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First order approximation of the stability regions of time delay systems

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Problem Formulation

- Delay Differential Equation (DDE) with multiple Delays, $oldsymbol{ au} = (au_1, au_2, \dots, au_\ell)^T.$
- Leads to ℓ -topples infinite dimensional spectrum due to ℓ number of delays.
- Stability boundary: The boundary separating STABLE vs. UNSTABLE in au.
- Analyzing *stability robustness* of DDE in presence of parametric uncertainties is a challenge.
- Some respectable techniques include: Hermite-Biehler Theorem, Zero-Exclusion Principle, Edge Theorem, generalization of Kharitonov's theorem.

Inherent Challenges

- Detection of the *stability boundary* is a challenge.
- Effects of parametric uncertainties $\mathbf{p} \in \mathbb{R}^n$ complicates the analysis further.
- Complications in existing techniques in treating *singular points* on the stability boundary.

Objective

For a nominal $\mathbf{p} = \mathbf{p}_0$ and given $\boldsymbol{\tau} = \boldsymbol{\tau}_0$ on the stability boundary of the characteristic function,

$$g(\lambda, \mathbf{p}, \boldsymbol{\tau}) = \sum_{k=0}^{M} \bar{a}_k(\mathbf{p}, e^{-\lambda \tau_1}, \dots e^{-\lambda \tau_\ell}) \lambda^k,$$
 (1)

investigate the *geometry* of the boundary in the *uncertainty space* $\mathbf{p} \in \mathbb{R}^n$ around the point $\mathbf{p} = \mathbf{p}_0$.

Stability Boundaries without Delays

A recent work by A.A. Mailybaev in 2000 presents a *systematic* approach to the geometry of the stability boundaries of

$$P(\lambda, \mathbf{p}) = \sum_{k=0}^{M} a_k(\mathbf{p})\lambda^k.$$
 (2)

• Notice that $P(\lambda, \mathbf{p}) = g(\lambda, \mathbf{p}, \boldsymbol{\tau} = \mathbf{0}).$

• Assume non-vanishing leading term.

Root continuity: For the delay systems of *retarded* type, root continuity of the characteristic roots with respect to the imaginary axis holds, see Datko 1978 and the well-known Neimark's D-decomposition method.

Crossing boundary: The set of parameters **p** for which the corresponding characteristic function has at least one root on the imaginary axis, $\lambda = j\omega$. This defines a decomposition of p space with respect to the arising number of strictly unstable roots of the characteristic function.

Mailybaev's Work - Geometric Approach

$$P(\lambda + \lambda_j, \mathbf{p}) = \sum_{k=0}^{M} a_k(\mathbf{p})(\lambda + \lambda_j)^k = \sum_{k=0}^{M} b_k(\lambda_j)\lambda^k = 0.$$
(3)

By differentiating (3) *l* times with respect to λ and setting $\lambda = 0$, we have

$$b_l = \frac{1}{l!} \frac{\partial^l P}{\partial \lambda^l} \bigg|_{\lambda=0} = \sum_{t=0}^{M-l} C_{M-t}^l a_{M-t} \lambda_j^{M-l-t}, \qquad (4)$$

Theorem: Geometry of the Stability Cone: If the system of vectors

М

is linearly independent, then the tangent cone to the

stability domain at the point $\mathbf{p} = \mathbf{p}_0$ consists of vectors e satisfying the conditions in (10) for imaginary roots λ_i , $j=1,\ldots,s$.

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Next, a differential operator is introduced,

$$\nabla = \left(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}\right),\tag{5}$$

where the partial derivatives are taken at point $\mathbf{p} = \mathbf{p}_0$. Following from (4), the expression $\nabla b_l(\lambda_i)$ is given by

$$\nabla b_l(\lambda_j) = \sum_{t=0}^{M-l} C_{M-t}^l \lambda_j^{M-l-t} \nabla a_{M-t}.$$
 (6)

Define the *n*-dimensional real vectors $\mathbf{f}_l(\lambda_i)$ and $\mathbf{g}_l(\lambda_i)$ that correspond to λ_i

$$\mathbf{f}_l(\lambda_j) + i\mathbf{g}_l(\lambda_j) = \nabla b_l(\lambda_j) \ l = 0, \dots, m_j - 3$$
(7)

$$\mathbf{f}_{m_j-2}(\lambda_j) + i\mathbf{g}_{m_j-2}(\lambda_j) = \nabla b_{m_j-2}(\lambda_j) / b_{m_j}^0(\lambda_j), \quad (8)$$

$$\mathbf{f}_{m_j-1}(\lambda_j) = \Re[(b_{m_j}^0(\lambda_j)\nabla b_{m_j-1}(\lambda_j) - b_{m_j+1}^0(\lambda_j)\nabla b_{m_j-2}(\lambda_j))/(b_{m_j}^0(\lambda_j))^2],$$
(9)

where $b_l^0(\lambda_j)$ is the value of $b_l(\lambda_j)$ for $\mathbf{p} = \mathbf{p}_0$.

Definition: Stability Cone – First-order Approximation Let a curve $\mathbf{p} = \mathbf{p}(\epsilon)$ originates from the point \mathbf{p}_0 on the stability boundary in the direction of $\mathbf{e} = (d\mathbf{p}/d\epsilon)|_{\epsilon=0}$. The complete set of vectors **e** of the curves $\mathbf{p} = \mathbf{p}(\epsilon)$ which reside in the stability domain for $\epsilon > 0$ is called a *tangent cone to the stability domain* at the point \mathbf{p}_0 .

Theorem: Vector Directions in Stability Regions For the curve $\mathbf{p} = \mathbf{p}(\epsilon)$ originating from the point \mathbf{p}_0 in the direction of $\mathbf{e} = (d\mathbf{p}/d\epsilon)|_{\epsilon=0}$ to lie in the stability domain for $\epsilon > 0$, the necessary condition is

$$(\mathbf{f}_{l}(\lambda_{j}), \mathbf{e}) = (\mathbf{g}_{l}(\lambda_{j}), \mathbf{e}) = 0, \ l = 0, \dots, m_{j} - 3,$$

$$(\mathbf{f}_{m_{j}-2}(\lambda_{j}), \mathbf{e}) \ge 0,$$

$$(\mathbf{g}_{m_{i}-2}(\lambda_{i}), \mathbf{e}) = 0, \ (\mathbf{f}_{m_{i}-1}(\lambda_{i}), \mathbf{e}) \ge 0,$$
(10)

for all λ_j , $j = 1, \ldots s$.

$$V_1 \cup \ldots \cup V_s, \tag{11}$$

$$V_{j} = \{\mathbf{f}_{l}(\lambda_{j}), \mathbf{g}_{t}(\lambda_{j}); l = 0, \dots, m_{j} - 1,$$

$$t = 0, \dots, m_{j} - 2\}, j = 1, \dots, s,$$
(12)



Main Result

Outline of approach

- domain T.

Pseudo-delay domain, T

- Introduce
- exact

$$k_{\omega,j} = \frac{\partial T_j}{\partial \omega}\Big|_{\omega,\tilde{\mathbf{p}}} \in \mathbf{R}, \ j = 1 \dots \ell.$$

When $k_{\omega,j} \neq 0$ the geometry of the stability boundary under the transformation (14) is *distorted*. The Approach with Delays

Step 1. Differentiating (15) one time (l = 1) for $\mathbf{p} = \mathbf{p}_0$,

Using $k_{\omega,j}$, it becomes

• Mailybaev's approach is not constructed to consider presence of delays in the characteristic function.

• Mailybaev's work is adapted to cover the cases $\tau \neq 0$. • This is a *non-trivial* task to accomplish.

• Introduce a new domain, namely the pseudo-delay

• Adapt Mailybaev's approach to pseudo-delay domain with uncertainties, (\mathbf{T}, \mathbf{p}) .

• Via a non-linear mapping, back transform the analysis to the domain of interest $(\boldsymbol{\tau}, \mathbf{p})$.

$$e^{-\lambda\tau_j} = \frac{1 - \lambda T_j}{1 + \lambda T_j}, \quad j = 1, \dots, \ell,$$
(13)

where $\mathbf{T} \in \mathbf{R}^{\ell}_+$ defines the pseudo delay domain, see Rekasius, Niculescu, Olgac, Sipahi for applications. • On the imaginary axis, $\lambda = i\omega$, bilinear transformation is

$$T_j = \frac{1}{\omega} \tan(\frac{\tau_j \omega}{2}), \quad j = 1, \dots, \ell,$$
 (14)

• After the transformation in (13),

$$h(\lambda, \mathbf{p}, \mathbf{T}) = \sum_{k=0}^{\tilde{M}} \tilde{a}_k(\mathbf{p}, \mathbf{T}) \lambda^k = 0, \qquad (15)$$

• Eq.(15) is *exactly* in the same form as defined in Mailybaev's work, see Eq.(2).

Property 1 (One-to-one mapping property) Given ω and τ_j information on the stability boundary, there exists a unique T_j *corresponding to this information as per (14).*

Lemma 1 (Invariance of root crossing set) For a fixed $\boldsymbol{\tau} = \boldsymbol{\tau}_0$, the imaginary root set of (1) along the stability *boundary,* Ω_{τ_0} *, is invariant under the transformation (13).*

Connection with Analysis of DDE - *Key point to adapt Mailybaev's approach to DDE.*

From Eq. (14), for $\boldsymbol{\tau} = \boldsymbol{\tau}_0$ fixed, one has

$$dh = \frac{\partial h}{\partial \lambda} d\lambda + \sum_{j=0}^{\ell} \frac{\partial h}{\partial T_j} dT_j, \qquad (16)$$

$$dh = \left[i\frac{\partial h}{\partial \lambda} + \sum_{j=0}^{\ell} k_{\omega,j}\frac{\partial h}{\partial T_j}\right]d\omega.$$
 (17)

The Approach with Delays, cont'd.

Step 2. When $k_{\omega,j} \neq 0$, the approach of Mailybaev extends to the analysis of DDE. In such a case, the approach modifies b_1 as b_1),

Comprehensive study for all ℓ is left for future work.

Example – Regular Points

Tangency Lines on Stability Boundary



Example – Singular Points

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$$\tilde{b}_1 = b_1 - i \sum_{j=0}^{\ell} k_{\omega,j} \frac{\partial h}{\partial T_j},$$
(18)

Characteristic function: $f(\lambda) = \lambda^2 + \lambda(-1 + \alpha) + \beta = 0$. Nominal PI gains: $(\alpha_0, \beta_0) = (1.5, 2)$.

Characteristic function: $g(\lambda, \mathbf{p}, \tau) = \lambda^2 + \alpha \lambda + 10 - \beta e^{-\tau \lambda} = 0$ Singular point at $(\alpha, \beta) = (2, 7.44025)$, for $\tau \approx 3.0293$.





Tangency Lines at Singular Points



