} \leq n$), and since $d_q \geq d_{t+1}$, we have $\delta_{s+1} \leq n - d_{t+1} + s + 1$. From the definition of τ, $\tau_{s+1} = p > n - d_{t+1} + s + 1$, therefore $\delta_{s+1} < p$ and thus $\theta_{s+1} = p$. We can continue this argument for each element of $\{\delta_{s+1}, \ldots, \delta_{d_q}\}$, until we conclude that $\theta = \tau$. This is a contradiction, therefore τ must be join irreducible. $\square$

Our next lemma will show that $J(H_Q)$ consists only of elements like τ above. As discussed, it is enough to show that for an element $\zeta \in J(I_{d_t,n})$, ζ is in $J(H_Q)$ if and only if ζ can be written as τ above. Let $\zeta = ([1, s], [\zeta_{s+1}, \zeta_{s+1} + d_t - s - 1])$, where

$0 \leq s < d_t$, and $s+1 < \zeta_{s+1} \leq n - d_{t+1} + s + 1$. We are assuming that ζ is of type $d_t \neq d_r$. (All join irreducible elements of $I_{d_r,n}$ are in $J(H_Q)$.)

LEMMA 8.3. *The element ζ as defined above is not join irreducible.*

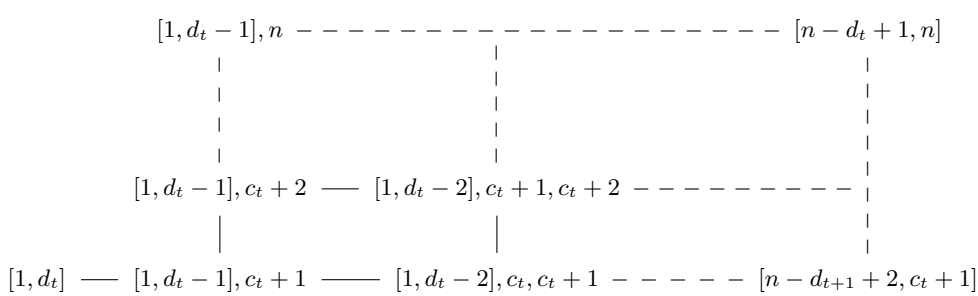
PROOF. Let $\theta = (1, \ldots, d_t)$, and $\delta = ([1, s], [\zeta_{s+1}, \zeta_{s+1} + d_{t+1} - s - 1])$. Note that δ is of type d_{t_1}, and by our conditions on ζ_{s+1}, we have $\zeta_{s+1} + d_{t+1} - s - 1 \leq n$, therefore δ is a valid element of H_Q. Clearly $\theta \vee \delta = \zeta$, and the result follows. □

COROLLARY 8.4. *The elements of $J(H_Q)$ of type d_t are of the form*

$$\tau = (1, \ldots, s, p, \ldots, p + d_t + s - 1)\,,\ 0 \leq s \leq d_t,\ p > n - d_{t+1} + s + 1,$$

(with condition on p only if $d_t < d_r$).

Now we will show that $J(H_Q)$ is a grid lattice. First, we see that $J(H_Q)_{d_t} = \{\tau \in J(H_Q) \mid \tau \text{ of type } d_t\}$ is independently a grid lattice for each d_t, $1 \leq t \leq r$. We know that $J(H_Q)_{d_r} = J(I_{d_r,n})$ is a grid lattice because $I_{d_r,n}$ is a minuscule lattice. Now take $d_t \neq d_r$. To make notation easier, let $c_t = n - d_{t+1} + d_t$. Here we give $J(H_Q)_{d_t}$ (drawn horizontally):



We have an identification of $J(H_Q)_{d_t}$ with a subset of $\mathbb{Z} \times \mathbb{Z}$ by sending $([1, d_t])$ to the point $(0, 1)$, and sending $([1, d_t - a], [c_t + b - a, c_t + b])$ to the point (a, b). Thus, as a sublattice of $\mathbb{Z} \times \mathbb{Z}$, $J(H_Q)_{d_t}$ is isomorphic to the interval from $(1, 1)$ to $(d_t, d_{t+1} - d_t)$ in union with the point $(0, 1)$.

The only thing remaining to show is that the union $\bigcup_{i=1}^{r} J(H_Q)_{d_i}$ is still a grid lattice. This fact will be made clear by examining the union $J(H_Q)_{d_t} \cup J(H_Q)_{d_{t-1}}$.

The partial order relations on this union can be seen by examining the intervals $[(1,\ldots,d_t-1,n)\,,\,(n-d_t+1,\ldots,n)] \subset J(H_Q)_{d_t}$ and
$[(1,\ldots,d_{t-1})\,,\,(n-d_t+2,\ldots,n-d_t+d_{t-1}+1)] \subset J(H_Q)_{d_{t-1}}$.

$$\begin{tikzcd}
 & {[1,d_{t-1}]} \arrow[r, dashed, no head] \arrow[d, no head] & {1,[n-d_t+3,c_{t-1}+1]} \arrow[r, no head] \arrow[d, no head] & {[n-d_t+2,c_{t-1}+1]} \arrow[d, no head] \\
{[1,d_t-1],n} \arrow[r, dashed, no head] & {[1,d_{t-1}],[c_{t-1}+1,n]} \arrow[r, dashed, no head] & {1,[n-d_t+2,n]} \arrow[r, no head] & {[n-d_t+1,n]}
\end{tikzcd}$$

From this, it is clear that the union of these two grid lattices forms a grid lattice. Since $J(H_Q)$ can be constructed as a series of these unions (one on top of another), the main result of the section is clear.

THEOREM 8.5. *The Young Lattice* $H_Q = \bigcup_{i=1}^{r} I_{d_i,n}$, $1 \leq d_1 < d_2 < \ldots < d_r \leq n-1$ *is a lattice with JIGL.*

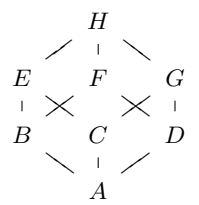
EXAMPLE 8.6. Let $SL_n/Q = SL_5/B$, where $B = \bigcap_{i=1}^{4} P_i$. Thus, (drawn horizontally) $J(H_Q) =$

$$\begin{tikzcd}
 & & & (1) \arrow[r, no head] \arrow[d, no head] & (5) \arrow[d, no head] \\
 & & (1,2) \arrow[r, no head] \arrow[d, no head] & (1,5) \arrow[r, no head] \arrow[d, no head] & (4,5) \arrow[d, no head] \\
 & (1,2,3) \arrow[r, no head] \arrow[d, no head] & (1,2,5) \arrow[r, no head] \arrow[d, no head] & (1,4,5) \arrow[r, no head] \arrow[d, no head] & (3,4,5) \arrow[d, no head] \\
(1,2,3,4) \arrow[r, no head] & (1,2,3,5) \arrow[r, no head] & (1,2,4,5) \arrow[r, no head] & (1,3,4,5) \arrow[r, no head] & (2,3,4,5)
\end{tikzcd}$$

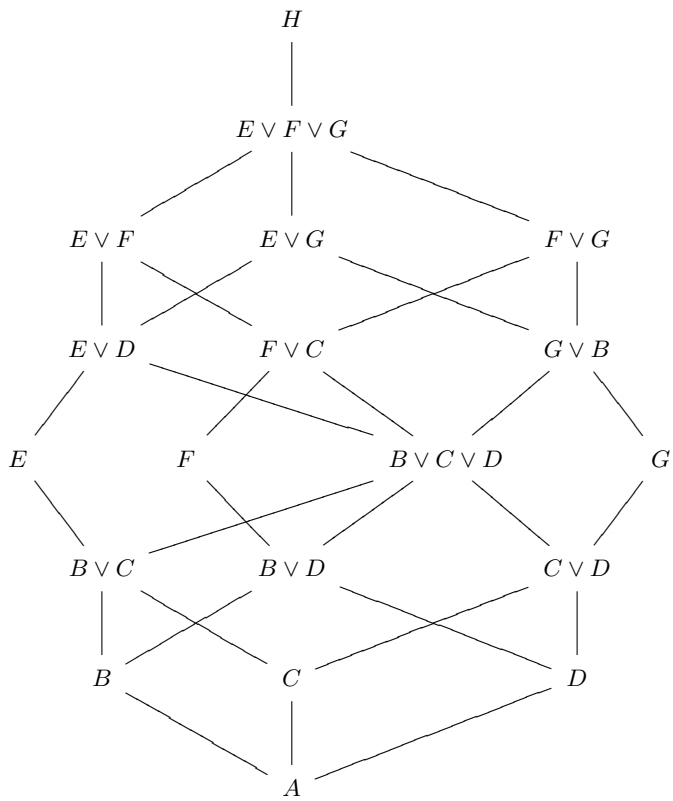
9. A counter example

It seems natural to ask whether or not the methods and results from the previous sections (specifically Theorem 6.14) apply to a larger class of Hibi toric varieties than just those based upon a lattice with JIGL. We provide the following example to show that in fact, neither the methods nor results apply.

Let $\mathcal{L}$ be a distributive lattice. If $J(\mathcal{L})$ is a sublattice of $\mathbb{N}^n$, then we call $J(\mathcal{L})$ an *n-grid lattice*. For example, we let $\mathcal{L}$ be a lattice such that $J(\mathcal{L})$ is a cube.



$J(\mathcal{L})$

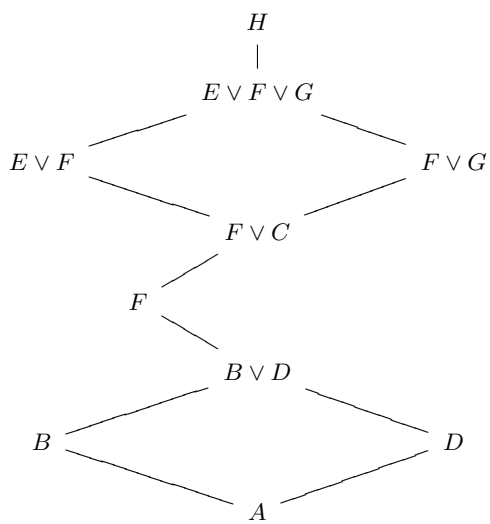


$\mathcal{L}$

First, we note that taking a skew pair of join-meet irreducibles α, β and setting

$$\mathcal{L}_{\alpha,\beta} = \mathcal{L} \setminus [\alpha \wedge \beta, \alpha \vee \beta]$$

does not necessarily give us an embedded sublattice. In the example above, let our skew pair be E, G. Clearly $E \wedge G = C$ and thus $\mathcal{L}_{E,G}$ is the following sublattice:



Here, we have that $F \vee (B \vee C \vee D), F \wedge (B \vee C \vee D)$ are in $\mathcal{L}_{E,G}$, however, the element $B \vee C \vee D$ is not $\mathcal{L}_{E,G}$. Therefore, the sublattice $\mathcal{L}_{E,G}$ is not embedded. Since the sublattice is not embedded, it does not coincide with a face of the cone associated to the toric variety $X(\mathcal{L})$. Thus, we can assert that Theorem 6.15 cannot be extended to this generalization of a lattice with join irreducibles forming an n-grid lattice.

Moreover, the singular locus in this example is not "pure" of codimension 3. To illustrate this, we first note that the singular locus does have a codimension 3 piece similar to the JIGL case (see Theorem 6.6). The sublattice of $\mathcal{L}$ given by the interval $[E, H]$ corresponds to the face of the cone with generators $\{e_A - e_B,\, e_A - e_C,\, e_B - e_E,\, e_C - e_E\}$. On the other hand, we note that the singular locus also has a codimension 5 piece not contained in any codimension 3 piece. The sublattice of $\mathcal{L}$ given by the three elements $\{A, E \vee F \vee G, H\}$ corresponds to the face of the cone with six generators $\{e_B - e_E,\, e_B - e_F,\, e_C - e_E,\, e_C - e_G,\, e_D - e_F,\, e_D - e_G\}$. We have an identification of the (open) affine piece in $X(L)$ corresponding to this face with

the product $D_2 \times (K^*)^3$, where D_2 is the determinantal variety with defining ideal given by all 2-minors on a 3×3 generic matrix.

This example, and other examples in which $J(\mathcal{L})$ is an n-grid lattice have led us to the following conjecture.

Conjecture 9.1. *Let $\mathcal{L}$ be a distributive lattice such that $J(\mathcal{L})$ is an n-grid lattice. Then the singular locus of $X(\mathcal{L})$ has irreducible components of codimension $3, 5, \ldots,$ and $2n-1$.*

10. Multiplicity formulae for G-H toric varieties

In this section, we restrict our attention to a specific class of B-H toric varieties: specifically $X(\mathcal{L})$ for $\mathcal{L}$ the Weyl group of the traditional Grassmannian. It is a well known result (cf. [**2**]) that

$$\mathcal{L} = I_{d,n} = \{\underline{i} = (i_1, \ldots, i_d) \mid 1 \leq i_1 < i_2 < \ldots < i_d \leq n\} \text{ for } 1 \leq d < n.$$

We will sometimes use the notation $X_{d,n}$ to denote $X(I_{d,n})$, and refer to $X_{d,n}$ as a *Grassmann-Hibi toric variety*, or G-H toric variety, for short.

For $1 \leq i \leq n-d-1$, $1 \leq j \leq d-1$, let

$$\mu_{ij} = (1, \ldots j,\, i+j+1, \ldots i+d)\,, \text{ and}$$

$$\lambda_{ij} = (i+1, \ldots i+j, n+1+j-d, \ldots n)\,.$$

Define

$$\mathcal{L}_{ij} = \mathcal{L} \setminus [\mu_{ij}, \lambda_{ij}].$$

Proposition 10.1. *From* [**13**, §8]*, we have the following facts about the distributive lattice $I_{d,n}$:*

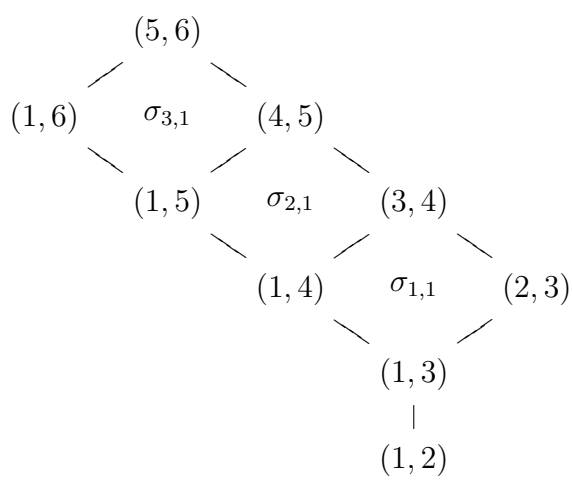
(1) *The element* $\tau = (i_1, \ldots, i_d)$ *is irreducible if and only if either* τ *is a segment, or* τ *consists of two disjoint segments* (μ, ν)*, with* μ *starting with* 1 *and* ν *ending with* n*.*

(2) *For any incomparable pair of irreducibles* (α, β)*, there exists* $1 \leq i \leq n-d-1$*,* $1 \leq j \leq d-1$ *such that* $\alpha \vee \beta = \lambda_{ij}$*,* $\alpha \wedge \beta = \mu_{ij}$*.*

Thus, we have that $\mathcal{L}_{ij}$ defined above is an embedded sublattice, playing the role of $\mathcal{L}_{\alpha,\beta}$ from Definition 5.7. Let $\sigma_{i,j}$ denote the singular face of σ corresponding to $\mathcal{L}_{ij}$.

10.2. Multiplicities of singular faces of $X_{2,n}$. In this section, we take $\mathcal{L} = I_{2,n}$, determine the multiplicity of $X_{2,n}$ at P_τ for certain of the singular faces of $X_{2,n}$, and deduce a product formula. Above we defined $\mathcal{L}_{ij}$ and the corresponding face $\sigma_{i,j}$ for $1 \leq j \leq d-1$, $1 \leq i \leq n-d-1$; thus for $I_{2,n}$, we need to consider only $\mathcal{L}_{i,1}$ for $1 \leq i \leq n-3$.

Example 10.3. Below is the poset of join irreducibles for $I_{2,6}$. We write $\sigma_{i,1}$ inside each diamond because the four segments surrounding it represent the four generators of the face.



To go from the join irreducibles of $I_{2,6}$ to $I_{2,7}$, we just add $(1,7)$ and $(6,7)$ to the poset above, forming $\sigma_{4,1}$. We will see that this makes the calculation of the multiplicities of singular faces of $I_{2,n}$ much easier.

In the sequel, we shall denote the set of join irreducibles of $I_{2,n}$ by $J_{2,n}$; also, as in the previous sections, σ will denote the polyhedral cone corresponding to $X_{2,n}$.

10.4. Mult$_{P_\sigma}$ $X_{2,n}$. Now $X_{d,n}$ being of cone type (i.e., the vanishing ideal is homogeneous), we have a canonical identification of $T_{P_\sigma}X_{d,n}$ (the tangent cone to $X_{d,n}$ at P_σ) with $X_{d,n}$. Hence by Theorem 3.8, we have that $\mathrm{mult}_{P_\sigma} X_{d,n}$ equals the number of maximal chains in $I_{d,n}$. So we begin by counting the number of maximal chains in $I_{2,n}$.

As we move through a chain from $(1,2)$, at any point (i,j) we have at most two possibilities for the next point: $(i+1,j)$ or $(i,j+1)$. For each cover in our chain, we assign a value: for a cover of type $((i,j+1),(i,j))$ assign $+1$; for a cover of type $((i+1,j),(i,j))$ assign -1.

A maximal chain C in $I_{2,n}$ contains $2n-3$ lattice points, and thus every chain can be uniquely represented by a $(2n-4)$-tuple of 1's and -1's; let us denote this $(2n-4)$-tuple by $n_C = \langle a_1, \ldots, a_{2n-4}\rangle$.

Of course we have restrictions on n_C. First it is clear that 1 and -1 occur precisely $n-2$ times. Secondly, we can see that $a_1 = +1$, and for any $1 \le k \le 2n-4$, if $\{a_1, \ldots, a_k\}$ contains more -1's than $+1$'s, then we have arrived at a point (i,j) with $i > j$, which is not a lattice point. Thus, we must have $a_1 + \ldots + a_k \ge 0$ for every $1 \le k \le 2n-4$.

THEOREM 10.5 (cf. Corollary 6.2.3 in [**32**]). *The Catalan number*

$$Cat_n = \frac{1}{n+1}\binom{2n}{n}, \ (n \ge 0)$$

counts the number of sequences $a_1, \ldots, a_{2n}$ *of* 1*'s and* -1*'s with*

$$a_1 + \ldots + a_k \geq 0, \ (k = 1, 2, \ldots, 2n)$$

and $a_1 + \ldots + a_{2n} = 0$.

COROLLARY 10.6. *The multiplicity of* $X_{2,n}$ *at* P_σ *is equal to the Catalan number*

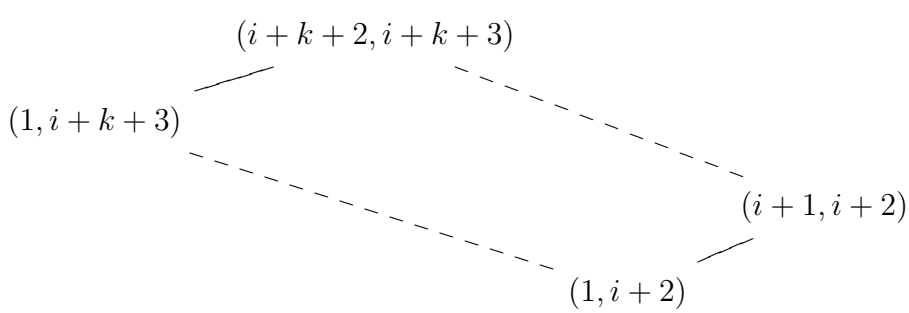
$$Cat_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

10.7. Mult$_{P_\tau}$ $X_{2,n}$. In this section, we determine $\mathrm{mult}_{P_\tau} X_{2,n}$, for τ being of block type (see Definition 10.10 below). Let τ be a face of σ, such that the associated (embedded sublattice) D_τ is of the form:

$$D_\tau = [(1,2),(i,i+1)] \cup [(i+k+2, i+k+3),(n-1,n)] = I_1 \cup I_2,$$

where $I_1 = [(1,2),(i,i+1)]$, $I_2 = [(i+k+2, i+k+3),(n-1,n)]$, $1 \leq i \leq n-3$, $0 \leq k \leq n-i-3$.

We shall now determine $W(\tau)$ (see Definition 6.2). Let A_τ denote the interval $[(1, i+2), (i+k+2, i+k+3)]$ in $J_{2,n}$:



LEMMA 10.8. *With* τ *as above, we have* $W(\tau) = \{e_{y'} - e_y \mid (y, y')$ *is a cover in* $A_\tau\}$.

PROOF. Clearly $e_{(n-1,n)}$ (the element in $W(\sigma)$ corresponding to the unique maximal element $(n-1,n)$ in $J_{2,n}$) is not in $W(\tau)$ (since $(n-1,n) \in D_\tau$). Let us denote

$$\theta = (i+k+2, i+k+3), \delta = (i, i+1)$$

Claim 1: For a cover (y, y') in A_τ, $f_{I_\alpha}(e_{y'} - e_y) = 0, \forall \alpha \in D_\tau$.

The claim follows in view of the facts that for a cover (y, y') in A_τ, we have,

(i) $y, y' \in I_\theta$, and hence $y, y' \in I_\alpha, \forall \alpha \in I_2$.

(ii) $y, y' \notin I_\delta$, and hence $y, y' \notin I_\alpha, \forall \alpha \in I_1$.

Claim 2: For a cover (y, y') in $J_{2,n}$ not contained in A_τ, there exists an $\alpha \in D_\tau$ such that $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

Note that a cover in $J_{2,n}$ is one of the following three types:

Type I: $((1, j), (1, j-1)), 3 \leq j \leq n$

Type II: $((j-1, j), (j-2, j-1)), 4 \leq j \leq n$

Type III: $((j-1, j), (1, j)), 3 \leq j \leq n$

Let now (y, y') be a cover not contained in A_τ.

If (y, y') is of Type I, then $(y, y') = ((1, j), (1, j-1))$, where either $j \leq i+2$ or $j \geq i+k+4$. Letting

$$\alpha = \begin{cases} (1, j-1), & \text{if } j \leq i+2 \\ (j-2, j-1), & \text{if } j \geq i+k+4 \end{cases}$$

we have, $\alpha \in D_\tau$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

If (y, y') is of Type II, then $(y, y') = ((j-1, j), (j-2, j-1))$, where either $j \leq i+2$ or $j \geq i+k+4$. Letting $\alpha = (j-2, j-1)$, we have, $\alpha \in D_\tau$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

If (y, y') is of Type III, then $(y, y') = ((j-1, j), (1, j))$, where either $j \leq i+1$ or $j \geq i+k+4$. Letting

$$\alpha = \begin{cases} (1, j), & \text{if } j \leq i+1 \\ (j-2, j), & \text{if } j \geq i+k+4 \end{cases}$$

we have, $\alpha \in D_\tau$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

The required result follows from claims 1 and 2. $\square$

COROLLARY 10.9. *With τ as in Lemma 10.8, we have*

$$\tau = \sigma_{i,1} \cup \sigma_{i+1,1} \cup \ldots \cup \sigma_{i+k,1}.$$

DEFINITION 10.10. We define a face τ as in Lemma 10.8 as a *J-block* (namely, τ is an union of consecutive $\sigma_{i,1}$'s).

REMARK 10.11. Note that in general a union of faces need not be a face.

DEFINITION 10.12. For an integer $r \geq 3$, let $\widetilde{I_{2,r}}$ denote the distributive lattice $I_{2,r} \setminus \{(1,2),(r-1,r)\}$. We define $Z_{2,r}$ to be the Hibi variety associated to $\widetilde{I_{2,r}}$.

In view of Theorem 3.8, we have

$$\mathrm{mult}_{\mathbf{0}} Z_{2,r} = \mathrm{mult}_{P_\sigma} X_{2,r} = Cat_{r-2}.$$

(Here, $\mathbf{0}$ denotes the origin.)

THEOREM 10.13. *Let τ be a face of σ which is a "J-block" of $k+1$ consecutive $\sigma_{i,1}$'s (as in Definition 10.10). We have an identification of X_τ (the open affine piece of X_σ corresponding to τ) with $Z_{2,k+4} \times (K^*)^m$ where $m = \mathrm{codim}_\sigma \tau = 2(n-k)-6$.*

PROOF. In view of 1.4 and Proposition 3.15, we have

$$\mathrm{codim}_\sigma \tau = dim\, X(D_\tau) = \#\{\text{elements in a maximal chain in } D_\tau\};$$

from this it is clear that $\mathrm{codim}_\sigma \tau$ equals $2(n-k)-6$. Next, in view of Lemma 10.8 and Definition 10.12, we obtain an identification of X_τ with $Z_{2,k+4} \times (K^*)^m$ (m being as in the Theorem). □

THEOREM 10.14. *Let τ be as in Theorem 10.13.*

(1) *We have an identification of $TC_{P_\tau} X_\sigma$ with $Z_{2,k+4} \times (K^*)^m$ where $m = \mathrm{codim}_\sigma \tau = 2(n-k)-6$; further, $TC_{P_\tau} X_\sigma$ is a toric variety.*

(2) $mult_{P_\tau} X_{2,n} = Cat_{k+2}$.

PROOF. X_τ being open in X_σ, we may identify $TC_{P_\tau} X_\sigma$ with $TC_{P_\tau} X_\tau$ which in turn coincides with X_τ (since X_τ is of cone type, with P_τ being identified with the origin); assertion (1) follows from this in view of Theorem 10.13 (and the fact that X_τ is a toric variety).

Assertion (2) follows from (1) and Corollary 10.6. □

10.15. A product formula. In this subsection, we give a product formula for $mult_{P_\tau} X_{2,n}$ where τ is a union of pairwise non-intersecting and non-consecutive J-blocks (see Remark 10.18 below).

Let τ be a face of σ, such that the associated (embedded sublattice) D_τ is of the form $D_\tau = J_1 \cup J_2 \cup J_3$:

$$J_1 = [(1,2),(i_1,i_1+1)] \quad J_2 = [(i_1+k_1+2, i_1+k_1+3),(i_2,i_2+1)]$$

$$J_3 = [(i_2+k_2+2, i_2+k_2+3),(n-1,n)],$$

where $i_1 + k_1 + 1 < i_2$.

Consider the following sublattices in $J_{2,n}$ (the set of join-irreducibles in $I_{2,n}$):

$$A = [(1, i_1+2),(i_1+k_1+2, i_1+k_1+3)], B = [(1, i_2+2),(i_2+k_2+2, i_2+k_2+3)].$$

LEMMA 10.16. *With τ as above, we have*

$$W(\tau) = \{e_{y'} - e_y \mid (y,y') \text{ is a cover in } A \text{ or } B\}.$$

PROOF. We proceed as in the proof of Lemma 10.8. As in that proof, we have $e_{(n-1,n)}$ is not in $W(\tau)$ (since $(n-1,n) \in D_\tau$). Let us denote

$$\theta_1 = (i_1+k_1+2, i_1+k_1+3), \theta_2 = (i_2+k_2+2, i_2+k_2+3), \delta_1 = (i_1, i_1+1), \delta_2 = (i_2, i_2+1).$$

For any cover (y, y') in A or B, we clearly have $y, y' \in I_{\theta_2}$ and hence $y, y' \in I_\alpha, \forall \alpha \in J_3$; also, $y, y' \notin I_{\delta_1}$ and hence $y, y' \notin I_\alpha, \forall \alpha \in J_1$. Thus we obtain that

$$(*) \qquad f_{I_\alpha}(e_{y'} - e_y) = 0, \forall \alpha \in J_1 \cup J_3$$

Next, if (y, y') is a cover in A, then $y, y' \in I_{\theta_1}$ and hence $y, y' \in I_\alpha, \forall \alpha \in J_2$; also, if (y, y') is a cover in B, then $y, y' \notin I_{\delta_2}$, and hence $y, y' \notin I_\alpha, \forall \alpha \in J_2$ (note that θ_1 (resp. δ_2) is the smallest (resp. largest) element in J_2). Thus we obtain that

$$(**) \qquad f_{I_\alpha}(e_{y'} - e_y) = 0, \forall \alpha \in J_2$$

Now $(*)$ and $(**)$ imply the inclusion "$\supseteq$". We shall prove the inclusion "$\subseteq$" by showing that if a cover (y, y') is not contained in A or B, then there exists an $\alpha \in D_\tau$ such that $f_{I_\alpha}(e_{y'} - e_y) \neq 0$. This proof again runs on similar lines as that of Lemma 10.8. Let then (y, y') be a cover in $J_{2,n}$ not contained in A or B. It would be convenient to introduce the following sublattices in $J_{2,n}$:

$$\begin{aligned} P &= [(1,2),(i_1+1, i_1+2)] \\ Q &= [(1, i_1+k_1+3),(i_2+1, i_2+2)] \\ R &= [(1, i_2+k_2+3),(n-1, n)]. \end{aligned}$$

We distinguish the following cases:

Case 1: Let (y, y') be of type I (cf. proof of Lemma 10.8), say, $((1, j), (1, j-1))$.

(i) If (y, y') is contained in P, then $j \leq i_1 + 2$. We let $\alpha = (1, j-1)$. Note that $\alpha \in J_1$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

(ii) If (y, y') is contained in Q (resp. R), then $i_1 + k_1 + 4 \leq j \leq i_2 + 2$ (resp. $i_2 + k_2 + 4 \leq j \leq n$). We let $\alpha = (j-2, j-1)$. Note that $\alpha \in J_2$ (resp. J_3) and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

Case 2: Let (y, y') be of type II, say, $((j-1, j), (j-2, j-1))$.

Then, $3 \leq j \leq i_1 + 2$, $i_1 + k_1 + 4 \leq j \leq i_2 + 2$ or $i_2 + k_2 + 4 \leq j \leq n$ accordingly as (y, y') is contained in P, Q or R. We let $\alpha = (j-2, j-1)$. Note that $\alpha \in J_1$, J_2 or J_3, accordingly as (y, y') is contained in P, Q or R; and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

Case 3: Let (y, y') be of type III, say, $((j-1, j), (1, j))$.

(i) If (y, y') is contained in P, then $j \leq i_1 + 1$. We let $\alpha = (1, j)$. Note that $\alpha \in J_1$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

(ii) If (y, y') is contained in Q (resp. R), then $i_1 + k_1 + 4 \leq j \leq i_2 + 1$ (resp. $i_2 + k_2 + 4 \leq j \leq n$). We let $\alpha = (j-2, j)$. Note that $\alpha \in J_2$ or J_3, accordingly as (y, y') is contained in Q or R; and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$. □

As an immediate consequence of Lemma 10.16 and Corollary 10.9, we have

Corollary 10.17. *Let τ be as in Lemma 10.16. Then $\tau = \tau_1 \cup \tau_2$, where*

$$\tau_1 = \sigma_{i_1,1} \cup \ldots \cup \sigma_{i_1+k_1,1},$$

$$\tau_2 = \sigma_{i_2,1} \cup \ldots \cup \sigma_{i_2+k_2,1};$$

Remark 10.18. We refer to a pair (τ_1, τ_2) of faces as in Corollary 10.17 as *non-intersecting J-blocks.*

Theorem 10.19. *Let $\tau = \tau_1 \cup \tau_2$, where τ_1 and τ_2 are two non-intersecting J-blocks (see Corollary 10.17). We have an identification of X_τ (the open affine piece of X_σ corresponding to τ) with $Z_{2,k_1+4} \times Z_{2,k_2+4} \times (K^*)^m$ where $m = \operatorname{codim}_\sigma \tau = 2(n - k_1 - k_2) - 9$.*

Proof is similar to that of Theorem 10.14 (using Lemma 10.16). As an immediate consequence, we have

Theorem 10.20. *Let $\tau = \tau_1 \cup \tau_2$, where τ_1 and τ_2 are two non-intersecting J-blocks (see Corollary 10.17) Then*

(1) *We have an identification of* $TC_{P_\tau}X_\sigma$ *with* $Z_{2,k_1+4} \times Z_{2,k_2+4} \times (K^*)^m$ *where* $m = \operatorname{codim}_\sigma \tau = 2(n-k_1-k_2)-9$*; in particular,* $TC_{P_\tau}X_\sigma$ *is a toric variety.*

(2) $mult_{P_\tau}X_{2,n} = \left(mult_{P_{\tau_1}}X_{2,n}\right) \cdot \left(mult_{P_{\tau_2}}X_{2,n}\right)$.

Proof is similar to that of Theorem 10.14 (using Theorem 10.19).

Remark 10.21. It is clear that we can extend this multiplicative property to $\tau = \tau_1 \cup \ldots \cup \tau_s$, a union of s pairwise non-intersecting, non-consecutive J-blocks.

10.22. A multiplicity formula for $X_{d,n}$. In this section, we give a formula for $mult_{P_\sigma}X_{d,n}$. By Theorem 3.8, we have that $mult_{P_\sigma}X_{d,n}$ equals the number of maximal chains in $I_{d,n}$. We give below an explicit formula for the number of maximal chains in $I_{d,n}$. Notice that the number of chains in $I_{d,n}$ from $(1,2,\ldots,d)$ to $(n-d+1,\ldots,n)$ is the same as the number of chains from $(0,0,\ldots 0)$ to $(n-d,n-d,\ldots n-d)$ such that for any $(i_1,\ldots i_d)$ in the chain, we have $i_1 \geq i_2 \geq \ldots i_d \geq 0$. Now, set

$$\mu = (\mu_1,\mu_2,\ldots,\mu_d) = (n-d,n-d,\ldots n-d). \qquad (*)$$

For any $\lambda \vdash m$, let $f^\lambda = K_{\lambda,1^m}$, i.e., the number of standard Young tableau of shape λ (cf. [**32**]).

Proposition 10.23 (cf. Proposition 7.10.3, [**32**]). *Let λ be a partition of m. Then the number f^λ counts the lattice paths $0 = v_0, v_1, \ldots, v_m$ in $\mathbb{R}^l$ (where $l = l(\lambda)$) from the origin v_0 to $v_m = (\lambda_1,\lambda_2,\ldots,\lambda_l)$, with each step a coordinate vector; and staying within the region (or cone) $x_1 \geq x_2 \geq \ldots \geq x_l \geq 0$.*

Thus, for μ as described in $(*)$ above, we have that the number of maximal chains in $I_{d,n}$ is equal to f^μ.

Corollary 7.21.5 of [**32**] gives an explicit description of f^λ.

PROPOSITION 10.24. *Let $\lambda \vdash m$. Then*

$$f^{\lambda} = \frac{m!}{\prod_{u \in \lambda} h(u)}.$$

The statement above refers to $u \in \lambda$ as a box in the Young tableau of λ, and $h(u)$ being the "hook length" of u. The hook length is easily defined as the number of boxes to the right, and below, of u, including u once.

Let us take, for example, $I_{3,6}$. Then $\mu = (3,3,3)$, and the Young tableau of shape μ with hook lengths given in their corresponding boxes is

5	4	3
4	3	2
3	2	1

Therefore

$$f^{\mu} = \frac{9!}{5 \cdot 4^2 \cdot 3^3 \cdot 2^2 \cdot 1} = 42.$$

In fact, in the scenario of $I_{d,n}$ our derived partition μ (given by $(*)$) will always be a rectangle; and we can deduce a formula for f^{μ} which does not require the Young tableau. The top left box of μ will always have hook length $(n-d)+d-1 = n-1$. Then, the box directly below it, and the box directly to the right of it will have length $n-2$. For any box of μ, the box below and the box to the right will have hook length one less than that of the box with which we started.

Since the posets $I_{d,n}$, and $I_{n-d,n}$ are isomorphic, we can assume that $d \leq n-d$. Then we have $\prod_{u \in \mu} h(u) =$

$$(n-1)(n-2)^2 \cdots (n-d)^d (n-d-1)^d \cdots (d)^d (d-1)^{d-1} \cdots (2)^2 (1).$$

Thus we arrive at the following.

THEOREM 10.25. *The multiplicity of $X_{d,n}$ at P_σ is equal to*

$$\frac{(d\,(n-d))!}{(n-1)(n-2)^2\cdots(n-d)^d(n-d-1)^d\cdots(d)^d(d-1)^{d-1}\cdots(2)^2(1)}.$$

CONJECTURE 10.26. *There exists a multiplicative formula in $X_{d,n}$ similar to Theorem 10.20 for a face which has the form as an union of several disjoint (and non-consecutive) faces.*

The generating set $W(\tau)$ of a face τ consists of $\{e_{y'} - e_y\}$, for certain covers (y, y') in $J(\mathcal{L})$ (assuming that $\hat{1} \in D_\tau$ so that $e_{\hat{1}}$ is not in $W(\tau)$). Thus $W(\tau)$ determines a sub set $H(\tau) := \dot{\bigcup} H(\tau)_i$ of $J(\mathcal{L})$, such that $W(\tau)$ consists of all the covers in the $H(\tau)_i$'s. In §10.15, if $\tau = \tau_1 \cup \tau_2$ for τ_1, τ_2 a pair of non-consecutive, non-intersecting J-blocks, $H(\tau) = H(\tau_1)\dot{\cup}H(\tau_2)$.

Theorem 10.14 implies that $mult_{P_{\tau_1}} X_{2,n} = mult_{P_{\tau_2}} X_{2,n}$ if both τ_1 and τ_2 are J-blocks of the same length; in particular, $H(\tau_1)$ and $H(\tau_2)$ are isomorphic. Guided by this phenomenon we make the following conjecture.

CONJECTURE 10.27. *For a face τ of any Hibi toric variety $X(\mathcal{L})$, $mult_{P_\tau} X(\mathcal{L})$ is determined by the poset $H(\tau)$. By this we mean that if τ, τ' are such that $H(\tau)$, $H(\tau')$ are isomorphic posets, then the multiplicities of $X(\mathcal{L})$ at the points P_τ, $P_{\tau'}$ are the same.*

CHAPTER 2

Arithmetically Gorenstein Schubert varieties in a minuscule G/P

1. Hodge algebras

In this section, we recall the definition (cf. [**7, 14**]) of *Hodge algebras* or also known as *algebras with straightening laws*, abbreviated ASL.

Let H be a finite poset and N be the set of non-negative integers. A *monomial* $\mathcal{M}$ on H is a map from H to N. The *support* of $\mathcal{M}$ is the set $\mathrm{Supp}(\mathcal{M}) = \{x \in H \mid \mathcal{M}(x) \neq 0\}$; $\mathcal{M}$ is *standard* if $\mathrm{Supp}(\mathcal{M})$ is a chain in H (a chain is a totally ordered subset of a poset).

If $\mathcal{R}$ is a commutative ring, and we are given an injection $\varphi : H \hookrightarrow \mathcal{R}$, then to each monomial $\mathcal{M}$ on H we may associate

$$\varphi(\mathcal{M}) := \prod_{x \in H} \varphi(x)^{\mathcal{M}(x)} \in \mathcal{R}.$$

Definition 1.1. Let $\mathcal{R}$ be a commutative K-algebra. Suppose that H is a finite poset with an injection $\varphi : H \hookrightarrow \mathcal{R}$. Then we call $\mathcal{R}$ a *Hodge algebra* or also an *algebra with straightening laws* (abbreviated ASL) on H over K if the following conditions are satisfied:

ASL-1 The set of standard monomials is a basis of the algebra $\mathcal{R}$ as a vector space over K.

ASL-2 If τ and ϕ in H are incomparable and if

$$\tau\phi = \sum_i a_i \, \gamma_{i1}\gamma_{i2}\cdots\gamma_{it_i},$$

(where $0 \neq a_i \in K$ and $\gamma_{i1} \leq \gamma_{i2} \leq \cdots \gamma_{it_i}$) is the unique expression for $\tau\phi \in \mathcal{R}$ as a linear combination of distinct standard monomials guaranteed by ASL-1, then $\gamma_{i1} \leq \tau, \phi$ for every i.

1.2. Schubert varieties and Hodge algebras. Let G be a semisimple algebraic group. Given a maximal parabolic subgroup P with ω as the associated fundamental weight, let L be the ample generator of $\operatorname{Pic} G/P (\cong \mathbb{Z})$, the isomorphism classes of line bundles on G/P. For the canonical projective embedding

$$X(w) \hookrightarrow G/P \hookrightarrow Proj(H^0(G/P, L))$$

($X(w)$ being a Schubert variety in G/P), let $R(w)$ denote the homogeneous coordinate ring of $X(w)$. We have a canonical identification (see [**16**] for example) of $H^0(G/P, L)$ with $(V(\omega))^*$, $V(\omega)$ being the Weyl module associated to ω. In particular, we have that in the T-module $H^0(G/P, L)$, $-\theta(\omega)$ for $\theta \in W/W_P$ occurs as a weight with multiplicity 1; let us fix a generator p_θ for the corresponding (one-dimensional) weight space.

Now taking P to be minuscule (for the definition of minuscule, see Chapter 1 §4.5), we have (cf. [**30**]) that $\{p_\theta, \theta \in W/W_P\}$ is a K-basis for $H^0(G/P, L)$. It is shown in [**30**] that the *standard monomials* $\{p_{\theta_1} \cdots p_{\theta_r}, w \geq \theta_1 \geq \ldots \geq \theta_r, r \in \mathbb{Z}_+\}$ give a K-basis for $R(w)$; further it is shown in [**30**] that in the expression for a non-standard monomial $p_\tau p_\phi$ as a sum of standard monomials $\sum c_{\alpha,\beta} p_\alpha p_\beta$, $\alpha \geq \beta$, $c_{\alpha,\beta} \in K$, one has that in each term $c_{\alpha,\beta} p_\alpha p_\beta$, $\alpha \gneq$ both τ and ϕ, and $\beta \lneq$ both τ and ϕ. As a consequence, $R(w)$ acquires the structure of a graded Hodge algebra, $R(w)$ having a set of algebra generators indexed by $H(w)$, the Bruhat poset of Schubert subvarieties of $X(w)$.

Proposition 1.3. *The homogeneous coordinate ring $R(w)$ for a Schubert variety $X(w)$ in a minuscule G/P is an ASL on $H(w)$ over K.*

1.4. Cohen-Macaulay & Gorenstein properties: Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $k = R / \mathfrak{m}$.

DEFINITION 1.5. The local ring $(R, \mathfrak{m})$ is *Cohen-Macaulay* if

$$\operatorname{Ext}^i_R(k, R) = 0, \text{ for } i < \dim R;$$

it is *Gorenstein* if in addition, we have, $\operatorname{Ext}^{\dim R}_R(k, R) = k$.

DEFINITION 1.6. An algebraic variety X is *Cohen-Macaulay at a point* (resp. *Gorenstein at a point*) $x \in X$, if the stalk $\mathcal{O}_{X,x}$ is Cohen-Macaulay (resp. Gorenstein); X is *Cohen-Macaulay* (resp. *Gorenstein*), if it is Cohen-Macaulay (resp. Gorenstein) at all $x \in X$.

DEFINITION 1.7. A projective variety $X = \operatorname{Proj} S$ is *arithmetically Cohen-Macaulay* (resp. *arithmetically Gorenstein*), if $\widehat{X}(= \operatorname{Spec} S)$, the cone over X, is Cohen-Macaulay (resp. Gorenstein).

REMARK 1.8. Note that the cone $\widehat{X}$ is Cohen-Macaulay (resp. Gorenstein), if and only if it is so at its vertex. Also note that if $\widehat{X}$ is Cohen-Macaulay (resp. Gorenstein), then so is X.

1.9. The Gorenstein property for an ASL. Let $\mathcal{L}$ be a distributive lattice, and $\mathcal{R}$ a graded ASL domain on $\mathcal{L}$ over a field K. Further, let

$$\deg(\alpha) + \deg(\beta) = \deg(\alpha \vee \beta) + \deg(\alpha \wedge \beta)$$

for all $\alpha, \beta \in \mathcal{L}$. We have the following characterization of the Gorenstein property for $\mathcal{R}$:

THEOREM 1.10 (cf. §3 of [**14**]). *$\mathcal{R}$ is Gorenstein if and only if $J(\mathcal{L})$ is a ranked poset.*

REMARK 1.11. The above theorem is a result of Stanley (cf. [**31**, Theorem 4.4]).

2. Arithmetically Gorenstein Schubert varieties in a Grassmannian

Using the above Theorem and Proposition 1.3, we shall give a characterization of the Gorenstein property for the cones over Schubert varieties in a Grassmannian; as a byproduct, we obtain an alternate proof of the result of [**34, 36**] on Gorenstein Schubert varieties in a Grassmannian.

Fix two positive integers d, n such that $d < n$; let $G_{d,n}$ be the Grassmann variety consisting of the d-dimensional subspaces of K^n. As seen in Chapter 1, §10, the Schubert varieties in $G_{d,n}$ are indexed by the distributive lattice $I_{d,n}$. For $\tau \in I_{d,n}$, let $X(\tau)$ be the associated Schubert variety in $G_{d,n}$, and let $H(\tau)$ be the Bruhat poset of Schubert subvarieties of $X(\tau)$. Note that $H(\tau)$ is a distributive sublattice of $I_{d,n}$. For the Plücker embedding

$$X(\tau) \hookrightarrow G_{d,n} \hookrightarrow \operatorname{Proj}(\Lambda^d K^n)^*$$

let $R(\tau)$ be the homogeneous co-ordinate ring of $X(\tau)$. Then by Proposition 1.3, we have that $R(\tau)$ is a graded domain which is an ASL on $H(\tau)$ over K. Hence by Theorem 1.10, $X(\tau)$ is arithmetically Gorenstein if and only if $J(\tau)$ (the poset of join-irreducibles in $H(\tau)$) is a ranked poset.

PROPOSITION 2.1. *Let $\tau \in I_{d,n}$. Then $X(\tau)$ is arithmetically Gorenstein if and only if τ consists of intervals $I_1, I_2, \cdots, I_s$ where*

$$I_t = [x_t, y_t], 1 \leq t \leq s,\ x_{t+1} - y_t = y_t + 2 - x_t,\ 1 \leq t \leq s-1$$

(Here, $[x_t, y_t]$ denotes the set $\{x_t, x_{t+1}, \cdots, y_{t-1}, y_t\}$)

The proof is similar to (and simpler than) that of Lemma 4.6 in §4.

2.2. Outer-corner description. (cf. [**36**]) To a $\tau \in I_{d,n}$, we associate a Young diagram (or also a partition) λ^τ as follows: Let $\tau = (\tau_1, \cdots, \tau_d)$ (as a d-tuple). Set

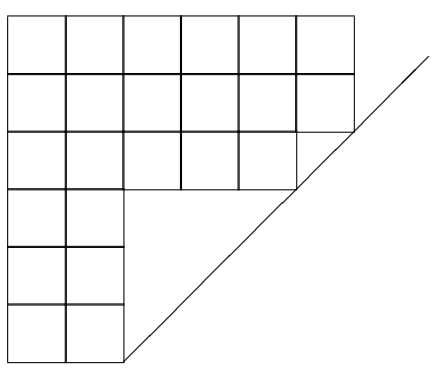
$$\lambda^\tau_r = \tau_r - r, \ \forall\ 1 \leq r \leq d.$$

Thus we write $\lambda^\tau = (\lambda^\tau_d, \cdots, \lambda^\tau_1)$; when there is no room for confusion, we drop the superscript and just write λ.

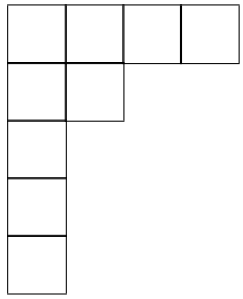
Let us write λ as a Young diagram, and place the bottom left corner of the first row (λ_1) at $(0,0)$ on the grid; then each block is a square of unit 1. Thus our diagram will be d-units high, and λ_d-units wide.

Definition 2.3. The partition λ satisfies the *outer corner condition* if all of the outer corners lie on a line of slope 1; we also refer to this as "the outer corners of λ lie on the same antidiagonal" (same terminology as in [**36**]).

Example 2.4. Let $n = 14, d = 6$, $\tau = (3,4,5,9,11,12)$, and thus $\lambda^\tau = (6,6,5,2,2,2)$. We write λ^τ as a diagram:



We can see that λ^τ satisfies the outer corner condition. Now let $\tau' = (1,3,4,5,7,10)$, thus $\lambda^{\tau'} = (4,2,1,1,1,0)$:



We can see that $\lambda^{\tau'}$ does not satisfy the outer corner condition.

As an immediate consequence of Theorem 1.10, Proposition 2.1 and Definition 2.3, we obtain the following.

THEOREM 2.5. *Let $\tau \in I_{d,n}$. Then $X(\tau)$ is arithmetically Gorenstein if and only if the outer corners of λ^{τ} lie on the same anti-diagonal.*

REMARK 2.6. Note that the above theorem gives a stronger result than that of [**34, 36**], namely, while in [**34, 36**], one has a characterization of Gorenstein Schubert varieties, the above theorem gives a characterization for Gorenstein property even for the cones over Schubert varieties. As a by-product, we obtain that a Schubert variety in the Grassmannian is arithmetically Gorenstein (for the Plücker embedding) if and only if it is geometrically Gorenstein

2.7. The minuscule G/P_1 in types C, D:. Using Theorem 1.10 and Proposition 1.3, the Gorenstein property for $R(w)$ is immediate if G is either $SP(2n)$ or $SO(2n)$, and $P = P_1$ (the maximal minuscule parabolic subgroup corresponding to the left end root of the Dynkin diagram, following the indexing of the Dynkin diagram as in [**5**]). In the former case, $G/P \cong \mathbb{P}^{2n-1}$ and $H(w)$ is clearly a chain for all $w \in W/W_P$.

In the latter case, we have, $\dim G/P = 2n-2$; further, for $0 \leq i \leq 2n-2$, there is precisely one Schubert variety in dimension $i \neq n-1$, and two Schubert varieties

in dimension $n-1$:

$$\begin{tikzcd}
 & X_{2n-2} \arrow[d, dashed, no head] & \\
 & X_n \arrow[dl, no head] \arrow[dr, no head] & \\
X_{n-1} \arrow[dr, no head] & & X'_{n-1} \arrow[dl, no head] \\
 & X_{n-2} \arrow[d, dashed, no head] & \\
 & X_0 &
\end{tikzcd}$$

Viewing the above diagrammatic representation as a distributive lattice, the only lattice element which is not join-irreducible is X_n, thus X_n is the only Schubert variety for which the Gorenstein property requires checking. On the other hand, one can easily see that the poset of Schubert subvarieties of X_n is ranked. From this it easily follows that the poset of join-irreducibles in $H(w)$ is a ranked poset for all $w \in W/W_P$.

In the next two sections we present results for the orthogonal Grassmannian.

3. The lattice of Schubert varieties in an orthogonal Grassmannian

Let G be the special orthogonal group $SO(m)$, and let P be the maximal parabolic subgroup corresponding to the right end root (resp. one of the right end roots if m is even) in the Dynkin diagram (following the indexing of the Dynkin diagram as in [**5**]). Then G/P is the *orthogonal Grassmannian*. Since P is minuscule, the Bruhat-poset of Schubert varieties in G/P is a minuscule lattice, and hence a distributive lattice (see Chapter 1, Remark 4.7). Further, we have isomorphisms of the following

minuscule lattices:

$$\mathbf{B_{n-1}}\left(\omega_{n-1}\right) \cong \mathbf{D_n}\left(\omega_{n-1}\right) \cong \mathbf{D_n}\left(\omega_n\right)$$

Hence for the rest of this section, we shall suppose that $G = SO(2n+1)$ and $P = P_n$ (the maximal parabolic subgroup with ω_n as the associated fundamental weight). This will allow us to identify the associated minuscule lattice with the subset $\mathcal{L} \subset I_{n,2n}$ as described below.

Description of $\mathcal{L}$ (cf. §2 of [**24**]): For two integers $r, s \in \mathbb{N}, r \leq s$, let $I_{r,s} = \{(i_1, \ldots, i_r) \mid 1 \leq i_1 < \ldots < i_r \leq s\}$, with partial order $\leq$ given by

$$(i_1, \ldots, i_r) \leq (j_1, \ldots, j_r) \Leftrightarrow i_1 \leq j_1, \ldots, i_r \leq j_r.$$

We have an identification (cf. [**24**]) of $\mathcal{L}$ as a sublattice of $I_{n,2n}$:

$$\mathcal{L} = \left\{ (i_1, \ldots, i_n) \in I_{n,2n} \;\middle|\; \begin{array}{l} \text{for } 1 \leq j \leq n, \text{ precisely one of} \\ \{j, 2n+1-j\} \in \{i_1, \ldots, i_n\} \end{array} \right\}.$$

Notation. We will use the notation $[i, j]$ to represent the string of consecutive integers $\{i, i+1, \ldots, j-1, j\}$. Whenever we refer to a segment $[i, j]$ such that $i > j$, the segment is understood to be empty.

LEMMA 3.1. $J(\mathcal{L}) = \{([1, j], [n+1-i, n], [n+1+i, 2n-j]) \mid 0 \leq j \leq n, 0 \leq i \leq n-j-1\}$. *Note that when $j = n$, we have the minimal element in $\mathcal{L}$, namely,* $(1, \ldots, n)$.

PROOF. For $w = (w_1, \ldots, w_n) \in \mathcal{L}$, let $1 \leq d \leq n$ be such that $w_d \leq n$ and $w_{d+1} \geq n+1$. Notice that w is determined by $(w_1, \ldots, w_d)$. In this way, we can project $\mathcal{L}$ onto $\bigcup_{1 \leq d \leq n} I_{d,n}$. This map is bijective, and order is preserved within each $I_{d,n}$. Further, if w does not project onto a join irreducible element in $I_{d,n}$, then w is not join irreducible in $\mathcal{L}$.

Therefore, we check to see if join irreducible elements in $I_{d,n}$ are join irreducible in $\mathcal{L}$. We have (from §8 of [**13**]),

$$J(I_{d,n}) = \{(1, \ldots, j, j+i+1, \ldots, d+i) \mid 0 \leq j \leq d,\ 1 \leq i \leq n-d\}.$$

Note that the element $(1, \ldots, j, j+i+1, \ldots, d+i) \in I_{d,n}$ corresponds to $([1,j], [j+i+1, d+i], [n+1, 2n-d-i], [2n-j-i+1, 2n-j]) \in \mathcal{L}$. In the case when $i \neq n-d$, this element covers two distinct elements,

$$([1,j], j+i, [j+i+2, d+i], [n+1, 2n-d-i], 2n-j-i, [2n-j-i+2, 2n-j]),$$

$$([1,j], [j+i+1, d+i], n, [n+2, 2n-d-i], [2n-j-i+1, 2n-j]).$$

Thus we only need to consider elements of the form $(1, \ldots, j, n-d+j+1, \ldots n, n+d-j+1, \ldots, 2n-j)$, for all d, and these elements clearly cover only one element each. The result follows. □

Remark 3.2. By the results of Chapter 1, we may identify $J(\mathcal{L})$ with a grid lattice (Chapter 1, Definition 4.1). This implies that $J(\mathcal{L})$ is a distributive lattice, and thus is ranked.

$J(\mathcal{L})$ being a ranked poset, each element of $J(\mathcal{L})$ has a well-defined level: let $level_w$ be the number of elements in a maximal chain from $\hat{0}$ to w minus one.

Lemma 3.3. *Let $w \in J(\mathcal{L})$, $w = (1, \ldots, j, n+1-i, \ldots, n, n+1+i, \ldots, 2n-j)$. Then $level_w = 2n-2j-i-1$.*

Proof. We give $J(\mathcal{L})$ a grid lattice structure. Namely, we send w to $(n-j, n-j-i) \in \mathbb{N} \times \mathbb{N}$. For book-keeping purposes, let $\hat{0} = (1, \ldots, n) \mapsto (0,1)$. Therefore, in this sublattice of $\mathbb{N} \times \mathbb{N}$, $level_{(a,b)} = a+b-1$, as we can see by taking the maximal chain from $(0,1)$ to $(a,1)$, and then from $(a,1)$ to (a,b). Therefore, $level_w = n-j+n-j-i-1$. The result follows. □

4. Arithmetically Gorenstein Schubert varieties in an orthogonal Grassmannian

For $\beta \in \mathcal{L}$, let $\mathcal{L}_\beta$ be the sublattice $[\hat{0}, \beta]$; then $R(\beta)$ is an ASL on $\mathcal{L}_\beta$. Setting $\mathcal{R} = R(\beta)$, we have $\deg(\alpha) = 1$ for all $\alpha \in \mathcal{L}_\beta$.

As stated in the previous section, we know $\beta = (\beta_1, \ldots, \beta_n)$ is determined by $(\beta_1, \ldots, \beta_d)$ where $\beta_d \leq n$ and $\beta_{d+1} \geq n+1$. For now, we will be primarily concerned with $(\beta_1, \ldots, \beta_d)$. We will now break β into its segments, thus we denote

$$(\beta_1, \ldots, \beta_d) = \left([1, j_0], [\beta_{(j_0)+1}, \beta_{j_1}], [\beta_{(j_1)+1}, \beta_{j_2}], \ldots, [\beta_{(j_{s-1})+1}, \beta_{j_s}]\right),$$

where $\{j_0, \ldots, j_s\}$ is a subset of $\{0, \ldots, d\}$, (we may have $j_0 = 0$, but we must have $j_s = d$), and $\beta_{(j_i)+1} - \beta_{j_i} \geq 2$ for $0 \leq i \leq s-1$.

We may write β as the join of non-comparable join irreducibles:

$$(*) \qquad \beta = w_1 \vee \ldots \vee w_s \, (\vee w_0),$$

where the w_i's are join-irreducible and mutually non-comparable. Specifically,

$$w_i = \left([1, j_{i-1}], [\beta_{(j_{i-1})+1}, n]\right), \ 1 \leq i \leq s,$$

and

$$w_0 = ([1, j_s]).$$

We only give those integers in w_i less than or equal to n. Note that in the case where $\beta_{j_s} = n$, w_0 is unnecessary, in fact $w_0 \leq w_s$, which is why we have listed w_0 in parentheses in $(*)$.

Example 4.1. Let $n = 5$, and $\beta = (2, 4, 6, 8, 10)$; we have

$$(2, 4, 6, 8, 10) = (2, 3, 4, 5, 10) \vee (1, 4, 5, 8, 9) \vee (1, 2, 6, 7, 8).$$

REMARK 4.2. With notation as above, $J(\mathcal{L}_\beta) = \bigcup_{0\leq i\leq s} [\hat{0}, w_i]$. Then as a result of Theorem 1.10, we have that the Schubert variety $X(\beta)$ is arithmetically Gorenstein if and only if $level_{w_i} = level_{w_j}$ for all $0 \leq i, j \leq s$.

REMARK 4.3.

$$\begin{aligned} level_{w_i} &= n - 2j_{i-1} + \beta_{(j_{i-1})+1} - 2, \text{ for } 1 \leq i \leq s; \\ level_{w_0} &= 2n - 2j_s - 1. \end{aligned}$$

REMARK 4.4. Note that $\beta_{(j_i)+1} = \beta_{(j_{i+1})} - (j_{i+1} - j_i - 1)$, by construction.

DEFINITION 4.5. We say that an element $\beta \in \mathcal{L}$ satisfies *condition A*, if β satisfies

$$\begin{aligned} \beta_{(j_i)+1} - \beta_{j_i} &= j_i - j_{i-1} + 1, \ \forall\ 1 \leq i \leq s-1, \text{ and} \\ n - \beta_{j_s} &= j_s - j_{s-1}, \text{ if } \beta_{j_s} \neq n. \end{aligned}$$

LEMMA 4.6. *$X(\beta)$ is arithmetically Gorenstein if and only if β satisfies condition A.*

PROOF. **Case 1:** Let $\beta_{j_s} = n$.

Let i be such that $1 \leq i \leq s-1$, and for convenience of notation, let $k = i - 1$.

Let $X(\beta)$ be arithmetically Gorenstein. Then, by Remarks 4.2 and 4.3, we have

$$n - 2j_i + \beta_{(j_i)+1} - 2 = n - 2j_k + \beta_{(j_k)+1} - 2.$$

Thus $\beta_{(j_i)+1} - \beta_{(j_k)+1} = 2j_i - 2j_k$; this together with Remark 4.4 (applied to $\beta_{(j_k)+1}$) implies that

$$\beta_{(j_i)+1} - (\beta_{j_i} - j_i + j_k + 1) = 2j_i - 2j_k.$$

Therefore $\beta_{(j_i)+1} - \beta_{j_i} = j_i - j_k + 1$, and thus β satisfies Condition A.

Now let β satisfy Condition A. Hypothesis implies that $\beta_{(j_i)+1} - \beta_{j_i} = j_i - j_k + 1$ for every i such that $1 \leq i \leq s-1$, where $k = i-1$. From Remark 4.4, $\beta_{j_i} = \beta_{(j_k)+1} + j_i - j_k - 1$, therefore

$$\beta_{(j_i)+1} - \beta_{(j_k)+1} = 2j_i - 2j_k; \Rightarrow \beta_{(j_i)+1} - 2j_i = \beta_{(j_k)+1} - 2j_k.$$

Thus we get that $level_{w_{i+1}} = level_{w_{k+1}}$, and since this is true for any i, we get $level_{w_i} = level_{w_j}$ for all $1 \leq i, j \leq s$. Therefore $X(\beta)$ is arithmetically Gorenstein.

This completes the proof for Case 1.

Case 2: Let $\beta_{j_s} \neq n$ (making w_0 relevant).

Let $X(\beta)$ be arithmetically Gorenstein. As in Case 1, we have, $\beta_{(j_i)+1} - \beta_{j_i} = j_i - j_k + 1, 1 \leq i \leq s-1 (k = i-1)$. It remains to show that $n - \beta_{j_s} = j_s - j_{s-1}$. Hypothesis implies that $level_{w_0} = level_{w_s}$; i.e.,

$$2n - 2j_s - 1 = n - 2j_{s-1} + \beta_{(j_{s-1})+1} - 2.$$

We apply Remark 4.4 (for $i = s-1$), and obtain

$$2n - 2j_s - 1 = n - 2j_{s-1} + \beta_{j_s} - (j_s - j_{s-1} - 1) - 2.$$

Therefore $n - \beta_{j_s} = j_s - j_{s-1}$, as desired.

Now let β satisfy Condition A. Hypothesis implies that

$$\beta_{j_{i+1}} - \beta_{j_i} = j_i - j_k + 1, 1 \leq i \leq s-1 (k = i-1),\ n - \beta_{j_s} = j_s - j_{s-1}.$$

We must show that $X(\beta)$ is arithmetically Gorenstein, equivalently, we must show that $level_{w_i} = level_{w_j}, 0 \leq i, j \leq s$. As in Case 1, hypothesis implies that $level_{w_i} = level_{w_j}, 1 \leq i, j \leq s$. We shall now show that $level_{w_0} = level_{w_s}$, thus completing the proof in this case. In view of Remark 4.3, it suffices to check that $2n - 2j_s - 1 =$

$n - 2j_{s-1} + \beta_{(j_{s-1})+1} - 2$. We begin with Remark 4.4:

$$\begin{aligned} \beta_{j_s} - \beta_{j_{s-1}+1} &= j_s - j_{s-1} - 1, \Rightarrow \\ n - \beta_{j_{s-1}+1} &= (n - \beta_{j_s}) + j_s - j_{s-1} - 1 \end{aligned}$$

By hypothesis, this implies

$$n - \beta_{j_{s-1}+1} = (j_s - j_{s-1}) + j_s - j_{s-1} - 1$$

Hence, it follows that $n - 2j_s = -2j_{s-1} + \beta_{j_{s-1}+1} - 1$.

This completes the proof for Case 2. □

4.7. Outer-corner condition: As in §2.2, for $\beta := (\beta_1, \cdots, \beta_n) \in \mathcal{L}$ we associate a Young diagram $\lambda^\beta = \left(\lambda_n^\beta, \ldots, \lambda_1^\beta\right)$, where

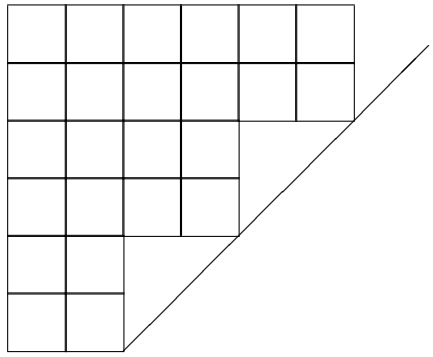
$$\lambda_i^\beta = \beta_i - i, \ \forall\ 1 \leq i \leq n$$

When there is no room for confusion, we drop the superscript and just write λ.

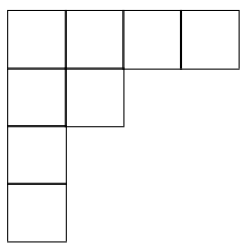
As in §2.2, we place the bottom left corner of the first row (λ_1) at $(0, 0)$ on the grid; then each block is a square of unit 1. Thus our diagram will be n-units high, and λ_n-units wide.

Definition 4.8. We follow the terminology "outer corner condition" for λ as in Definition 2.3, namely if all of the outer corners of λ lie on a line of slope 1.

Example 4.9. Let $n = 8$, $\beta = (1, 2, 5, 6, 9, 10, 13, 14)$, and thus $\lambda^\beta = (6, 6, 4, 4, 2, 2, 0, 0)$. We write λ^β as a diagram:



We can see that λ^β satisfies the outer corner condition. Now let $\beta' = (1, 2, 3, 4, 6, 7, 9, 12)$, thus $\lambda^{\beta'} = (4, 2, 1, 1, 0, 0, 0, 0)$:



We can see that $\lambda^{\beta'}$ does not satisfy the outer corner condition.

We now return to a general $\beta \in \mathcal{L}$; we preserve the notation from §4. From our definition of $\{j_0, \ldots j_s\} \subseteq \{0, \ldots, d\}$, we have $\lambda_{(j_i)+1} = \ldots = \lambda_{(j_{i+1})}$. Note that $\lambda_1 = \ldots = \lambda_{j_0} = 0$, so the first outer corner takes place at the bottom right of the row assigned to $\lambda_{(j_0)+1}$; specifically, the point $(\beta_{j_0+1} - j_0 - 1, j_0)$. Similarly, the first s outer corners are at the points

$$\left(\lambda_{(j_{i-1})+1} - j_{i-1} - 1, j_{i-1}\right) \;\; 1 \leq i \leq s.$$

If $\beta_{j_s} \neq n$, we have the $s+1$ outer corner at $(n - j_s, j_s)$.

Note that the diagram is necessarily "self dual," (cf. [**24**]). The dual of the partition λ is given by λ', where $\lambda'_i = \#\{\lambda_j \mid \lambda_j \geq i\}$, i.e. the rows of λ' are given by the columns of λ. Thus, for λ derived from $\beta \in \mathcal{L}$, we have $\lambda = \lambda'$. In the case of the grid, this implies that the diagram reflects over the line between points $(0, n)$

and $(n, 0)$. Thus, if there is an outer corner at the point (i, j), there will be an outer corner at the point $(n - j, n - i)$.

Note that in the case where $\beta_{j_s} \neq n$, the $(s + 1)^{th}$ outer corner corresponds to itself, it actually lies on the line from $(0, n)$ to $(n, 0)$. Therefore, the first s outer corners fall below the line, and correspond to another set of s outer corners above the line. Therefore we get $2s$ outer corners given in pairs: for $1 \leq i \leq s$,

$$\left(\beta_{(j_{i-1})+1} - j_{i-1} - 1, j_{i-1}\right), \ \left(n - j_{i-1}, n - \beta_{(j_{i-1})+1} + j_{i-1} + 1\right);$$

with one more outer corner whenever $\beta_{j_s} \neq n$: $(n - j_s, j_s)$.

Lemma 4.10. *If the first $s+1$ outer corners of λ satisfy the outer corner condition, then λ satisfies the outer corner condition.*

Proof. Note that two corners (x_1, y_1) and (x_2, y_2) fall on a line of slope 1 whenever $y_1 - x_1 = y_2 - x_2$. Note that this always holds true for the pair

$$\left(\beta_{(j_{i-1})+1} - j_{i-1} - 1, j_{i-1}\right), \ \left(n - j_{i-1}, n - \beta_{(j_{i-1})+1} + j_{i-1} + 1\right).$$

Therefore, each of the first s outer corners lies on a line of slope 1 with its dual counterpart. If we now assume that the first $s + 1$ outer corners are on a line of slope 1, (including the special case if $\beta_{j_s} \neq n$), then all the outer corners satisfy the outer corner condition. The result follows. □

Lemma 4.11. *The lattice point $\beta \in \mathcal{L}$ satisfies condition A if and only if the partition λ^β satisfies the outer corner condition.*

Proof. We begin with the outer corners for i and $i + 1$:

$$\left(\beta_{(j_{i-1})+1} - j_{i-1} - 1, j_{i-1}\right), \ \left(\beta_{(j_i)+1} - j_i - 1, j_i\right).$$

If we begin letting $\beta_{j_s} = n$, by Lemma 4.10, the outer corner condition is satisfied if and only if for every $1 \leq i \leq s-1$,

$$\begin{aligned} &\beta_{(j_{i-1})+1} - 2j_{i-1} - 1 = \beta_{j_i+1} - 2j_i - 1, \quad \Leftrightarrow \\ &\beta_{j_i} + 1 - \beta_{(j_{i-1})+1} = 2j_i - 2j_{i-1}. \end{aligned}$$

By Remark 4.4, this is if and only if

$$\begin{aligned} &\beta_{j_i+1} - (\beta_{j_i} - j_i + j_{i-1} + 1) = 2j_i - 2j_{i-1} \quad \Leftrightarrow \\ &\beta_{j_i+1} - \beta_{j_i} = j_i - j_{i-1} + 1, \end{aligned}$$

which holds if and only if β satisfies condition A.

Now, if $\beta_{j_s} \neq n$, we must show that the $(s+1)$-th outer corner is on line with the others if and only if $n - \beta_{j_s} = j_s - j_{s-1}$. We have that the corners $\left(\beta_{(j_{s-1})+1} - j_{s-1} - 1, j_{s-1}\right)$, $(n - j_s, j_s)$ are on line if and only if

$$n - 2j_s = \beta_{(j_{s-1})+1} - 2j_{s-1} - 1$$

By Remark 4.4, we have $\beta_{(j_{s-1})+1} = \beta_{j_s} - j_s - j_{s-1} - 1$, thus

$$\begin{aligned} &n - 2j_s = \beta_{j_s} - j_s + j_{s-1} + 1 - 2j_{s-1} - 1 \quad \Leftrightarrow \\ &n - \beta_{j_s} = j_s - j_{s-1}. \end{aligned}$$

The result follows. □

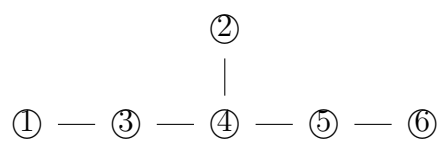
The Lemmas from §3, §4 (together with Theorem 1.10 and Proposition 1.3) lead to the characterization of the arithmetically Gorenstein property for Schubert varieties in the orthogonal Grassmannian:

THEOREM 4.12. *The Schubert variety $X(\beta)$ in the orthogonal Grassmannian G/P is arithmetically Gorenstein if and only if λ^β satisfies the outer corner condition.*

5. The exceptional groups

In this section, we list the arithmetically Gorenstein minuscule Schubert varieties in types E_6, E_7. We shall index the simple roots as in [**5**]:

5.1. $\mathbf{E}_6$. In the root system of type $\mathbf{E}_6$, we have two minuscule weights: ω_1 and ω_6.



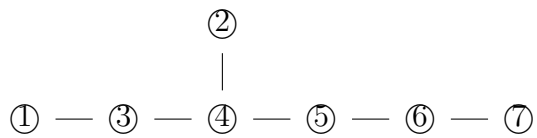
For the results below, we let P be the minuscule parabolic subgroup associated to the root ω_1. Similar results can be found for P associated to ω_6 simply by performing the appropriate permutation of simple roots.

We list the Gorenstein, as well as the non-Gorenstein Schubert varieties in G/P. For a simple root $\alpha_i, 1 \leq i \leq 6$, s_i will denote the reflection with respect to α_i. For convenience of notation, we will denote an element $s_4 s_3 s_1 W_P \in W/W_P$ by just 431; in fact, we will denote the associated Schubert variety also by just 431 (since there is no room for confusion).

All results can be checked by constructing the Bruhat poset of Schubert varieties, and using Theorem 1.10.

Gorenstein	Non - Gorenstein
e (identity element)	652431
1	63452431
31	163452431
431	5613452431
5431	245613452431
65431	
2431	
52431	
452431	
3452431	
6452431	
13452431	
54652431	
563452431	
4563452431	
24563452431	
45613452431	
345613452431	
2345613452431	
42345613452431	
542345613452431	
6542345613452431	

5.2. $\mathbf{E}_7$. In the root system of type $\mathbf{E}_7$, we have one minuscule weight: ω_7. We use the indexing of simple roots found in [**5**]:



For the results below, we let P be the minuscule parabolic subgroup associated to the root ω_7. As with the $\mathbf{E}_6$ case, we categorize the Schubert varieties in G/P as Gorenstein or non-Gorenstein.

Gorenstein	Non - Gorenstein
e (identity element)	1234567
7	541234567
67	6541234567
567	76541234567
4567	65341234567
24567	765341234567
34567	7645341234567
234567	25645341234567
134567	6725645341234567
4234567	7425645341234567
41234567	73425645341234567
54234567	673425645341234567
654234567	173425645341234567
7654234567	1673425645341234567
341234567	15673425645341234567
5341234567	1245673425645341234567

(The table is continued on the next page.)

Gorenstein	Non - Gorenstein
45341234567	
245341234567	
645341234567	
5645341234567	
25645341234567	
72645341234567	
75645341234567	
675645341234567	
725645341234567	
425645341234567	
3425645341234567	
67425645341234567	
13425645341234567	
567425645341234567	
5673425645341234567	
45673425645341234567	
245673425645341234567	
1456734256453412345 67	
31456734256453412345 67	
312456734256453412345 67	
4312456734256453412345 67	
54312456734256453412345 67	
654312456734256453412345 67	
7654312456734256453412345 67	

6. The arithmetically Gorenstein property for the flag variety

In this section, we present results for the (partial) flag varieties $SL(n)/Q$.

Let us denote by X the flag variety $SL(n)/B$ (here, we take B to be the Borel subgroup in $SL(n)$ consisting of upper triangular matrices). The Picard group of X is free abelian of rank $\ell := n-1$, having the line bundles L_d (the ample generator of Pic G/P_d, where P_d is the maximal parabolic subgroup associated to the fundamental weight ω_d) as a $\mathbb{Z}$-basis. For λ in the weight lattice, we shall denote the associated line bundle on X by $L(\lambda)$. We have (see [**16**] for example), $H^0(X, L(\lambda))$ is non-zero if and only if λ is dominant (i.e., λ is a non-negative integral combination of the fundamental weights). Also, G/P_d is simply the Grassmannian $G_{d,n}$ of the d-dimensional subspaces of K^n; further, for $\tau \in W/W_{P_d}$, the extremal weight vectors p_τ's in $H^0(G_{d,n}, L(\omega_d))$ are simply the Plücker coordinates, and they give a basis for $H^0(G_{d,n}, L(\omega_d))$ (for details see [**18**]). Let us denote the multi-homogeneous coordinate ring of X by R. (Note that R is just the K-algebra generated by the extremal weight vectors in $H^0(G_{d,n}, L(\omega_d))$, $1 \le d \le n-1$.) We have a canonical inclusion

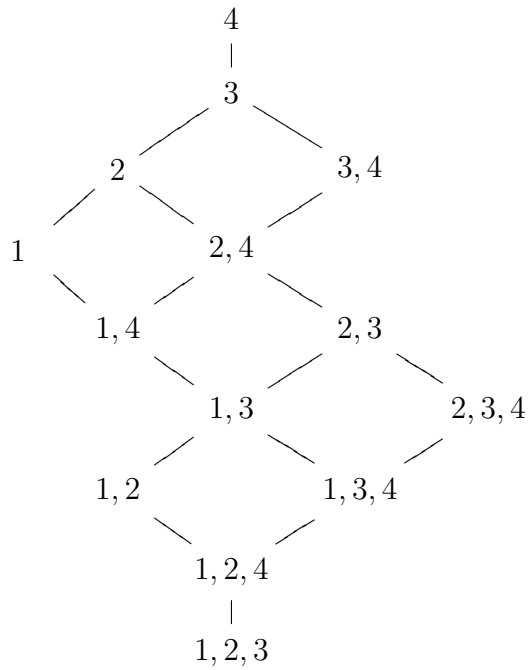
$$R \hookrightarrow \bigoplus_{\{\lambda \text{ dominant}\}} H^0(X, L(\lambda)).$$

As in §1.9, let $I_{d,n} = \{(1 \le i_1 < i_2 < \cdots < i_d \le n)\}$, and let

$$H_n = \bigcup_{d=1}^{n-1} I_{d,n}\,.$$

We define a partial order on H_n as follows. Let τ, $w \in H_n$, $\tau = (i_1, \cdots, i_r)$, $w = (j_1, \cdots, j_s)$. Define $\tau \ge w$ if $r \le s$ and $i_1 \ge j_1, \cdots, i_r \ge j_r$. With this partial order, H_n is in fact a distributive lattice (see [**12**] for a proof).

EXAMPLE 6.1. For SL_4/B, the distributive lattice H_4 is given below.



REMARK 6.2. In the literature, the distributive lattice H_n is called a *Young lattice*. We introduced the Young lattice H_n (and H_Q, defined below) in Chapter 1, §8.

As noted above, R has a set of K-algebra generators consisting of $\{p_\tau, \tau \in H_n\}$.

DEFINITION 6.3. Define a monomial $p_{\tau_1} \cdots p_{\tau_m}, \tau_i \in H_n, 1 \leq i \leq m$ of multi-degree $(m_1, \cdots, m_{n-1})$ (where $m_d = \#\{i \,|\, \tau_i \in I_{d,n}\}$) to be *standard on* X if $\tau_1 \geq \cdots \geq \tau_m$.

Let $\lambda = \sum_{d=1}^{n-1} m_d \omega_d$. By the results of [**23**], we have the following

PROPOSITION 6.4.

(1) $R = \bigoplus_{\{\lambda \text{ dominant}\}} H^0(X, L(\lambda))$.

(2) *Standard monomials on* X *of multi-degree* $(m_1, \cdots, m_{n-1})$ *form a basis for* $H^0(X, L(\lambda))$.

(3) *Let τ, ϕ be two non-comparable elements of H_n. Then in the expression*

$$p_\tau p_\phi = \sum c_{\alpha,\beta} p_\alpha p_\beta,\ c_{\alpha,\beta} \in K$$

where the right hand side is a sum of standard monomials, we have that any $\alpha >$ both τ and ϕ, and any $\beta <$ both τ and ϕ.

THEOREM 6.5. *The multicone $\operatorname{Spec} R$ is Gorenstein.*

PROOF. By the above Proposition, we have that R is an ASL over H_n. Hence the result will follow from Theorem 1.10 once we check that $J(H_n)$ (the poset of join-irreducibles in H_n) is a ranked poset. In Chapter 1 §8, it was shown that $J(H_n)$ is a grid lattice. Therefore $J(H_n)$ is a distributive lattice, and therefore $J(H_n)$ is ranked. □

More generally, for a parabolic subgroup $Q \supset B$ with $S_Q = S \setminus \{\alpha_{d_1}, \cdots, \alpha_{d_r}\}$ (S being the set of simple roots), we denote

$$H_Q = \bigcup_{t=1}^{r} I_{d_t,n}$$

Considering H_Q as a sublattice of H_n, we have that H_Q is a distributive lattice. The Picard group of G/Q is free abelian of rank r, having the line bundles $L_{d_t} :=$ the ample generator of $\operatorname{Pic} G/P_{d_t}$ as a $\mathbb{Z}$-basis. We define standard monomials in the p_τ's, $\tau \in H_Q$ as in Definition 6.3. Let R_Q be the multi-homogeneous coordinate ring of G/Q. (R_Q is just the K-algebra generated by the extremal weight vectors in $H^0(G_{d_t,n})$, $1 \le t \le r$.) Let $\lambda = \sum_{t=1}^{r} m_t \omega_{d_t}, m_t \in \mathbb{Z}_+$; we shall refer to such a λ as *Q-dominant*. We have a canonical inclusion

$$R_Q \hookrightarrow \bigoplus_{\{\lambda,\, Q-\text{dominant}\}} H^0(G/Q, L(\lambda)).$$

We have (by [**23**]) the following:

PROPOSITION 6.6.

(1) $R_Q = \bigoplus_{\{\lambda, Q-dominant\}} H^0(G/Q, L(\lambda))$.

(2) *Standard monomials on G/Q of multi-degree $(m_1, \cdots, m_r)$ form a basis for $H^0(X, L(\lambda))$.*

(3) *Let τ, ϕ be two non-comparable elements of H_Q. Then in the expression*

$$p_\tau p_\phi = \sum c_{\alpha,\beta} p_\alpha p_\beta,\ c_{\alpha,\beta} \in K$$

where the right hand side is a sum of standard monomials, we have that any $\alpha >$ both τ and ϕ, and any $\beta <$ both τ and ϕ.

THEOREM 6.7. *The multicone $\operatorname{Spec} R_Q$ is Gorenstein.*

PROOF. This proof follows from the fact that $J(H_Q)$ is a grid lattice (Chapter 1 §8), and thus ranked. □

CHAPTER 3

Wahl's conjecture for a minuscule G/P

The main goal of this chapter is the proof of Wahl's conjecture for a minuscule G/P, over a base field with positive characteristic.

Let k be the base field which we assume to be algebraically closed of positive characteristic; note that if Wahl's conjecture holds in infinitely many positive characteristics, then it holds in characteristic zero also, for the Gaussian is defined over the integers. Let G be a simple algebraic group over k (if G is the special orthogonal group, then characteristic of k will be assumed to be different from 2). Let T be a maximal torus in G, and R the root system of G relative to T. We fix a Borel subgroup $B, B \supset T$; let S be the set of simple roots in R relative to B, and let R^+ be the set of positive roots in R. We shall follow [**5**] for indexing the simple roots. Let W be the Weyl group of G; then the T-fixed points in G/B (for the action given by left multiplication) are precisely the cosets $e_w := wB, w \in W$. For $w \in W$, we shall denote the associated Schubert variety (the closure of the B-orbit through e_w) by $X(w)$.

1. Frobenius splittings

Let X be a scheme over k, separated and of finite type. Denote by F the *absolute Frobenius* map on X: this is the identity map on the underlying topological space X and is the p-th power map on the structure sheaf $\mathcal{O}_X$. We say that X is *Frobenius split*, if the p-th power map $F^{\#} : F_*\mathcal{O}_X \leftarrow \mathcal{O}_X$ splits as a map of $\mathcal{O}_X$-modules (see [**26**, §1, Definition 2], [**6**, Definition 1.1.3]). A splitting $\sigma : F_*\mathcal{O}_X \to \mathcal{O}_X$ *compatibly splits*

a closed subscheme Y of X if $\sigma(F_*\mathcal{I}_Y) \subseteq \mathcal{I}_Y$ where $\mathcal{I}_Y$ is the ideal sheaf of Y (see [**26**, §1, Definition 3], [**6**, Definition 1.1.3]).

Now let X be a non-singular projective variety, and K its canonical bundle. Using Serre duality (and the observation that $F^*L \cong L^p$ for an invertible sheaf L on X), we get a (k-semilinear) isomorphism of $H^0(X, \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X))$ $(= \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X))$ with $H^0(X, K^{1-p})$ (see [**26**, Page 32], [**6**, Lemma 1.2.6 and §1.3]). Thus to find splittings of X, we are led to look at a σ in $H^0(X, K^{1-p})$ such that the associated homomorphism $F_*\mathcal{O}_X \to \mathcal{O}_X$ is a splitting of $F^\#$; in the sequel, following [**26**], we shall refer to this situation by saying *the element $\sigma \in H^0(X, K^{1-p})$ splits X.*

Remark 1.1. By local computations, it can be seen easily that if $\sigma \in H^0(X, K^{1-p})$ vanishes to order greater than $d(p-1)$ along a subvariety Y of codimension d for some $1 \leq d \leq dim\, X - 1$, then σ is not a splitting of X. Hence we say that *a subvariety Y is compatibly split by σ with maximum multiplicity* if σ is a splitting of X which compatibly splits Y and which vanishes to order $d(p-1)$ generically along Y.

We will often use the following Lemma:

Lemma 1.2. *Let $f : X \to Y$ be a morphism of schemes such that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

(1) *If X is Frobenius split, then so is Y.*
(2) *If Z is compatibly split, then so is the scheme-theoretic image of Z in Y.*

For a proof, see [**6**, Lemma 1.1.8] or [**26**, Proposition 4].

1.3. The section $\sigma \in H^0(X, K_X^{-1})$. As above, let $X = G/B$. In [**26**], a section $s' \in H^0(X, K_X^{1-p})$ giving a splitting for X is obtained by inducing it from a section $s \in H^0(Z, K_Z^{1-p})$ which gives a splitting for Z, the Bott-Samelson variety. It turns out that up to a non-zero scalar multiple, s' equals σ^{p-1} where $\sigma \in H^0(X, K_X^{-1})$. In

fact, one has an explicit description of σ: We have, $K_X^{-1} = L(2\rho)$ where ρ denotes half the sum of positive roots (here, for an integral weight $\lambda, L(\lambda)$ denotes the associated line bundle on X). Let f^+, f^- denote respectively a highest, lowest weight vector in $H^0(X, L(\rho))$ (note that f^+, f^- are unique up to scalars). Then σ is the image of $f^+ \otimes f^-$ under the map

$$H^0(X, L(\rho)) \otimes H^0(X, L(\rho)) \to H^0(X, L(\rho))$$

given by multiplication of sections.

See [**6**, §2.3] for details.

2. Splittings and blow-ups

Let Z be a non-singular projective variety and σ a section of K^{1-p} (where K is the canonical bundle) that splits Z. Let Y be a closed non-singular subvariety of Z of codimension c. Let $ord_Y\sigma$ denote the order of vanishing of σ along Y. Let $\pi : \tilde{Z} \to Z$ denote the blow up of Z along Y and E the exceptional divisor (the fiber over Y) in $\tilde{Z}$.

A splitting $\tilde{\tau}$ of $\tilde{Z}$ induces a splitting τ on Z, in view of Lemma 1.2 (since, $\pi_*\mathcal{O}_Z \to \mathcal{O}_{\tilde{Z}}$ is an isomorphism). We say that σ *lifts* to a splitting of $\tilde{Z}$ if it is induced thus from a splitting $\tilde{\sigma}$ of $\tilde{Z}$ (note that the lift of σ to $\tilde{Z}$ is unique if it exists, since $\tilde{Z} \to Z$ is birational and two global sections of the locally free sheaf $\mathrm{Hom}_{\mathcal{O}_{\tilde{Z}}}(F_*\mathcal{O}_{\tilde{Z}}, \mathcal{O}_{\tilde{Z}})$ that agree on an open set must be equal).

Proposition 2.1. *With notation as above, we have*

(1) $ord_Y\sigma \leq c(p-1)$.

(2) *If* $ord_Y\sigma = c(p-1)$ *then* Y *is compatibly split.*

(3) $ord_Y\sigma \geq (c-1)(p-1)$ *if and only if* σ *lifts to a splitting* $\tilde{\sigma}$ *of* $\tilde{Z}$*; moreover,* $ord_Y\sigma = c(p-1)$ *if and only if the splitting* $\tilde{\sigma}$ *is compatible with* E.

PROOF. Assertion(1) follows in view of Remark 1.1 (since σ is a splitting). Assertion (2) follows from the local description as in [**26**, Proposition 5]. For a proof of assertion (3), see [**19**], Proposition 2.1. □

Now let $Z = G/P \times G/P$, and Y the diagonal copy of G/P in Z. We have:

THEOREM 2.2 (cf.[**19**]). *Assume that the characteristic p is odd. If E is compatibly split in $\tilde{Z}$, or, equivalently, if there is a splitting of Z compatibly splitting Y with maximal multiplicity, then the Gaussian map is surjective for $X = G/P$.*

Let us recall (cf.[**19**]) the following conjecture:

LMP-Conjecture For any G/P, there exists a splitting of Z that compatibly splits the diagonal copy of G/P with maximal multiplicity.

3. Steps leading to a proof of LMP-conjecture for a minuscule G/P

Our proof of the LMP-conjecture for a minuscule G/P is in the same spirit as in [**25**]. We describe below a sketch of the proof.

I. **The splitting λ of $G \times^B G/B$** : For a Schubert variety X in G/B, using the B-action on X, we may form the twisted fiber space $G \times^B X = G \times X/(gb, bx) \sim (g, x),\ g \in G, b \in B, x \in X$. For $X = G/B$, we have a natural isomorphism

$$f : G \times^B G/B \cong G/B \times G/B,\ (g, xB) \mapsto (gB, gxB)$$

We have (cf.[**27**]) that there exists a splitting for $G \times^B G/B (\cong G/B \times G/B)$ compatibly splitting the G-Schubert varieties $G \times^B X$. In fact, by [**6**, Theorem 2.3.8], we have that this splitting is induced by σ^{p-1} (where σ is as in §1.3; as in that subsection, one identifies σ with $f^+ \otimes f^-$). We shall denote this splitting of $G \times^B G/B$ by λ.

II. **Order of vanishing of λ along $G \times^B P/B$:** Let P be a (standard) parabolic subgroup. From the description of λ, it is clear that the order of vanishing of λ along

$G \times^B P/B$ equals $(p-1)ord_{P/B}\sigma$, where $ord_{P/B}\sigma$ denotes the order of vanishing of σ along P/B. For simplicity of notation, let us denote this order by q.

III. **Reduction to computing the order of vanishing of σ along P/B:** Consider the natural surjection $\pi : G/B \times G/B \to G/P \times G/P, (g_1B, g_2B) \mapsto (g_1P, g_2P)$. Then under the identification $f : G \times^B G/B \cong G/B \times G/B$, we have that π induces a surjection

$$G \times^B P/B \to \Delta_{G/P}$$

where $\Delta_{G/P}$ denotes the diagonal in $G/P \times G/P$. We now recall the following lemma from [**19**].

LEMMA 3.1. *Let $f : X \to Y$ be a morphism of schemes such that $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Let X_1 be a smooth subvariety of X such that f is smooth (submersive) along X_1. If X_1 is compatibly split in X with maximum multiplicity, then the induced splitting of Y has maximum multiplicity along $f(X_1)$.*

Main Reduction: Hence the LMP-conjecture will hold for a G/P if we could show that q equals $(p-1)dim\, G/P$, equivalently that $ord_{P/B}\sigma$ equals $dim\, G/P$ $(= codim_{G/B} P/B)$.

DEFINITION 3.2. A fundamental weight ω is called *minuscule* if $\langle \omega, \beta \rangle$ $(= \frac{2(\omega,\beta)}{(\beta,\beta)}) \leq 1$ for all $\beta \in R^+$; the maximal parabolic subgroup associated to ω is called a *minuscule parabolic subgroup.*

In the following sections, we prove the above equality for a minuscule G/P. This is in fact the line of proof for the Grassmannian in [**25**], and for the symplectic and orthogonal Grassmannians in [**22**]; as already mentioned, our methods (for computing the order of vanishing of sections) differ from those of [**25, 22**]. In the following

section, we describe the main steps involved in our approach; but the reader may want to recall the list of minuscule fundamental weights from Chapter 1, §4.5.

4. Steps leading to the determination of $ord_{P/B}\sigma$

We first describe explicit realizations for f^+, f^-, and then describe the main steps involved in computing the the order of vanishing of σ along P/B.

Explicit realizations for f^+, f^-: We shall denote a maximal parabolic subgroup corresponding to omitting a simple root α_i by P_i; also, we follow the indexing of simple roots, fundamental weights etc., as in [**5**]. Let $\omega_1, \cdots, \omega_l$ be the fundamental weights (l being the rank of G). For $1 \leq d \leq l$, let $V(\omega_d)$ be the Weyl module with highest weight ω_d. One knows (see [**16**] for instance) that the multiplicity of ω_d (in $V(\omega_d)$) is 1. Then $w(\omega_d), w \in W$ give all the *extremal weights* in $V(\omega_d)$, and these weights again have multiplicities equal to 1; of course, it suffices to run w over a set of representatives of the elements of W/W_{P_d}. Given $w \in W$, let us fix representatives $w^{(d)}$, $1 \leq d \leq l$ for wW_{P_d}, $1 \leq d \leq l$, and let us fix a highest weight vector and denote it by $q_{e^{(d)}}$; denote $w^{(d)}q_{e^{(d)}}$ by $q_{w^{(d)}}$. Note in particular that $q_{w_0^{(d)}}$ is a lowest weight vector, w_0 being the element of largest length in W. Recall the following well-known fact (see [**16**])

$$H^0(G/P_d, L(\omega_d)) \cong V(\omega_d)^*$$

where $V(\omega_d)^*$ is the linear dual of $V(\omega_d)$. In particular, $H^0(G/P_d, L(\omega_d))$ may be identified with the Weyl module $V(i(\omega_d)), i$ being the Weyl involution (equal to $-w_0$, as an element of $\operatorname{Aut} R$), and thus the extremal weights in $H^0(G/P_d, L(\omega_d))$ are given by $-w^{(d)}(\omega_d)$. We may choose extremal weight vectors $p_{w^{(d)}}$ in $H^0(G/P_d, L(\omega_d))$, of weight $-w^{(d)}(\omega_d)$, in such a way that under the canonical G-invariant bilinear form $(,)$ on $H^0(G/P_d, L(\omega_d)) \times V(\omega_d)$, we have,

$$(p_{\theta^{(d)}}, q_{\tau^{(d)}}) = \delta_{\theta^{(d)},\tau^{(d)}}$$

As a consequence, we have, for $\tau \in W$,

$$(*) \qquad p_{\theta^{(d)}}|_{X(\tau)} \neq 0 \;\Leftrightarrow\; \theta^{(d)} \in X(\tau^{(d)})$$

Now ρ being $\omega_1 + \cdots + \omega_l$, we may take f^+ (resp. f^-) to be the image of $f_1^+ \otimes \cdots \otimes f_l^+$ (resp. $f_1^- \otimes \cdots \otimes f_l^-$) under the canonical map

$$H^0(G/B, L(\omega_1)) \otimes \cdots \otimes H^0(G/B, L(\omega_l)) \to H^0(G/B, L(\rho))$$

given by multiplication of sections. Hence we may choose

$$f^+ = \prod_{1 \leq d \leq l} p_{w_0^{(d)}}, \; f^- = \prod_{1 \leq d \leq l} p_{e^{(d)}}.$$

Thus, σ may be taken to be

$$\sigma = (\prod_{1 \leq d \leq l} p_{w_0^{(d)}})(\prod_{1 \leq d \leq l} p_{e^{(d)}}).$$

Now eB belongs to every Schubert variety, and hence in view of $(*)$, $p_{e^{(d)}}|_X \neq 0, 1 \leq d \leq l$, for any Schubert variety X. In particular,

$$p_{e^{(d)}}|_{P/B} \neq 0, 1 \leq d \leq l.$$

Hence we obtain

$$ord_{P/B}\,\sigma = ord_{P/B}(\prod_{1 \leq d \leq l} p_{w_0^{(d)}}) = \sum_{1 \leq d \leq l} ord_{P/B}\, p_{w_0^{(d)}}.$$

Thus we are reduced to computing $ord_{P/B}\, p_{w_0^{(d)}}$.

Computation of $ord_{P/B}\, p_{w_0^{(d)}}$: Let us denote the element of largest length in W_P by τ_P or just τ (P having been fixed). Since the B-orbit through τ (we are denoting e_τ by just τ) is dense open in P/B, we have

$$ord_{P/B}\, p_{w_0^{(d)}} = ord_\tau\, p_{w_0^{(d)}}$$

where the right hand side denotes the order of vanishing of $p_{w_0^{(d)}}$ at the point e_τ. Hence

$$ord_{P/B}\,\sigma = \sum_{1\le d\le l} ord_\tau\, p_{w_0^{(d)}}.$$

Thus, our problem is reduced to computing $ord_\tau\, p_{w_0^{(d)}}$; to compute this, we may as well work in G/P_d. We shall continue to denote the point $\tau\, P_d$ (in G/P_d) by just τ.

The affine space $\tau B^-\tau^{-1}\cdot\tau\, P_d$ (B^- being the Borel subgroup opposite to B) is open in G/P_d, and gives a canonical affine neighborhood for the point $\tau(=\tau\, P_d)$; further,the point $\tau\, P_d$ is identified with the origin. The affine coordinates in $\tau B^-\tau^{-1}\cdot\tau\, P_d$ may be indexed as $\{x_\gamma, \gamma\in\tau(R^-\setminus R^-_{P_d})\}$ (here, $R^-_{P_d}$ denotes the set of negative roots of P_d). We recall the following two well known facts:

Fact 1: For any $f\in H^0(G/P_d, L(\omega_d))$, the evaluations of $\frac{\partial f}{\partial x_\gamma}$ and $X_\gamma f$ at $\tau(=\tau P_d)$ coincide, X_γ being the element in the Chevalley basis of $Lie\, G$ (the Lie algebra of G).

Fact 2: For $f\in H^0(G/P_d, L(\omega_d))$, we have that $ord_\tau\, f$ is the degree of the leading form (i.e., form of smallest degree) in the local polynomial expression for f at τ.

In the sequel, for $f\in H^0(G/P_d, L(\omega_d))$, we shall denote the leading form in the polynomial expression for f at τ by $LF(f)$.

4.1. Computation of $LF(p_{w_0^{(d)}})$: Toward this computation, we first look for γ_i's in $\tau(R^-\setminus R^-_{P_d})$ such that $X_{\gamma_1}^{n_1}\cdots X_{\gamma_r}^{n_r}p_{w_0^{(d)}}$, the n_i's being positive integers ≥ 1, is a non-zero multiple of p_τ (note that any monomial in the local expression for $p_{w_0^{(d)}}$ arises from such a collection of γ_i's and n_i's, in view of Fact 1; also note that a γ_i could repeat itself one or more times in $X_{\gamma_1}^{n_1}\cdots X_{\gamma_r}^{n_r}p_{w_0^{(d)}}$). Consider such an equality:

$$X_{\gamma_1}^{n_1}\cdots X_{\gamma_r}^{n_r}p_{w_0^{(d)}} = cp_\tau, c\in k^*.$$

Weight considerations imply

$$\sum_{1\le j\le r} n_j\gamma_j + i(\omega_d) = -\tau(\omega_d),$$

i being the Weyl involution. Writing $\gamma_j = -\tau(\beta_j)$, for a unique $\beta_j \in R^+ \setminus R^+(P_d)$, we obtain

$$\tau(\omega_d) + i(\omega_d) = \sum_{1\leq j\leq r} n_j \tau(\beta_j),$$

i.e.,

$$\omega_d + \tau(i(\omega_d)) = \sum_{1\leq j\leq r} n_j \beta_j$$

(note that $\tau = \tau^{-1}$).

Also, using the facts that $p_\tau = c\tau p_{e^{(d)}}$ (for some non-zero scalar c), and $X_{\gamma_i} = \tau X_{\beta_i}\tau^{-1}$, we obtain that $X_{\gamma_1}^{n_1}\cdots X_{\gamma_r}^{n_r} p_{w_0^{(d)}}$ is a non-zero scalar multiple of p_τ if and only if $X_{\beta_1}^{n_1}\cdots X_{\beta_r}^{n_r} p_{\tau w_0^{(d)}}$ is a non-zero scalar multiple of $p_{e^{(d)}}$. Thus,

$$ord_\tau p_{w_0^{(d)}} = ord_e p_{\tau w_0^{(d)}}.$$

Now from the above discussion, we have that $ord_e p_{\tau w_0^{(d)}}$ equals $\min\{\sum_{1\leq j\leq r} n_j\}$ such that there exist $\{\beta_1, \cdots, \beta_r; n_1, \cdots, n_r,\ \beta_j \in R^+ \setminus R^+(P_d),\ n_j \geq 1\}$ with $X_{\beta_1}^{n_1}\cdots X_{\beta_r}^{n_r} p_{\tau w_0^{(d)}}$ being a non-zero scalar multiple of $p_{e^{(d)}}$.

Thus in the following sections, for each minuscule G/P, we carry out Steps 1 & 2 below. Also, in view of the results in [**25, 22**], we shall first carry out Steps 1 & 2 for the following minuscule G/P's:

I. G of Type $\mathbf{D}$, $P = P_1$

II. G of type $\mathbf{E}_6$, $P = P_1, P_6$

III. G of type $\mathbf{E}_7$, $P = P_7$.

Remark 4.2. We need not consider $Sp(2n)/P_1$ (which is minuscule), since it is isomorphic to $\mathbb{P}^{2n-1}$; further, as is easily seen, $\mathbb{P}^N \times \mathbb{P}^N$ has a splitting which compatibly splits the diagonal with maximum multiplicity (one may also deduce this from [**25**] by identifying $\mathbb{P}^N$ with the Grassmannian of 1-dimensional subspaces of

k^{N+1}). It should be remarked (as observed in [**25**]) that for $G = Sp(2n)$, σ (as above) does not have maximum multiplicity along P_1/B.

Step 1: For each $1 \leq d \leq l$, we find the expression $\sum_{1\leq j\leq l} c_j\alpha_j$, $c_j \in \mathbb{Z}^+$ for $(\omega_d) + \tau(i(\omega_d))$ as a non-negative integral linear combination of simple roots; in fact, as will be seen, we have that $c_j \neq 0, \forall j$.

Step 2: We show that $\min\{\sum_{1\leq j\leq r} n_j\}$ (with notation as above) is given as follows:

$$\mathbf{D}_n,\ \mathbf{E}_6: \qquad \min\{\sum_{1\leq j\leq r} n_j\} = c_1$$

$$\mathbf{E}_7: \qquad \min\{\sum_{1\leq j\leq r} n_j\} = c_7$$

Toward proving this, we observe that in $\mathbf{D}_n$, $\mathbf{E}_6$, coefficient of α_1 in any positive root is less than or equal to one, while in $\mathbf{E}_7$, the coefficient of α_7 in any positive root is less than or equal to one (see [**5**]). Hence for any collection $\{\beta_1, \cdots, \beta_r; n_1, \cdots, n_r,\ n_j \geq 1\}$ as above, we have

$$\sum_{1\leq j\leq r} n_j \geq \begin{cases} c_1, & \text{if type } \mathbf{D}_n \text{ or } \mathbf{E}_6 \\ c_7, & \text{if type } \mathbf{E}_7 \end{cases}$$

We then exhibit a collection $\{\beta_1, \cdots, \beta_r; n_j = 1, 1 \leq j \leq r\}$, $\beta_j \in R^+ \setminus R^+(P_d), 1 \leq j \leq r$ such that

(a) $\omega_d + \tau(i(\omega_d)) = \sum_{1\leq j\leq r} \beta_j (= \sum_{1\leq j\leq m_d} n_j\beta_j)$

(b) The reflections s_{β_j}'s (and hence the Chevalley basis elements $X_{-\beta_j}$'s) mutually commute.

(c) For any subset $\{\delta_1, \cdots, \delta_s\}$ of $\{\beta_1, ..., \beta_r\}$, $X_{\delta_1} \cdots X_{\delta_s} p_{\tau w_0^{(d)}}$ is an extremal weight vector (in $H^0(G/P_d, L(\omega_d))$), and $X_{\beta_1}^{n_1} \cdots X_{\beta_r}^{n_r} p_{\tau w_0^{(d)}}$ is a lowest weight vector (i.e., a non-zero scalar multiple of $p_{e^{(d)}}$).

(d)

$$\sum_{1\le j\le r} n_j = \begin{cases} c_1, & \text{if type } \mathbf{D}_n \text{ or } \mathbf{E}_6 \\ c_7, & \text{if type } \mathbf{E}_7 \end{cases}$$

(e) We then conclude (by the foregoing discussion) that

$$ord_\tau\, p_{w_0^{(d)}} = \begin{cases} c_1, & \text{if type } \mathbf{D}_n \text{ or } \mathbf{E}_6 \\ c_7, & \text{if type } \mathbf{E}_7 \end{cases}$$

Remark 4.3. Thus we obtain a nice realization for $ord_\tau\, p_{w_0^{(d)}}$ (the order of vanishing along P/B of $p_{w_0^{(d)}}$) being the length of the shortest path through extremal weights in the weight lattice connecting the highest weight (namely, $i(\omega_d)$ in $H^0(G/B, L(\omega_d))$ and the extremal weight $-\tau(\omega_d)$.

Remark 4.4. For the sake of completeness, we have given the details for the remaining G/P's in §8.

For the convenience of notation, we make the following definition.

Definition 4.5. Define m_d to be $ord_e\, p_{\tau w_0^{(d)}} (= ord_\tau\, p_{w_0^{(d)}})$.

We shall treat the cases I, II, III above, respectively in the following three sections. In the following sections, we will repeatedly use the following fact:

Fact 3: Suppose p_θ is an extremal weight vector in $H^0(G/P_d, L(\omega_d))$ of weight $\chi(= -\theta(\omega_d))$, and $\beta \in R$ such that $(\chi, \beta^*) = r$, for some positive integer r. Then $X^r_{-\beta} p_\theta$ is a non-zero scalar multiple of the extremal weight vector $p_{s_\beta \theta}$ (here, $(,)$ is a W-invariant scalar product on the weight lattice, and $(\chi, \beta^*) = \frac{2(\chi,\beta)}{(\beta,\beta)}$).

The above fact follows from $sl(2)$-theory (note that p_θ is a highest weight vector for the Borel subgroup $\theta B^- \theta^{-1}$, B^- being the Borel subgroup opposite to B).

5. The minuscule $SO(2n)/P_1$

Let the characteristic of k be different from 2. Let $V = k^{2n}$ together with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Taking the matrix of the form $(\cdot,\cdot)$ (with respect to the standard basis $\{e_1,\ldots,e_{2n}\}$ of V) to be E, the anti-diagonal $(1,\ldots,1)$ of size $2n\times 2n$. We may realize $G = SO(V)$ as the fixed point set $SL(V)^\sigma$, where $\sigma : SL(V)\to SL(V)$ is given by $\sigma(A) = E({}^tA)^{-1}E$. Set $H = SL(V)$.

Denoting by T_H (resp. B_H) the maximal torus in H consisting of diagonal matrices (resp. the Borel subgroup in H consisting of upper triangular matrices) we see easily that T_H, B_H are stable under σ. We set $T_G = {T_H}^\sigma,\ B_G = {B_H}^\sigma$. Then it follows that T_G is a maximal torus in G and B_G is a Borel subgroup in G. We have a natural identification of the Weyl group W of G as a subgroup of S_{2n} :

$$W = \{(a_1\cdots a_{2n})\in S_{2n}\mid a_i = 2n+1-a_{2n+1-i},\ 1\le i\le 2n,\ \textit{and } m_w \textit{ is even}\}$$

where $m_w = \#\{i\le n\,|\,a_i > n\}$. Thus $w = (a_1\cdots a_{2n})\in W_G$ is known once $(a_1\cdots a_n)$ is known. In the sequel, we shall denote such a w by just $(a_1\cdots a_n)$; also, for $1\le i\le 2n$, we shall denote $2n+1-i$ by i'. For details see [**21**].

Let $P = P_{\alpha_1}$. We preserve the notation from the previous section; in particular, we denote the element of largest length in W_P by τ. We have

$$\tau = \begin{cases} (12'3'\cdots(n-1)'n), & \text{if } n \text{ is even}\\ (12'3'\cdots(n-1)'n'), & \text{if } n \text{ is odd}\end{cases}$$

Steps 1 & 2 of §4.1: As in [**5**], we shall denote by $\epsilon_j, 1\le j\le n$, the restriction to T_G of the character of T_H, sending a diagonal matrix $diag\{t_1,\cdots,t_n\}$ to t_j.

Case 1: Let $d\le n-2$. Then $\omega_d = \epsilon_1+\cdots+\epsilon_d$ (cf. [**5**]). We have, $i(\omega_d) = \omega_d$, and $\tau(\omega_d) = \epsilon_1+\epsilon_{2'}+\cdots+\epsilon_{d'}$. Hence,

$$\omega_d + \tau(i(\omega_d)) = 2\epsilon_1 = 2(\alpha_1+\cdots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n$$

(note that an element in T_G is of the form $diag\{t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}\}$, and hence $\epsilon_{j'} = -\epsilon_j$; also, we follow [**5**] for denoting the simple roots). We let $\{\beta_1, \beta_2\} \subset R^+ \setminus R^+_{P_d}$ be any (unordered) pair of the form $\{\epsilon_1 - \epsilon_j, \epsilon_1 + \epsilon_j, d+1 \leq j \leq n\}$. Clearly, s_{β_1}, s_{β_2} commute (since, $(\beta_1, \beta_2^*) = 0$), and $\omega_d + \tau(i(\omega_d)) = \beta_1 + \beta_2$. Also,

$$(-\tau w_0^{(d)}(\omega_d), \beta_j^*) = (\tau(\omega_d), \beta_j^*) = 1, j = 1, 2;$$

$$(-\tau w_0^{(d)}(\omega_d) - \beta_j, \beta_m^*) = (\tau(\omega_d) - \beta_j, \beta_m^*) = 1, \ j, m \in \{1, 2\}, \text{ and } j, m \text{ distinct};$$

$$-\tau w_0^{(d)}(\omega_d) - \beta_1 - \beta_2 = \tau(\omega_d) - \beta_1 - \beta_2 = -\omega_d = -(\epsilon_1 + \cdots + \epsilon_d).$$

From this, (a)-(c) in Step 2 of §4.1 follow for the above choice of $\{\beta_1, \beta_2\}$; (d) in Step 2 is obvious. Hence $m_d = 2, 1 \leq d \leq n-2$ (recall m_d from Definition 4.5).

Case 2: $d = n-1$. We have, $i(\omega_{n-1})$ equals ω_{n-1} or ω_n, according as n is even or odd.

If n is even, then $\tau(i(\omega_{n-1})) = \tau(\omega_{n-1}) = \frac{1}{2}(\epsilon_1 + \epsilon_{2'} + \cdots + \epsilon_{(n-1)'} - \epsilon_n)$.

If n is odd, then $\tau(i(\omega_{n-1})) = \tau(\omega_n) = \frac{1}{2}(\epsilon_1 + \epsilon_{2'} + \cdots + \epsilon_{(n-1)'} + \epsilon_{n'})$

$= \frac{1}{2}(\epsilon_1 + \epsilon_{2'} + \cdots + \epsilon_{(n-1)'} - \epsilon_n)$.

Thus in either case, $\tau(i(\omega_{n-1})) = \frac{1}{2}(\epsilon_1 + \epsilon_{2'} + \cdots + \epsilon_{(n-1)'} - \epsilon_n)$. Hence

$$\omega_{n-1} + \tau(i(\omega_{n-1})) = \epsilon_1 - \epsilon_n = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1}$$

which is clearly a root in $R^+ \setminus R^+_{P_{n-1}}$. Hence taking β_1 to be $\epsilon_1 - \epsilon_n$, we find that $\{\beta_1\}$ (trivially) satisfies (a)-(c) in Step 2 of §4.1 follow; (d) in Step 2 is obvious. Hence $m_{n-1} = 1$.

Case 3: $d = n$. Proceeding as in Case 2, we have,

$$\omega_n + \tau(i(\omega_n)) = \epsilon_1 + \epsilon_n = \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n$$

which is clearly a root in $R^+ \setminus R^+_{P_{n-1}}$. As in case 2, we conclude that $m_n = 1$.

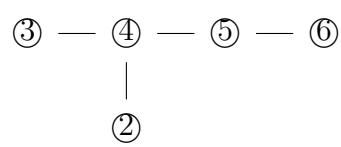
THEOREM 5.1. *The LMP conjecture holds for the minuscule G/P, G, P being as above.*

PROOF. From §3 (see "Main reduction" in that section), we just need to show that $\sum_{1\leq d\leq n} ord_\tau p_{w_0^{(d)}}$ equals $\mathrm{codim}_{G/B} P/B$. From the above computations, and the discussion in §4, we have $\sum_{1\leq d\leq n} ord_\tau p_{w_0^{(d)}}$ equals $\sum_{1\leq d\leq n} m_d = 2n-2$ which is precisely $\mathrm{codim}_{G/B} P/B$. □

6. Exceptional group $\mathbf{E}_6$

Let G be simple of type E_6. Let $P = P_1$. As in the previous sections, let τ be the unique element of largest length in W_P.

Step 1 & 2 of §4.1: Note that τ is the unique element of largest length inside the Weyl group of type D_5; we have that D_5 sits inside of E_6 as



Thus, for $2 \leq j \leq 6$, we have $\tau(\alpha_j) = -i(\alpha_j)$, i being the Weyl involution of D_5; in this case, we have

$$(*) \qquad \tau(\alpha_2) = -\alpha_3, \tau(\alpha_3) = -\alpha_2, \tau(\alpha_j) = -\alpha_j, j = 4, 5, 6.$$

Thus, using the Tables in [**5**] , to find $\tau(i(\omega_d)), 1 \leq d \leq 6$, as a linear sum (with rational coefficients) of the simple roots, it remains to find $\tau(\alpha_1)$.

Let $\tau(\alpha_1) = \sum_{1\leq j\leq 6} a_j\alpha_j, a_j \in \mathbb{Z}$. Since $\alpha_1 \notin R_P$ (the root system of P), we have that $\tau(\alpha_1) \notin R_P$. Hence $a_1 \neq 0$; further, $\tau(\alpha_1) \in R^+$ (since, clearly, $l(\tau s_{\alpha_1}) = l(\tau) + 1$). Hence, $a_1 > 0$; in fact, we have, $a_1 = 1$ (since any positive root in the root system of E_6 has an α_1 coefficient ≤ 1). Using $(*)$ above, and the following linear system, we

determine the remaining a_j's:

$$\begin{aligned}
2a_2 - a_4 &= \langle \tau(\alpha_1), \alpha_2^* \rangle = \langle \alpha_1, \tau(\alpha_2^*) \rangle = \langle \alpha_1, -\alpha_3^* \rangle = 1 \\
2a_3 - a_1 - a_4 &= \langle \tau(\alpha_1), \alpha_3^* \rangle = \langle \alpha_1, \tau(\alpha_3^*) \rangle = \langle \alpha_1, -\alpha_2^* \rangle = 0 \\
2a_4 - a_2 - a_3 - a_5 &= \langle \tau(\alpha_1), \alpha_4^* \rangle = \langle \alpha_1, \tau(\alpha_4^*) \rangle = \langle \alpha_1, -\alpha_4^* \rangle = 0 \\
2a_5 - a_4 - a_6 &= \langle \tau(\alpha_1), \alpha_5^* \rangle = \langle \alpha_1, \tau(\alpha_5^*) \rangle = \langle \alpha_1, -\alpha_5^* \rangle = 0 \\
2a_6 - a_5 &= \langle \tau(\alpha_1), \alpha_6^* \rangle = \langle \alpha_1, \tau(\alpha_6^*) \rangle = \langle \alpha_1, -\alpha_6^* \rangle = 0
\end{aligned}$$

Either one may just solve the above linear system or use the properties of the root system of type $\mathbf{E}_6$ to quickly solve for a_j's. For instance, we have, $a_6 \neq 0$; for, $a_6 = 0$ would imply (working with the last equation and up) that $a_4 = 0$ which in turn would imply (in view of the first equation) that $a_2 = \frac{1}{2}$, not possible. Hence $a_6 \neq 0$, and in fact equals 1 (for the same reasons as in concluding that $a_1 = 1$). Once again working backward in the linear system, $a_5 = 2, a_4 = 3$; hence from the first equation, we obtain, $a_2 = 2$. Now the second equation implies that $a_3 = 2$. Thus we obtain

$$\tau(\alpha_1) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \end{pmatrix}.$$

For $d \in \{1, \ldots, 6\}$, we shall now describe $\{\beta_1, \cdots, \beta_r, \mid \beta_i \in R^+ \setminus R^+_{P_d}\}$ which satisfies the conditions (a)-(d) in Step 2 of §4.1.

For convenience, we list the fundamental weights here:

$$\omega_1 = \frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 \\ & & 3 & & \end{pmatrix}, \ \omega_2 = \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \end{pmatrix}, \ \omega_3 = \frac{1}{3}\begin{pmatrix} 5 & 10 & 12 & 8 & 4 \\ & & 6 & & \end{pmatrix}$$

$$\omega_4 = \begin{pmatrix} 2 & 4 & 6 & 4 & 2 \\ & & 3 & & \end{pmatrix}, \ \omega_5 = \frac{1}{3}\begin{pmatrix} 4 & 8 & 12 & 10 & 5 \\ & & 6 & & \end{pmatrix}, \ \omega_6 = \frac{1}{3}\begin{pmatrix} 2 & 4 & 6 & 5 & 4 \\ & & 3 & & \end{pmatrix}$$

Case 1: $d = 1$.

We have, $i(\omega_1) = \omega_6$. Hence using (from above), the expression for ω_6 as a (rational) sum of simple roots, and the expressions for $\tau(\alpha_j), j = 1, \cdots, 6$, we obtain

$$\tau(i(\omega_1)) + \omega_1 = \begin{pmatrix} 2 & 2 & 2 & 1 & 0 \\ & & 1 & & \end{pmatrix}.$$

We let $\{\beta_1, \beta_2\}$ be the unordered pair of roots:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ & & 0 & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{pmatrix}.$$

Clearly β_1, β_2 are in $R^+ \setminus R^+_{P_1}$, and the reflections s_{β_1}, s_{β_2} commute (since $\beta_1 + \beta_2$ is not a root). Further, $\tau(i(\omega_1)) + \omega_1 = \beta_1 + \beta_2$. Also,

$$\left\langle -\tau w_0^{(1)}(\omega_1), \beta_j^* \right\rangle = \left\langle \tau(i(\omega_1)), \beta_j^* \right\rangle = \left\langle \beta_1 + \beta_2 - \omega_1, \beta_j^* \right\rangle = 1, \ j = 1, 2;$$

$$\left\langle -\tau w_0^{(1)}(\omega_1) - \beta_l, \beta_j^* \right\rangle = \left\langle \beta_j - \omega_1, \beta_j^* \right\rangle = 1, \ j, l \in \{1, 2\}, \text{ and } j, l \text{ distinct};$$

$$-\tau w_0^{(1)}(\omega_1) - \beta_1 - \beta_2 = -\omega_1.$$

From this, (a)-(c) in Step 2 of §4.1 follow for the above choice of $\{\beta_1, \beta_2\}$; (d) in Step 2 is obvious. Hence $m_1 = 2$ (cf. Definition 4.5).

Case 2: $d = 2$.

We have $i(\omega_2) = \omega_2$. As in case 1, using the expression for ω_2 as a (rational) sum of simple roots, and the expressions for $\tau(\alpha_j), j = 1, \cdots, 6$, we obtain

$$\tau(\omega_2) + \omega_2 = \begin{pmatrix} 2 & 2 & 3 & 2 & 1 \\ & & 2 & & \end{pmatrix}.$$

We let $\{\beta_1, \beta_2\}$ be the unordered pair of roots:

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ & & 1 & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ & & 1 & & \end{pmatrix}.$$

Clearly β_1, β_2 are in $R^+ \setminus R^+_{P_2}$, and the reflections s_{β_1}, s_{β_2} commute (since $\beta_1 + \beta_2$ is not a root). Further, $\tau(\omega_2) + \omega_2 = \beta_1 + \beta_2$. We have

$$\left\langle -\tau w_0^{(2)}(\omega_2), \beta_j^* \right\rangle = \left\langle \tau(\omega_2), \beta_j^* \right\rangle = \left\langle \beta_1 + \beta_2 - \omega_2, \beta_j^* \right\rangle = 1, \ j = 1, 2;$$

$$\left\langle -\tau w_0^{(2)}(\omega_2) - \beta_l, \beta_j^* \right\rangle = \left\langle \beta_j - \omega_2, \beta_j^* \right\rangle = 1, \ j, l \in \{1, 2\}, \text{ and } j, l \text{ distinct};$$

$$-\tau w_0^{(2)}(\omega_2) - \beta_1 - \beta_2 = -\omega_2.$$

As in case 1, (a)-(c) in Step 2 of §4.1 follow for the above choice of $\{\beta_1, \beta_2\}$; (d) in Step 2 is also clear. Hence $m_2 = 2$.

The discussion in the remaining cases are similar; in each case we will just give the expression for $\tau(i(\omega_d)) + \omega_d$ as an element in the root lattice, and the choice of $\{\beta_1, \cdots, \beta_r\}$ in $R^+ \setminus R^+_{P_d}$ which satisfy the conditions (a)-(d) in Step 2 of §4.1. Then deduce the value of m_d.

Case 3: $d = 3$.

We have $i(\omega_3) = \omega_5$. Further,

$$\tau(\omega_5) + \omega_3 = \begin{pmatrix} 3 & 4 & 4 & 2 & 1 \\ & & 2 & & \end{pmatrix}.$$

We let $\{\beta_1, \beta_2, \beta_3\}$ be the unordered triple of roots:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & & 0 & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 1 & 0 \\ & & 1 & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ & & 1 & & \end{pmatrix}.$$

Then we have $\tau(\omega_3) + \omega_3 = \beta_1 + \beta_2 + \beta_3$. Reasoning as in case 1, we conclude $m_3 = 3$.

Case 4: $d = 4$.

We have $i(\omega_4) = \omega_4$. Further,

$$\tau(\omega_4) + \omega_4 = \begin{pmatrix} 4 & 5 & 6 & 4 & 2 \\ & & 3 & & \end{pmatrix}.$$

We let $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ be the unordered quadruple of roots:

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 1 \\ & & 1 & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ & & 1 & & \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & & 1 & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ & & 0 & & \end{pmatrix}.$$

Then we have $\tau(\omega_4) + \omega_4 = \beta_1 + \beta_2 + \beta_3 + \beta_4$. Reasoning as in case 1, we conclude $m_4 = 4$.

Case 5: $d = 5$.

We have $i(\omega_5) = \omega_3$. Further

$$\tau(\omega_3) + \omega_5 = \begin{pmatrix} 3 & 4 & 5 & 4 & 2 \\ & & 2 & & \end{pmatrix}.$$

We let $\{\beta_1, \beta_2, \beta_3\}$ be the unordered triple of roots:

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 1 & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ & & 0 & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & & 1 & & \end{pmatrix}.$$

We have, $\tau(\omega_3) + \omega_3 = \beta_1 + \beta_2 + \beta_3$. We proceed as in case 1, and conclude $m_5 = 3$.

Case 6: $d = 6$.

We have, $i(\omega_6) = \omega_1$. Further, $\tau(\omega_1) + \omega_6 = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 \\ & & 2 & & \end{pmatrix}$. We let $\{\beta_1, \beta_2\}$ be the unordered pair of roots:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & & 0 & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \end{pmatrix}.$$

We have $\tau(\omega_1) + \omega_6 = \beta_1 + \beta_2$. Proceeding as in case 1, we conclude $m_6 = 2$.

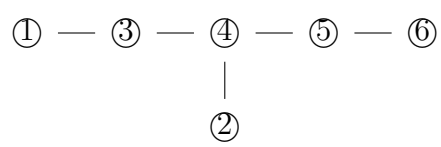
THEOREM 6.1. *The LMP conjecture holds for the minuscule G/P, G, P being as above.*

PROOF. Proceeding as in the proof of Theorem 5.1, we just need to show that $\sum_{1\leq d\leq 6} ord_\tau p_{w_0^{(d)}}$ equals $\mathrm{codim}_{G/B} P/B$. From the above computations, we have $\sum_{1\leq d\leq 6} ord_\tau p_{w_0^{(d)}}$ equals 16 which is precisely $\mathrm{codim}_{G/B} P/B$. □

7. Exceptional group $\mathbf{E}_7$

Let G be simple of type E_7. Let P be the maximal parabolic subgroup associated to the fundamental weight ω_7 (the only minuscule weight in E_7). We preserve the notation of the previous sections; in particular, τ will denote the unique element of largest length in W_P.

Step 1 & 2 of §4.1: Note that τ is the unique element of largest length inside the Weyl group of type $\mathbf{E}_6$; $\mathbf{E}_6$ sits inside $\mathbf{E}_7$ in the natural way:



Thus, for α_i, $1 \leq i \leq 6$, we have, $\tau(\alpha_i) = -i(\alpha_i)$, where i is the Weyl involution on E_6. To be very precise, we have,

$$(*) \qquad \tau(\alpha_1) = -\alpha_6,\ \tau(\alpha_3) = -\alpha_5,\ \tau(\alpha_i) = -\alpha_i, i = 2, 4.$$

Thus, using [**5**] Tables, to find $\tau(i(\omega_d)), 1 \leq d \leq 7$, as a linear sum (with rational coefficients) of the simple roots, it remains to find $\tau(\alpha_7)$. Toward computing $\tau(\alpha_7)$, we proceed as in §6. Let $\tau(\alpha_7) = \sum_{i=1}^{7} a_i\alpha_i$, where $a_i \in \mathbb{Z}$. Since $\alpha_7 \notin R_P$ (the root system of P), we have that $\tau(\alpha_7) \notin R_P$. Hence $a_7 \neq 0$; further, $\tau(\alpha_7) \in R^+$ (since, clearly, $l(\tau s_{\alpha_7}) = l(\tau)+1$). Hence, $a_7 > 0$; in fact, we have, $a_7 = 1$ (since any positive root in the root system of E_7 has an α_7 coefficient ≤ 1). Using $(*)$ above, and the

following linear system, we determine the remaining a_j's:

$$\begin{aligned}
2a_1 - a_3 &= \langle \tau(\alpha_7), \alpha_1^* \rangle = \langle \alpha_7, \tau(\alpha_1^*) \rangle = \langle \alpha_7, -\alpha_6^* \rangle = 1 \\
2a_2 - a_4 &= \langle \tau(\alpha_7), \alpha_2^* \rangle = \langle \alpha_7, \tau(\alpha_2^*) \rangle = \langle \alpha_7, -\alpha_2^* \rangle = 0 \\
2a_3 - a_4 - a_1 &= \langle \tau(\alpha_7), \alpha_3^* \rangle = \langle \alpha_7, \tau(\alpha_3^*) \rangle = \langle \alpha_7, -\alpha_5^* \rangle = 0 \\
2a_4 - a_2 - a_3 - a_5 &= \langle \tau(\alpha_7), \alpha_4^* \rangle = \langle \alpha_7, \tau(\alpha_4^*) \rangle = \langle \alpha_7, -\alpha_4^* \rangle = 0 \\
2a_5 - a_4 - a_6 &= \langle \tau(\alpha_7), \alpha_5^* \rangle = \langle \alpha_7, \tau(\alpha_5^*) \rangle = \langle \alpha_7, -\alpha_3^* \rangle = 0 \\
2a_6 - a_5 - a_7 &= \langle \tau(\alpha_7), \alpha_6^* \rangle = \langle \alpha_7, \tau(\alpha_6^*) \rangle = \langle \alpha_7, -\alpha_1^* \rangle = 0
\end{aligned}$$

The fact that $a_7 = 1$ together with the last equation implies $a_5 \neq 0$ (and hence $a_6 \neq 0$, again from the last equation; note that all $a_i \in \mathbb{Z}^+$). Similarly, from the first equation, we conclude $a_3 \neq 0$ (and hence $a_1 \neq 0$). From the first and third equations, we conclude $a_4 \neq 0$ (and hence $a_2 \neq 0$, in view of the second equation). Thus, all a_i's are non-zero. The fifth equation implies that a_4, a_6 are of the same parity, and are in fact both even (in view of the second equation); hence $a_6 = 2$. Now working with the last equation and up, we obtain

$$a_5 = 3, a_4 = 4, a_2 = 2, a_3 = 3, a_1 = 2.$$

Thus

$$\tau(\alpha_7) = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ & & 2 & & & \end{pmatrix}.$$

We proceed as in §6. Of course, the Weyl involution for $\mathbf{E}_7$ is just the identity map. For each maximal parabolic subgroup $P_d, 1 \leq d \leq 7$, we will give the expression for $\tau(i(\omega_d)) + \omega_d (= \tau(\omega_d) + \omega_d)$ as an element in the root lattice, and the choice of $\{\beta_1, \cdots, \beta_r\}$ in $R^+ \setminus R^+_{P_d}$ which satisfy the conditions (a)-(d) in Step 2 of §4.1. Then deduce the value of m_d.

For convenience, we list the fundamental weights here:

$$\omega_1 = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ & & 2 & & & \end{pmatrix},$$

$$\omega_2 = \frac{1}{2}\begin{pmatrix} 4 & 8 & 12 & 9 & 6 & 3 \\ & & 7 & & & \end{pmatrix}, \omega_3 = \begin{pmatrix} 3 & 6 & 8 & 6 & 4 & 2 \\ & & 4 & & & \end{pmatrix},$$

$$\omega_4 = \begin{pmatrix} 4 & 8 & 12 & 9 & 6 & 3 \\ & & 6 & & & \end{pmatrix}, \omega_5 = \frac{1}{2}\begin{pmatrix} 6 & 12 & 18 & 15 & 10 & 5 \\ & & 9 & & & \end{pmatrix},$$

$$\omega_6 = \begin{pmatrix} 2 & 4 & 6 & 5 & 4 & 2 \\ & & 3 & & & \end{pmatrix}, \omega_7 = \frac{1}{2}\begin{pmatrix} 2 & 4 & 6 & 5 & 4 & 3 \\ & & 3 & & & \end{pmatrix}.$$

Case 1: $d = 1$.

We have $\tau(\omega_1) + \omega_1 = \begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 2 \\ & & 2 & & & \end{pmatrix}$. We let $\{\beta_1, \beta_2\}$ be the unordered pair of roots:

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 1 & 1 \\ & & 2 & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & & 0 & & & \end{pmatrix}.$$

Then we have $\tau(\omega_1) + \omega_1 = \beta_1 + \beta_2$, and $m_1 = 2$.

Case 2: $d = 2$.

$Q = P_2$**.** We have $\tau(\omega_2) + \omega_2 = \begin{pmatrix} 2 & 4 & 6 & 5 & 4 & 3 \\ & & 3 & & & \end{pmatrix}$. We let $\{\beta_1, \beta_2, \beta_3\}$ be the unordered triple of roots:

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & 1 & & & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 1 \\ & & 1 & & & \end{pmatrix}.$$

Then we have $\tau(\omega_2) + \omega_2 = \beta_1 + \beta_2 + \beta_3$, and $m_2 = 3$.

Case 3: $d = 3$.

We have $\tau(\omega_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ & & 0 & & & \end{pmatrix}$; thus $\tau(\omega_3) + \omega_3 = \begin{pmatrix} 3 & 6 & 8 & 6 & 5 & 4 \\ & & 4 & & & \end{pmatrix}$.
We let $\{\beta_i, i = 1, .., 4\}$ be the unordered quadruple of roots:

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 1 \\ & & 1 & & & \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix}.$$

Then we have $\tau(\omega_3) + \omega_3 = \sum_{1 \le i \le 4} \beta_i$, and $m_3 = 4$.

Case 4: $d = 4$.

We have $\tau(\omega_4) + \omega_4 = \begin{pmatrix} 4 & 8 & 12 & 10 & 8 & 6 \\ & & 6 & & & \end{pmatrix}$. We let $\{\beta_i, i = 1, .., 6\}$ be the unordered 6-tuple of roots:

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 1 \\ & & 1 & & & \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix}.$$

Then we have $\tau(\omega_4) + \omega_4 = \sum_{1 \le i \le 6} \beta_i$, and $m_4 = 6$.

Case 5: $d = 5$.

We have $\tau(\omega_5) + \omega_5 = \begin{pmatrix} 3 & 6 & 10 & 9 & 7 & 5 \\ & & 5 & & & \end{pmatrix}$. We let $\{\beta_i, i = 1, .., 5\}$ be the unordered quintuple of roots:

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix}.$$

Then we have $\tau(\omega_5)+\omega_5 = \sum_{1\le i\le 5}\beta_i$, and $m_5 = 5$.

Case 6: $d = 6$.

We have $\tau(\omega_6)+\omega_6 = \begin{pmatrix} 2 & 5 & 8 & 7 & 6 & 4 \\ & & 4 & & & \end{pmatrix}$. We let $\{\beta_i, i = 1,..,4\}$ be the unordered quadruple of roots:

$$\begin{pmatrix} 1 & 2 & 3 & 2 & 2 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 1 \\ & & 1 & & & \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 2 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix}.$$

Then we have $\tau(\omega_6)+\omega_6 = \sum_{1\le i\le 4}\beta_i$, and $m_6 = 4$.

Case 7: $d = 7$.

We have $\tau(\omega_7)+\omega_7 = \begin{pmatrix} 2 & 4 & 6 & 5 & 4 & 3 \\ & & 3 & & & \end{pmatrix}$. We let $\{\beta_1, \beta_2, \beta_3\}$ be the unordered triple of roots:

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 2 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 1 \\ & & 1 & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 1 \\ & & 1 & & & \end{pmatrix}.$$

Then we have $\tau(\omega_7)+\omega_7 = \beta_1+\beta_2+\beta_3$, and $m_7 = 3$.

We have $\sum_{1\le d\le 7} m_d = 27$ which is equal to $\mathrm{codim}_{G/B} P/B$. Hence we obtain

THEOREM 7.1. *The LMP conjecture holds for the minuscule G/P, G, P being as above.*

8. The remaining minuscule G/P's

In this section, we give the details for the remaining G/P's along the same lines as in §5, §6, §7, thus providing an alternate proof for the results of [**25, 22**]. We fix a maximal parabolic subgroup P of G, denote (as in the previous sections), the element

of largest length in W_P by τ. Also, as in the previous sections (cf. Definition 4.5), we shall denote $ord_e\, p_{\tau w_0^{(d)}} (= ord_\tau\, p_{w_0^{(d)}})$ by $m_d, 1 \leq d \leq n$.

8.1. The simple root α. While computing m_d, as in §5, §6, §7, in each case, we work with a simple root α which occurs with a non-zero coefficient c_α in the expression for $\omega_d + \tau(i(\omega_d))$ (as a non-negative integral linear combination of simple roots) and which has the property that in the expression for any positive root (as a non-negative integral linear combination of simple roots), it occurs with a coefficient ≤ 1. This α will depend on the type of G, and we shall specify it in each case. Then as seen in §4.1, $m_d \geq c_\alpha$. We shall first exhibit a set of roots $\beta_1, \cdots, \beta_r, r = c_\alpha$ in $R^+ \setminus R^+_{P_d}$, satisfying (a)-(c) in Step 2 of §4.1, and then conclude that $m_d = c_\alpha$. It will turn out (as shown below) that in all cases, $\sum\limits_{1 \leq d \leq n} m_d$ equals $\operatorname{codim}_{G/B} P/B$, thus proving the LMP conjecture (and hence Wahl's conjecture).

8.2. Grassmannian. Let $G = SL(n)$. In this case, every maximal parabolic subgroup is minuscule. Let us fix a maximal parabolic subgroup $P := P_c$; we may suppose that $c \leq n - c$ (in view of the natural isomorphism $G/P_c \cong G/P_{n-c}$). Identifying the Weyl group with the symmetric group S_n, we have

$$\tau = (c\, c-1 \cdots 1\, n\, n-1 \cdots c+1).$$

Let $\epsilon_j, 1 \leq j \leq n$ be the character of T (the maximal torus consisting of diagonal matrices in G), sending a diagonal matrix to its j-th diagonal entry. Note that $\sum\limits_{1 \leq j \leq n} \epsilon_j = 0$ (writing the elements of the character group additively, as is customary); this fact will be repeatedly used in the discussion below. Also, for $1 \leq d \leq n-1$, we have $i(\omega_d) = \omega_{n-d}$. We observe that in the expression for a positive root (as a non-negative integral linear combination of simple roots), any simple root occurs with a coefficient ≤ 1. For each P_d, we shall take α to be α_d.

Case 1: Let $d < c$. We have (cf.[**5**])

$$\begin{aligned}
\omega_d + \tau(i(\omega_d)) &= \omega_d + \tau(\omega_{n-d}) \\
&= (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_1 + \cdots + \epsilon_c + \epsilon_n + \cdots + \epsilon_{c+d+1}) \\
&= \epsilon_1 + \cdots + \epsilon_d - (\epsilon_{c+1} + \cdots + \epsilon_{c+d}) \\
&= (\epsilon_1 - \epsilon_{c+d}) + (\epsilon_2 - \epsilon_{c+d-1}) + \cdots + (\epsilon_d - \epsilon_{c+1})
\end{aligned}$$

(note that $d < c \leq n - c$, and hence $n - d > c$). From the last expression, it is clear that $c_\alpha = d$ (c_α being as in §8.1); note that every one of the roots in the last expression belongs to $R^+ \setminus R^+_{P_d}$. We now let $\beta_1, \cdots, \beta_d$ be the unordered d-tuple of roots:

$$\epsilon_1 - \epsilon_{c+d}, \epsilon_2 - \epsilon_{c+d-1}, \cdots, \epsilon_d - \epsilon_{c+1}.$$

Then it is easily checked that the above β_j's satisfy (a)-(c) in Step 2 of §4.1, and we have $m_d = d$ (in fact any such grouping will also work).

Case 2: Let $c \leq d \leq n - c$. We have

$$\begin{aligned}
\omega_d + \tau(i(\omega_d)) &= (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_1 + \cdots + \epsilon_c + \epsilon_n + \cdots + \epsilon_{c+d+1}) \\
&= \epsilon_1 + \cdots + \epsilon_c - (\epsilon_{d+1} + \cdots + \epsilon_{d+c}) \\
&= (\epsilon_1 - \epsilon_{d+c}) + (\epsilon_2 - \epsilon_{d+c-1}) + \cdots + (\epsilon_c - \epsilon_{d+1}).
\end{aligned}$$

Hence $c_\alpha = c$. We now let $\beta_1, \cdots, \beta_c$ be the unordered c-tuple of roots:

$$\epsilon_1 - \epsilon_{d+c}, \epsilon_2 - \epsilon_{d+c-1}, \cdots, \epsilon_c - \epsilon_{d+1}.$$

Then it is easily checked that the above β_j's satisfy (a)-(c) in Step 2 of §4.1, and we have $m_d = c$.

Case 3: Let $n-c<d\leq n-1$. We have

$$\begin{aligned}\omega_d+\tau(i(\omega_d)) &= (\epsilon_1+\cdots+\epsilon_d)+(\epsilon_c+\epsilon_{c-1}+\cdots+\epsilon_{c+d+1-n})\\ &= -(\epsilon_{d+1}+\cdots+\epsilon_n)+(\epsilon_c+\cdots+\epsilon_{c+d+1-n})\\ &= (\epsilon_c-\epsilon_{d+1})+(\epsilon_{c-1}-\epsilon_{d+2})+\cdots+(\epsilon_{c+d+1-n}-\epsilon_n)\end{aligned}$$

(note that $c\leq n-c<d$). Hence $c_\alpha=n-d$. We now let $\beta_1,\cdots,\beta_{n-d}$ be the unordered $(n-d)$-tuple of roots:

$$\epsilon_c-\epsilon_{d+1},\epsilon_{c-1}-\epsilon_{d+2},\cdots,\epsilon_{c+d+1-n}-\epsilon_n.$$

Then it is easily checked that the above choice of β_j's satisfy (a)-(c) in Step 2 of §4.1, and we have $m_d=n-d$.

From the above computations, we have,

$$\sum_{1\leq d\leq n-1} m_d=(1+\cdots+c-1)+(n-2c+1)c+(1+\cdots+c-1)=c(n-c)=codim_{G/B}P/B.$$

8.3. Lagrangian Grassmannian. Let $V=K^{2n}$ together with a nondegenerate, skew-symmetric bilinear form $(\cdot,\cdot)$. Let $H=SL(V)$ and $G=Sp(V)=\{A\in SL(V)\mid A$ leaves the form $(\cdot,\cdot)$ invariant $\}$. Taking the matrix of the form (with respect to the standard basis $\{e_1,...,e_{2n}\}$ of V) to be

$$E=\begin{pmatrix}0 & J\\ -J & 0\end{pmatrix}$$

where J is the anti-diagonal $(1,\ldots,1)$ of size $n\times n$, we may realize $Sp(V)$ as the fixed point set of a certain involution σ on $SL(V)$, namely $G=H^\sigma$, where $\sigma:H\longrightarrow H$ is given by $\sigma(A)=E({}^tA)^{-1}E^{-1}$. Denoting by T_H (resp. B_H) the maximal torus in H consisting of diagonal matrices (resp. the Borel subgroup in H consisting of upper triangular matrices) we see easily that T_H,B_H are stable under σ. We set $T_G=T_H^\sigma,B_G=B_H^\sigma$. Then it can be seen easily that T_G is a maximal torus in G and

B_G is a Borel subgroup in G. We have a natural identification of the Weyl group W of G as a subgroup of S_{2n} :

$$W = \{(a_1 \cdots a_{2n}) \in S_{2n} \mid a_i = 2n+1-a_{2n+1-i},\ 1 \leq i \leq 2n\}.$$

Thus $w = (a_1 \cdots a_{2n}) \in W_G$ is known once $(a_1 \cdots a_n)$ is known. For details see [**21**].

In the sequel, we shall denote such a w by just $(a_1 \cdots a_n)$; also, for $1 \leq i \leq 2n$, we shall denote $2n+1-i$ by i'. The Weyl involution is the identity map. In type $\mathbf{C}_n$, we have that in the expression for a positive root (as a non-negative integral linear combination of simple roots), the simple root α_n occurs with a coefficient ≤ 1. For all $P_d, 1 \leq d \leq n$, we shall take α (cf. §8.1) to be α_n.

Let $P = P_n$. Then $\tau = (n \cdots 1)$. Let $1 \leq d \leq n$. We have

$$\omega_d + \tau(\omega_d) = (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_n + \cdots + \epsilon_{n+1-d}).$$

If $d < n+1-d$, then each ϵ_j in the above sum are distinct. Hence writing

$$\omega_d + \tau(\omega_d) = (\epsilon_1 + \epsilon_n) + (\epsilon_2 + \epsilon_{n-1}) + \cdots + (\epsilon_d + \epsilon_{n+1-d})$$

we have that each root in the last sum belongs to $R^+ \setminus R^+_{P_d}$ (since each of the roots clearly involves α_d); further, each of the roots involves α_n with coefficient equal to 1. Hence $c_\alpha = d$. We let $\beta_1, \cdots, \beta_d$ be the unordered d-tuple of roots:

$$\epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_{n-1}, \cdots, \epsilon_d + \epsilon_{n+1-d}.$$

If $d \geq n+1-d$, then

$$\begin{aligned} \omega_d + \tau(\omega_d) &= (\epsilon_1 + \cdots + \epsilon_{n-d}) + (\epsilon_n + \cdots + \epsilon_{d+1}) + (2\epsilon_{n-d+1} + \cdots + 2\epsilon_d) \\ &= (\epsilon_1 + \epsilon_n) + (\epsilon_2 + \epsilon_{n-1}) + \cdots + (\epsilon_{n-d} + \epsilon_{d+1}) + 2\epsilon_{n-d+1} + \cdots + 2\epsilon_d. \end{aligned}$$

Again, we have that each root in the last sum belongs to $R^+ \setminus R^+_{P_d}$, and involves α_n with coefficient equal to 1. Hence we obtain that $c_\alpha = d$. We let $\beta_1, \cdots, \beta_d$ be the

unordered d-tuple of roots:

$$\epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_{n-1}, \cdots, \epsilon_{n-d} + \epsilon_{d+1}, 2\epsilon_{n-d+1}, \cdots, 2\epsilon_d.$$

Then it is easily checked that the above β_j's satisfy (a)-(c) in Step 2 of §4.1, and we have $m_d = d$.

Hence

$$\sum_{1 \leq d \leq n} m_d = \sum_{1 \leq d \leq n} d = \binom{n+1}{2} = codim_{G/B} P/B.$$

8.4. Orthogonal Grassmannian. Since $SO(2n+1)/P_n \cong SO(2n+2)/P_{n+1}$, and $SO(2n)/P_n \cong SO(2n)/P_{n-1}$, we shall give the details for the orthogonal Grassmannian $SO(2n)/P_n$. Thus $G = SO(2n), P = P_n$, and $\tau = (n\, n-1 \cdots 1)$.

In type $\mathbf{D}_n$, we have that in the expression for a positive root (as a non-negative integral linear combination of simple roots), the simple root α_n occurs with a coefficient ≤ 1. For all $P_d, 1 \leq d \leq n$, we shall take α (cf. §8.1) to be α_n.

Case 1: Let $1 \leq d \leq n-2$. We have,

$$\omega_d + \tau(\omega_d) = (\epsilon_1 + \cdots + \epsilon_d) + (\epsilon_n + \cdots + \epsilon_{n+1-d}).$$

If $d < n+1-d$, then as in §8.3, we have

$$\omega_d + \tau(\omega_d) = (\epsilon_1 + \epsilon_n) + (\epsilon_2 + \epsilon_{n-1}) + \cdots + (\epsilon_d + \epsilon_{n+1-d}).$$

We have that each root in the last sum belongs to $R^+ \setminus R^+_{P_d}$, and involves α_n with coefficient equal to 1. Hence $c_\alpha = d$. We let $\beta_1, \cdots, \beta_d$ be the unordered d-tuple of roots:

$$\epsilon_1 + \epsilon_n, \epsilon_2 + \epsilon_{n-1}, \cdots, \epsilon_d + \epsilon_{n+1-d}.$$

If $d \geq n+1-d$, then

$$\begin{aligned}\omega_d + \tau(\omega_d) &= (\epsilon_1 + \cdots + \epsilon_{n-d}) + (\epsilon_n + \cdots + \epsilon_{d+1}) + (2\epsilon_{n-d+1} + \cdots + 2\epsilon_d) \\ &= (\epsilon_1 + \epsilon_{n+1-d}) + (\epsilon_2 + \epsilon_{n+2-d}) + \cdots + (\epsilon_d + \epsilon_n).\end{aligned}$$

Again we have that each root in the last sum belongs to $R^+ \setminus R^+_{P_d}$, and involves α_n with coefficient equal to 1. Hence we obtain that $c_\alpha = d$. We let $\beta_1, \cdots, \beta_d$ be the unordered d-tuple of roots:

$$(\epsilon_1 + \epsilon_{n+1-d}), (\epsilon_2 + \epsilon_{n+2-d}), \cdots, (\epsilon_d + \epsilon_n).$$

Then it is easily checked that the above β_j's satisfy (a)-(c) in Step 2 of §4.1, and we have $m_d = d$.

Case 2: $d = n-1$. We have,

$$\omega_{n-1} + \tau(i(\omega_{n-1})) = \begin{cases} (\epsilon_2 + \cdots + \epsilon_{n-1}), & \text{if } n \text{ is even} \\ (\epsilon_1 + \cdots + \epsilon_{n-1}), & \text{if } n \text{ is odd.} \end{cases}$$

Hence expressing $\omega_{n-1} + \tau(i(\omega_{n-1}))$ as a non-negative linear integral combination of positive roots, we obtain

$$\omega_{n-1} + \tau(i(\omega_{n-1})) = \begin{cases} \sum\limits_{2 \leq i \leq \frac{n-2}{2}} \epsilon_i + \epsilon_{n+1-i}, & \text{if } n \text{ is even} \\ \sum\limits_{1 \leq i \leq \frac{n-1}{2}} \epsilon_i + \epsilon_{n-i}, & \text{if } n \text{ is odd.} \end{cases}$$

Thus $\omega_{n-1} + \tau(i(\omega_{n-1}))$ is a sum of $\frac{n-2}{2}$ or $\frac{n-1}{2}$ roots in $R^+ \setminus R^+_{P_{n-1}}$, according as n is even or odd; further, each of them involves α_n with coefficient one. Hence

$$c_\alpha = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

We let $\beta_1, \cdots, \beta_r, r = c_\alpha$ be the unordered r-tuple of roots:

$$\beta_i = \begin{cases} \epsilon_i + \epsilon_{n+1-i}, 2 \leq i \leq \frac{n-2}{2}, & \text{if } n \text{ is even} \\ \epsilon_i + \epsilon_{n-i}, 1 \leq i \leq \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, the above β_j's satisfy (a)-(c) in Step 2 of §4.1

$$m_{n-1} = \begin{cases} \frac{n-2}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Case 3: $d = n$. Proceeding as in Case 2, we have,

$$\omega_n + \tau(i(\omega_n)) = \begin{cases} (\epsilon_1 + \cdots + \epsilon_n), & \text{if } n \text{ is even} \\ (\epsilon_2 + \cdots + \epsilon_n), & \text{if } n \text{ is odd.} \end{cases}$$

Hence we obtain

$$\omega_n + \tau(i(\omega_n)) = \begin{cases} \sum\limits_{1 \leq i \leq \frac{n}{2}} \epsilon_i + \epsilon_{n+1-i}, & \text{if } n \text{ is even} \\ \sum\limits_{2 \leq i \leq \frac{n+1}{2}} \epsilon_i + \epsilon_{n+2-i}, & \text{if } n \text{ is odd.} \end{cases}$$

Thus $\omega_{n-1} + \tau(i(\omega_{n-1}))$ is a sum of $\frac{n}{2}$ or $\frac{n-1}{2}$ roots in $R^+ \setminus R^+_{P_n}$, according as n is even or odd; further, each of them involves α_n with coefficient one. Hence

$$c_\alpha = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

We let $\beta_1, \cdots, \beta_r, r = c_\alpha$ be the unordered r-tuple of roots:

$$\beta_i = \begin{cases} \epsilon_i + \epsilon_{n+1-i}, 1 \leq i \leq \frac{n}{2}, & \text{if } n \text{ is even} \\ \epsilon_i + \epsilon_{n+2-i}, 2 \leq i \leq \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Clearly, the above choice of β_j's satisfies (a)-(c) in Step 2 of §4.1

$$m_n = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Combining cases 2 and 3, we obtain $m_{n-1} + m_n = n - 1$, in both even and odd cases. Combining this with the value of $m_d, 1 \leq d \leq n-2$ as obtained in case 1, we obtain

$$\sum_{1 \leq d \leq n} m_d = \sum_{1 \leq d \leq n-2} d + n - 1 = \binom{n}{2} = codim_{G/B} P/B.$$

Thus combining the results of §5, §6, §7, §8, we obtain

THEOREM 8.5. *The LMP conjecture and Wahl's conjecture hold for a minuscule G/P.*

REMARK 8.6. Thus in these cases again, we obtain a nice realization for $ord_\tau \, p_{w_0^{(d)}}$ (the order of vanishing along P/B of $p_{w_0^{(d)}}$) being the length of the shortest path through extremal weights in the weight lattice connecting the highest weight (namely, $i(\omega_d)$ in $H^0(G/B, L(\omega_d))$ and the extremal weight $-\tau(\omega_d)$.

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