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# Delay Modeling in Supply Network Dynamics; Performance, Synchronization, Stability<sup>\*</sup>

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**Abstract:** Recent research results in Industrial Engineering and Business disciplines specifically emphasize the critical role of delays in the functionality of supply networks. Delays are inevitable and pervasive; they arise due to the time needed for material deliveries, information flow and human perception towards adjusting to new decisions. Delays are endangering as they contaminate decision-making which ultimately leads to poor performance, synchronization problems and fluctuating inventory levels resulting in major economical losses. It is critical to state that inventory regulation and synchronization in supply networks even without delays is an open problem today due to large scale of the network, nonlinearities, randomness and uncertainties in partially artificial consumer demand. With delays, the problem is more dramatic and end-effects of decision-making are not thoroughly understood. This paper surveys mathematical models developed for supply network dynamics, connecting it via a system-level construct with standard delay models pertaining to material deliveries, information flows and human perception. Next, the analogy between such delay models arising in supply networks and other real-life applications is pointed out. It is foreseen that complexity of the problem requires multi-disciplinary research bridging the fields of Industrial Engineering, Business, Systems & Control Engineering and Mathematics. The paper concludes with an illustrative example and discussions of specific challenges to be encountered in future research along the same line.

Keywords: modeling, supply networks, delay, inventory regulation, stability

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## 1. INTRODUCTION

Supply networks, Forrester [1961], Helbing et al. [2006], Helbing [2003], Riddalls et al. [2002a,b], Sipahi et al. [2006a], Sterman [2000] can be seen as interconnected dynamics of customers, suppliers, manufacturing units, companies and sources, Figure 1. While supplies flow along the directed links of these networks to satisfy the changing demand of customers (solid lines in Figure 1), the information concerning the product orders flows in the opposite direction (dashed lines in Figure 1). One of the main objectives in *dynamic* supply network management is to control individual production rates such that inventories maintain their desired levels when responding to customer demands. Although this seems to be a simplistic proposition, supply network management is known to be a challenge, The Economist [2002].

What enables the desired functionality of the network is assured by the decision-making units which utilize the communication medium to relay certain orders to the source, factory, distributor and retailer branch. When placing these orders, however, it is crucial to possess the freshest information possible regarding the demand and supply rates so that the decisions relayed achieve a desired objective. On the other hand, there are numerous

constraints inherent to the physics of the supply network. First of all, decision-making units (managers) tend to wait enough time before they order more/less supplies when demand changes. This wait time naturally occurs due to collection of necessary data to conclude a decision and perception of human behavior towards deciding a new command, Sterman [1989b]. Second, the adaptation of supplies and their transportation are not instantaneous, but need certain period of time, *only* after which supplies can meet with customers Sterman [2000], Riddalls et al. [2002a], Sipahi et al. [2006a].

The above arguments are equivalent to the following. The effects of supplies leaving a retailer branch at any instant will be seen at customer end only after certain time. What is currently occurring in the network is the *after-effects* of what has happened earlier. Consequently, any decision based on what is currently observed in the network is actually cross-coupled with past events, and therefore it may not successfully achieve the desired functionality of the supply network.

The source of after-effects or *delays* is due to presence of (a) inventories, (b) transportation paths, (c) information flow and (d) decision-making in the network, Sterman [1989a], Riddalls et al. [2002a,b], Sipahi et al. [2006a], Sterman [2000]. Undesirable effects of delays are very well known in Operations Research and Business Management,

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Sterman [2000]. Delays induce *oscillations*, *limit cycles*, *overshoot* and *excessive levels* in inventories as well as *synchronization* (consensus) problems across parallel-running processes, Agrawal et al. [2001], Ceroni et al. [2005]. These effects may cost companies, as evident from Cisco’s \$2.2 billion inventory write-off in 2001 which occurred due to large orders placed in 1999-2000 based on large demand observations, CIO Magazine [2001]; or desynchronization in manufacturing and sales: unnecessary green cars produced by Volvo, Agrawal et al. [2001].

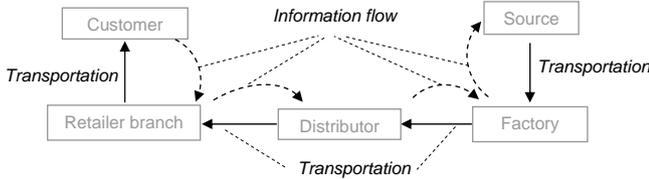


Fig. 1. A generic supply-demand flow from source to customers, inspired by beer distribution game, Sterman [2000].

Consideration of delays in mathematical models representing governing dynamics may dramatically change the expected dynamical behavior, even in linear models. Depending on the system structure and decision-making strategies, small delays may be the source of severe detrimental effects to system behavior. On the other hand, large delays may stabilize a system, which is not necessarily stable for smaller delays. Although counter-intuitive, such behavioral classifications are known to exist, Niculescu [1997], Hale [1993], Niculescu [2001], Niculescu et al. [2004], Stepan [1989]. Clearly, one can conclude that “intuition alone” may be misleading to explain the effects of “large” and “small” delays in dynamical behavior of systems. This can be seen as one of the reasons why research in the field of *time delay systems* has six decades of history with increasing intensity in the last twenty years.

In order to avoid the detrimental effects of delays, it becomes necessary to understand the dynamics of interactions between supply-demand points by developing mathematical models *considering delays*. Exploiting physics laws, inherent features of supply networks and engineering approaches, dynamics of supply networks are expressed in the form of differential equations, Sterman [2000], Riddalls et al. [2002a,b], Sipahi et al. [2006a]. In this paper, we survey ideas both for modeling of dynamic supply networks (Section 2) and of delays (Section 3) based on their source and physics. It is foreseen that this will establish the foundation towards research in the area of supply network management “with delays”. Furthermore, the interconnection between these models and system-level approach particularly in the field of *time delay systems* (TDS) will be established (Section 4). Numerous dynamical systems arising in different engineering applications will also be presented in this section to lay out the analogy between these applications and supply networks dynamics. An illustrative example will demonstrate (Section 5) how to connect Sections 2-4. In Section 6, we will conclude the paper by pointing out future research at the intersection of Business Management, Engineering and Mathematics.

**Notation.** Notations are standard. We use  $\mathbb{R}$  for real numbers, where an additional + or – sign as a sub-

script indicates the positive and negative real numbers, respectively. Complex plane is represented by  $\mathbb{C}$  and the imaginary axis of  $\mathbb{C}$  is denoted by  $j\mathbb{R}$ , where  $j = \sqrt{-1}$ . We use  $s \in \mathbb{C}$  for the Laplace variable, whose values on the imaginary axis are denoted by  $s = j\omega$  for  $\omega \in \mathbb{R}$ . We use bold face for vectors and matrices. The eigenvalues of the matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  are represented by  $\lambda_i(\mathbf{A})$ ,  $i = 1, \dots, n$ .

## 2. MATHEMATICAL MODELING OF SUPPLY NETWORKS WITHOUT DELAYS

In this section, we briefly present some ideas for mathematical modeling of supply networks developed in the literature, Helbing et al. [2004b], Sterman [1989a], Riddalls et al. [2002a]. The following main components of the network play role in the development of these models,

- Inventories,
- Communication medium,
- Decision-making & human-in-the-loop dynamics,
- Production/Supplies,
- Transportation medium,

where inventories and decision-making are the main components giving rise to mathematical models, as it is often the case in the literature. The remaining components, as we shall discuss in Section 3, will reveal further details on the supply network dynamics especially concerning the *delays*.

### 2.1 Helbing’s model, Helbing et al. [2004b]

This model considers a supply network of  $n$  suppliers  $i$  delivering products to other suppliers  $\mu$  or to costumers, Figure 2. The rate at which supplier  $i$  delivers products to and consumes product from supplier  $\mu$  is given by  $d_{\mu i}X_i(t)$  and  $c_{i\mu}X_i(t)$ , respectively, where  $X_i(t) > 0$  denotes the production rate. The coefficients  $c_{i\mu}$  define an input matrix  $\mathbf{C}$  and  $d_{i\mu}$  an output matrix  $\mathbf{D}$  with  $0 \leq d_{i\mu}, c_{i\mu} \leq 1$ .

*Inventories.* The inventory level  $N_i(t)$  represented by a bowl at supplier  $i$  changes at the rate

$$\frac{d}{dt}N_i = \sum_{\mu=1}^{n_1} (d_{i\mu} - c_{i\mu}) X_{\mu}(t) + Y_i(t), \quad i = 1, \dots, n_2, \quad (1)$$

where the external demand is denoted by  $Y_i(t)$ . In order to keep the inventory at some desired level  $\bar{N}_i$ , any changes in the demand  $Y_i(t)$  require an *adaptation* of the production rates  $X_i(t)$ .

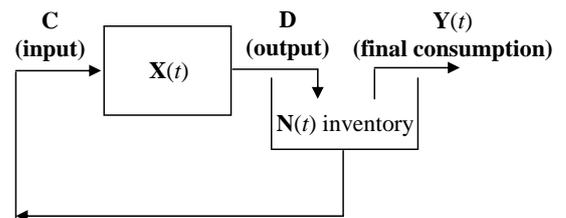


Fig. 2. Supply network model proposed by Helbing et al. [2004b].

*Decision-making.* The adaptation is represented by the time constant  $T_\mu$ , which defines the measure of speed of actual production rate  $X_\mu(t)$  converging to a desired one defined by a non-linear function  $W_\mu(t)$  dependent in general on *all* the inventories  $N_i(t)$  and/or their change in time  $dN_i(t)/dt$ . Mathematically, this corresponds to

$$\frac{dX_\mu(t)}{dt} = \frac{1}{T_\mu} (W_\mu(\{N_i(t)\}, \{dN_i(t)/dt\}) - X_\mu(t)), \quad (2)$$

which concludes the foundation of Helbing's model (1)-(2).

## 2.2 Sterman's model, Sterman [1989a]

Among various models of Sterman, a fundamental one arising in a stock acquisition system is given below. Different than Helbing's model, Sterman<sup>1</sup> utilizes two *sequential* bowls representing the supply line  $SL$  and inventory (or stock),  $S$ . The two bowls are accumulation components, thus they are mathematically integrator dynamics. The rate at which supplies being delivered from  $SL$  to  $S$  and out from  $S$  are called acquisition rate,  $A$ , and loss rate,  $L$ , respectively.

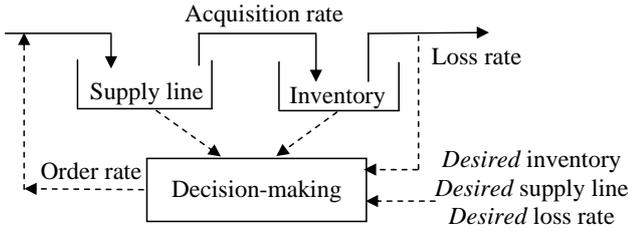


Fig. 3. Stock (inventory) acquisition model proposed by Sterman [1989a].

*Inventories.* The equations defining the dynamics of the inventory and supply line form as follows

$$S(t) = \int_{t_0}^t (A(\kappa) - L(\kappa))d\kappa + S(t_0), \quad (3)$$

$$SL(t) = \int_{t_0}^t (O(\kappa) - A(\kappa))d\kappa + SL(t_0). \quad (4)$$

*Decision-making.* The decision-making utilizes the information concerning the levels at  $SL$  and  $S$  as well as  $L$ . Furthermore, desired supply line,  $SL^*$ , desired inventory  $S^*$  and expected loss rate  $\hat{L}$ , which can be constant or time-varying, are used for comparison with  $SL$ ,  $S$  and  $L$ , respectively. This comparison is necessary to re-formulate the order rate  $O$  to correct the actual  $SL$ ,  $S$  and  $L$  towards maintaining the desired physical entities  $SL^*$ ,  $S^*$  and  $\hat{L}$ . The order rate strategy is formulated as,

<sup>1</sup> We also note that delays are a part of Sterman's model, however, to maintain the coherency of this section, delays will be discussed in the following section.

$$O(t) = \max(0, IO(t)), \quad (5)$$

$$IO(t) = \hat{L} + AS(t) + ASL(t), \quad (6)$$

$$AS(t) = \alpha_S(S^*(t) - S(t)), \quad (7)$$

$$ASL(t) = \alpha_{SL}(SL^*(t) - SL(t)), \quad (8)$$

$$\hat{L} = L^*, \quad (9)$$

where  $IO(t)$  is the indicated order rate, 'maximum' between zero and  $IO(t)$  assures that  $O(t)$  is nonnegative,  $\alpha_S$  is the stock adjustment parameter,  $\alpha_{SL}$  is the fractional adjustment rate for the supply line,  $AS$  is the actual stock and  $ASL$  is the actual supply line. Furthermore, other than modeling expected loss rate  $\hat{L}$  as constant,  $\hat{L} = L^*$ , i.e. static expectations, one can also deploy regressive, adaptive and extrapolative expectations, see Sterman [1989a].

## 2.3 Riddalls's model, Riddalls et al. [2002a]

The work in Riddalls explicitly incorporates pure time delays in mathematical modeling of supply networks. In this work, the inventory dynamics is taken as in (3), while assuming  $A(t) = O(t)$  thus disregarding (4). In modeling decision-making, Riddalls utilizes (5)-(8) as they are, but modifies (9) by proposing the following formulation for expectations<sup>2</sup> (short term forecast/trend detector)

$$\hat{L} = \frac{1}{T} \int_{t-T}^t L(\kappa)d\kappa, \quad (10)$$

where  $T$  is a period of time. The above equation suggests that the expectation is the average of the integration of loss rate over the period  $T$ . In order to maintain a consistent flow here, we will discuss the presence of time delays in Riddalls's formulation later in a devoted section. Riddalls's model, *without delays*, becomes

$$\frac{dO(t)}{dt} = \alpha_S(L(t) - O(t)), \quad (11)$$

Notice that, similar mathematical models can also be derived from various supply chain models which differ only in their degree of detail concerning the considerations of forecasts, Riddalls et al. [2002a,b], Nagatani et al. [2004], price dynamics, Helbing [2003], Helbing et al. [2004a], or lack of materials, Helbing [2003], Sterman [2000].

## 2.4 Supply Network at Equilibrium - Linear Dynamics

As it is often implicitly needed in queuing theory, dynamics of the supply network operating at an equilibrium is of interest. The corresponding linear dynamics can trivially be obtained from the non-linear one. A convenient way to unify the linear form of all the possible models under one umbrella is to re-write the coupled equations in the form of  $n$  number of first-order linear differential equations, which will result in the following vector form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{F}(t). \quad (12)$$

Depending on the type of model utilized, the matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ , vectors  $\mathbf{x}(t) \in \mathbb{R}^{n \times 1}$  and  $\mathbf{F}(t) \in \mathbb{R}^{n \times 1}$  will be different. In Helbing's model for instance,  $n \geq n_1 + n_2$ ,  $\mathbf{x}(t)$  comprises  $\delta X_\mu(t)$  and  $\delta N_i(t)$  and their derivatives (if  $n > n_1 + n_2$ ),  $\mathbf{F}(t)$  contains the terms  $\delta Y_i(t)$  and matrix

<sup>2</sup> In order to enable easier comparison, we adopt Sterman's notation to express Riddalls work.

$\mathbf{A}$  carries  $c_{i\mu}$ ,  $d_{i\mu}$  terms as well as local derivatives of the non-linearities, where  $\delta(\bullet)$  denotes small variations on  $(\bullet)$  around its equilibrium. Notice also that matrix  $\mathbf{A}$  defines the topology of the network that shows which one of the supply-demand units interact with each other.

*Local Stability.* It is trivial to study the local stability of the supply network dynamics by analyzing the spectrum of the matrix  $\mathbf{A}$ . If eigenvalues of  $\mathbf{A}$ ,  $\lambda_i(\mathbf{A})$ , reside in open left half of complex plane, then local stability is guaranteed, Ogata [2002]. In the case of instability, further analysis is not of interest.

*Performance.* If the local stability holds, the minimum of the distances of the eigenvalues to the imaginary axis, i.e.  $\min(|\Re(\lambda_i)|)$  is proportional to disturbance rejection speed of the dynamics. Smaller  $\min(|\Re(\lambda_i)|)$  is, longer it takes to damp out the effects of disturbances.

*Randomness/Bullwhip Effects.* Assume that a production unit exhibits periodic demand with a frequency  $\omega$  and an amplitude  $v_d > 0$ ,  $v_d \sin(\omega t)$ . For linear dynamics, the output of the production unit will create a supply response in the form of  $v_s \sin(\omega t + \phi)$  where  $v_s > 0$  is the amplitude of the supply, and  $\phi$  is the relative phase difference between what is demanded and what is supplied. If  $v_s > v_d$  then a chain of production units will create supplies with amplifying amplitudes. This phenomenon is known as *Bullwhip Effects*<sup>3</sup>. Obviously, such a destructive situation should be avoided by appropriately designing the network<sup>4</sup> so that  $v_s < v_d$  holds for any excitation frequency  $\omega$  and for all production units.

### 3. MATHEMATICAL MODELING OF DELAYS IN SUPPLY NETWORKS

For accurate understanding of supply network dynamics, it is crucial to model the delays based on the physics they originate from. In the field of Industrial Engineering and Business, presence of delays in the context of supply networks is mainly due to transportation of materials/supplies, Sterman [2000], Sipahi et al. [2006a]; flow/distribution of energy, communication with technological constraints, Sterman [2000]; lead times for machine set-up Riddalls et al. [2002a,b]; and human behavior, Sterman [1989a]. As we explain in the sequel, the underlying physics of these components and ultimately their delay modeling are different. While, for instance, lead time will interrupt a process during a period of time, delivery of products to different destinations may exhibit a distribution of different delivery times.

It is critical to state that various dynamical systems with different delay models are also studied in the field of *time delay systems*. Therefore, we also aim to point out the interconnection between supply networks and time delay systems. To establish this requires delay modeling in the context of supply networks. In the following, inspired by Sterman [2000, 1989a], Riddalls et al. [2002a,b], Sipahi et al. [2006a], we will discuss these models.

<sup>3</sup> In the literature, bullwhip phenomenon is called as ‘slinky effects’ in control theory, Niculescu [2001]; ‘chain/string stability’ in traffic flow research, Treiber et al. [2006] and ‘randomness’ in Business, Sterman [2000].

<sup>4</sup> See also network topology induced performance in Section 4.

#### 3.1 Constant delay model:

This model assumes that delay  $\tau > 0$  is constant. In Business, constant delay is also known as *pipeline delay*, while in time delay systems it is called as *discrete delay*. An inflow  $i(t)$  through a constant delay model will create an outflow  $o(t)$ , where  $o(t)$  is a pure translation of the inflow  $i(t)$  along time axis at an amount of  $\tau$ . In other words,  $o(t) = i(t - \tau)$  holds, Figure 4, which indicates that the output identically mimics the input but it is shifted  $\tau$  amount in time. This class of delay model may represent

- human as decision-maker: waiting the trends in the network before a new decision, updating of beliefs, adjusting towards a new decision,
- communication, data collection and measurement times,
- machine set-up lead times,
- material flow in assembly lines.

Constant delay models FIFO (first in, first out) behavior, but it does not consider any *mixing*, which might be needed in biological systems and chemical processes. Incorporating the effects of mixing requires the utilization of distribution functions, as we explain below.

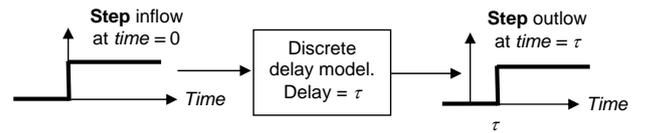


Fig. 4. Discrete delay modeling and its effects between an input and an output.

#### 3.2 Distributed delay model:

In many cases, this type of delay models is used for

- material delivery delays,
- mixing of materials,
- diffusion in social networks,
- chemical and biological systems,
- energy flow delays.

“Distributed” indicates that materials being delivered do not arrive to their destination all at once, but rather in a distributed fashion along the time. Some examples are exponential, gamma ( $\gamma$ ) and Erlang distributions. An example where distributed delay can easily be seen occurs in mass mailing, which corresponds to a pulse, while delivery of these mails to various destinations will not be at the same time, thus they will exhibit a distribution with respect to delivery time, similar to those given in Figure 5. Furthermore, notice in this figure that distribution functions are depicted with a dead-time  $h$ , which is nothing but a discrete delay after which deliveries start to arrive at their destinations. We notice that when dead-time is zero, one will recover those distributed delay models presented in the cited references. For instance,  $n^{\text{th}}$  order Erlang distribution, which is closely related to a gamma distribution, is given by

$$p(\kappa) = \frac{(n/D)^n}{(n-1)!} \kappa^{n-1} e^{-\frac{n}{D}\kappa}, \quad (13)$$

where  $\kappa > 0$  is the delivery time and  $D$  is the outflow average (mean of the distribution) in Figure 5; and when  $n = 1$ , Eq.(13) becomes an exponential distribution.

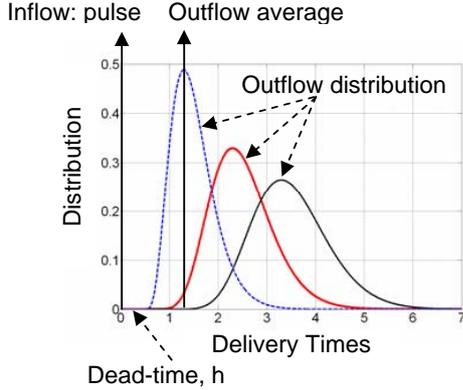


Fig. 5. Distributed delay modeling and its effects between an input pulse and output.

We finally wish to remark that, when  $h = 0$ , the outflow *average* in Figure 5 is also called as pipeline delay, which is what this distribution converges to as its variance becomes zero.

### 3.3 Other models:

Other delay models comprise time-varying and state-dependent delays. *Time-varying* delay  $\tau = \tau(t)$  creates an outflow  $o(t) = i(t - \tau(t))$  for an inflow  $i(t)$ . The time dependency in the delay may take into account the *uncertainties in delivery times, possible changes in delivery routes and interruption of deliveries*, Riddalls et al. [2002a]. *State-dependent* delays can be seen as delays associated with what the inventories, acquisition rates, loss rates, etc. For instance, it takes time to load materials on a truck, which will add delay in the process, however, this delay is dependent on the loss rate of the inventory and the instantaneous available capacity for loading.

Finally, we state that delays may also act as multipliers of system states. For instance, the desired supply line  $SL^*(t)$  in (8) requires an adaptation, which depends on the desired throughput  $\Phi^*$  and an expected delay  $\hat{\tau}$  as  $SL^*(t) = \Phi^* \hat{\tau}$ . This indicates that if a retailer wants to receive 100 items of a product per day and delivery takes 5 days, then 500 items should always be on order so that the retailer does not experience any interruption of deliveries at the desired rate, Sterman [1989a].

### 3.4 Hybrid delay models:

Based on the discussions regarding various delay models arising due to different physics laws, it seems it makes sense to consider *hybrid delay models* comprising constant, distributed, time-varying and state-dependent delays. Such an approach will improve the accuracy of the representation of the physics and enable reliable design of the supply network. Although this is quite beneficial, in the following sections we will also point out major complications that arise in stability, performance, robustness analysis in presence of delays.

## 4. CONNECTION WITH SYSTEM-LEVEL APPROACH

Although the first sight might think that Operations Research and Business Management fields independently progress from Control theory, this is not the case. One can find numerous successful “systems thinking” approaches for understanding supply network dynamics, see Dejonckheere et al. [2002], Sterman [2000, 1989a], Helbing et al. [2004b], Riddalls et al. [2002a] and the references therein. At system-level, one can consider the supply-network dynamics as a connection of block diagrams representing suppliers (feed-forward line) along with transportation lines for material deliveries and information flow forming the *feedback* line, Figure 6.

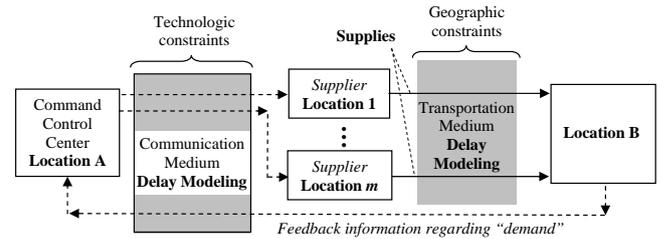


Fig. 6. Supply network with feed-forward and feedback lines. Interpretation from system-level perspective.

Along the *feed-forward* path of the network, it is clear that communication and transportation medium will lag material and information deliveries and induce *phase* causing synchronization problems. On the other hand, in the *feedback loop*, any information available to Location A at any instant concerning the demand of Location B is not fresh but delayed. Hence, decisions made at Location A for responding to demand at Location B is cross-coupled with delayed information. This coupling will have dramatic effects on the functionality of the network: the inventories at Location B will become under-damped, oscillatory and even unstable. Although such effects seem to be similar to those detected in Helbing et al. [2004b], the qualitative and quantitative conditions that give rise to these similarities with and without delays are quite different.

### 4.1 Stability

Stability analysis of differential equations with delays is still an active research topic in control theory. Details on this topic is beyond the scope of this paper. We direct interested readers to Richard [2003], Gu [2005], Michiels [2002], Olgac et al. [2002], Sipahi et al. [2005], Stepan [1989]. The main challenge in stability analysis, even in the linear case (local), arises due to delays which induce *infinite dimensionality* to the dynamics even if it possesses finite degrees of freedom. The infinite dimensionality reflects itself in the characteristic function  $f(s, \tau)$  of the system, which exhibits infinitely many characteristic roots  $s$  on the complex plane (where  $\tau$  here generically represents delays). Asymptotic stability supply network dynamics with delays is assured if all the *infinitely* many roots  $s$  reside on open left half of the complex plane.

*Sequential/Parallel Running Processes* Given the particular application of the supply network, some processes

should run sequentially, while some others will run in parallel. For instance, the supply line and the inventory in Serman's model are both integrator dynamics which occur one after the other, whereas the well-known Proportional-Integral-Derivative (PID) control law runs in parallel. Assume two sequential discrete delay models are connected and let the input to the first one is an impulse at  $time = 0$ ,  $\delta(t)$ . Then the output of the first model will be  $o_1(t) = \delta(t - \tau_1)$ , which is the input to the second block,  $i_2(t) = o_1(t)$ . Consequently, the output of the second block will be  $o_2(t) = i_2(t - \tau_2)$ , which is  $o_2(t) = \delta(t - \tau_1 - \tau_2)$ . On the other hand, assume the delay models are in parallel and their outputs add up (similar to a PD controller), let each model receives an impulse input at  $time = 0$ ,  $i_1(t) = i_2(t) = \delta(t)$ , then summation of the outputs will be  $o_1(t) + o_2(t) = \delta(t - \tau_1) + \delta(t - \tau_2)$ . Laplace transform of the sequential case between  $i_1(t)$  and  $o_2(t)$  will then be  $O_2(s)/I_1(s) = e^{-(\tau_1 + \tau_2)s}$ , where  $O_2(s)$  and  $I_1(s)$  are the laplace transforms of  $i_1(t)$  and  $o_2(t)$ , respectively. In the case of parallel models, the laplace of the summation of the individual outputs will be  $O(s) = O_1(s) + O_2(s) = e^{-\tau_1 s} + e^{-\tau_2 s}$ . Notice that, for the sequential case, the additive property of delays in argument of time becomes multiplicative in laplace domain (multiplication of two exponential function), while the additive property of the delayed states in the parallel case remains additive in laplace domain (addition of two exponential functions). When the parallel/sequential processes connected with a transfer function  $G(s)$  in their output are closed by a unity feedback loop, the *arising transfer functions and their stability features will be different*. For the sequential case, characteristic function becomes  $f(s, \tau_1, \tau_2) = 1 + G(s)e^{-(\tau_1 + \tau_2)s} = 0$ , while in the parallel case, we obtain  $f(s, \tau_1, \tau_2) = 1 + G(s)(e^{-\tau_1 s} + e^{-\tau_2 s}) = 0$ . See Niculescu [2001], Michiels [2002], Niculescu et al. [2004], Sipahi et al. [2005], Gu [2005] for details on the stability features of similar types of characteristic functions.

#### 4.2 Delay Induced Performance

One of the criteria indicating the performance of the network dynamics, as defined earlier, is the minimum of the distances of all the characteristic roots  $s$  to the imaginary axis,  $\min(|\Re(s)|)$ , where  $s$  is the root of the characteristic function  $f(s, \tau)$  corresponding to linear dynamics with delays. Thus,  $\min(|\Re(s)|)$  is also a function of the delays  $\tau$ . There are two major challenges in performance assessment in presence of delays. First of all, local stability analysis discussed earlier may not be trivial, but should be guaranteed first, and secondly it is also non-trivial to compute  $\min(|\Re(s)|)$  for a given non-zero delay among all the infinitely many roots  $s$ . Research along this lines can be found in the work of Breda et al. [2005], Michiels [2002] and the references therein.

#### 4.3 Network Topology Induced Performance

Intuitively, at *local level*, each member of the network can be designed to exhibit stable over-damped response characteristics. On the other hand, the topology of the network, i.e., the way these members are interconnected plays a crucial role in determining the performance of the network at *global level*. It is recently shown, without

considering delays, that networks with stable over-damped members may behave dramatically differently, e.g. in a low-damped, oscillatory or unstable regime, Helbing et al. [2004b]. Consequently, the structure of the topology can be seen as a source of loosing robustness against uncertainties as well as deteriorating attenuation capabilities against *bullwhip effects*.

In Control theory, the bullwhip/randomness/slinky effects correspond to *frequency response analysis*, Ogata [2002]. This analysis is performed in steady state, thus no effects of transient dynamics or initial conditions. In summary, one first obtains the transfer function  $TF(s) = N(s)/D(s)$  between an input (demand) and an output (supply) in terms of the Laplace variable  $s$ . Next, the transfer function is studied in frequency domain  $\omega$  by setting  $s = j\omega$  in  $TF(s)$ . The magnitude of the transfer function  $|TF(j\omega)|$  with respect to  $\omega$  will show if bullwhip effects occur or not. If  $|TF(j\omega)| < 1$  for *any* given  $\omega$ , then bullwhip effects do not occur. Otherwise, bullwhip effects occur within interval(s) of  $\omega$ . The term  $\omega$  is nothing but the frequency of excitation of the input, while  $|TF(j\omega)|$  corresponds to the ratio  $v_s/v_d$  discussed earlier. Similar guidelines should be followed in presence of delays, where  $|TF(j\omega)|$  should be expressed in terms of parameters defining the delays, Sipahi et al. [2006a].

#### 4.4 Connection with other Applications

Mathematical models with time delays arise in many real life applications. Similar to those in supply networks, delays create qualitatively the same problems in these applications; oscillations, limit cycles, poor performance, over shoot.

Discrete time-varying and constant delay models are widely seen in traffic flow behavior, Treiber et al. [2006], Sipahi et al. [2006c]; machine tool chatter, Stepan [1989]; multi-agent consensus/synchronization problems, Ren et al. [2005]; tele-operation, Anderson et al. [1989]; active vibration suppression, Olgac et al. [1994]; whereas distributed delays arise in biology, Kuang [1993]; machine tool chatter, Stepan [1989], and in chemical process control Niculescu [2001]. In Sipahi et al. [2007], authors study traffic flow in which drivers' decision-making is based on what is retained in their "memory", which is modeled using uniform and  $\gamma$  distributions. Furthermore, in this work, it was shown that system stability may be weak even against the presence of a small dead-time,  $h$ , in  $\gamma$  distribution.

## 5. CASE STUDY

In order to present the outcomes of utilizing system-level approach, we borrow the mathematical modeling from Riddalls et al. [2002a], see also above the subsection devoted to this model. Recall that this model takes into account discrete delay  $h$  which arises due to production lead times. We modify this model slightly by considering that acquisition rate  $A(t)$ <sup>5</sup> does not instantaneously affect the inventory. Such an assumption takes into account the transportation times,  $\tau$ , that will arise between the

<sup>5</sup> Recall that we adapt Serman's notation to maintain easier comparison among models.

geographic locations of production and inventory. This time is the delay; what is observed about inventory levels exhibits the effects of acquisition rate that occurred  $\tau$  time units ago. Our aim is to analyze the robust stability of the inventory level against such transportation delay,  $\tau$ . Although  $\tau$  is in the feed-forward path of the supply chain, it will induce instability since the information regarding the inventory will be fed-back, coupling with the decision-making. Consequently, the governing dynamics comprise the equations (3), (6)-(8), (10) and additionally the acquisition rate  $A(t)$  in (3) will be modified as  $A(t - \tau)$ .

In summary, we wish to perform stability analysis with respect to delays, which is a part of the analysis in the work of Riddalls et al. [2002a]. The main difference here is the additional parameter  $\tau$  that will complicate the stability analysis. As opposed to a single delay  $h$ , the stability analysis here should be performed in a two-dimensional delay parameter space  $h$  versus  $\tau$ . Without going into details, we give in the following the homogenous part of the governing dynamics (delay differential equation) over which the stability should be studied with respect to  $h$  and  $\tau$ ,

$$\frac{dO(t)}{dt} = -\alpha_S O(t - \tau - h) - \alpha_{SL} (O(t) - O(t - h)). \quad (14)$$

From (14), the characteristic equation is obtained as,

$$f(s, h, \tau) = s + \alpha_S e^{-(\tau+h)s} + \alpha_{SL} (1 - e^{-hs}) = 0. \quad (15)$$

We remark that the complete stability analysis of the characteristic function above is not trivial due to presence of two delays  $\tau$  and  $h$ . Starting from 1989s, various analytical techniques corresponding to necessary and sufficient conditions of stability in the delay parameter space have been developed, Stepan [1989], Hale et al. [1993], Gu [2005], Sipahi et al. [2005], Sipahi [2007]. We by-pass the details on these techniques and direct the readers to the cited references. Utilizing the ideas given in the work of Sipahi [2007], we compute the stability *regions* of supply network dynamics on the first quadrant of  $h$  versus  $\tau$  plane, Figure 7, for the choice of  $\alpha_S = 0.2$  and  $\alpha_{SL} = 0.2$ .

In Figure 7, the shaded region is the stability region, while the remaining regions indicate instability. Furthermore, the curves separating stable and unstable behavior correspond to the locations in  $\tau$  and  $h$  for which the dynamics becomes a perfect oscillator, see Gu [2005], Sipahi et al. [2005] for further discussions on these curves. In this figure, along the  $h$ -axis, i.e. when  $\tau = 0$ , one recovers the study in Riddalls et al. [2002a], while for  $\tau \neq 0$  the effects of transportation delays can be seen. We wish to point out that the stability region has an intricate geometry which may serve counter-intuitive. For instance, when  $h = 18$  and  $\tau = 20$ , the supply chain dynamics is stable, while *decreasing*  $\tau$  down to  $\tau = 10$  will create *instability*. Furthermore, it is interesting to observe that for  $\tau < 7.38$ , the dynamics is stable independent of the choice of  $h$ , while on the other hand for any  $\tau > 7.38$ , the supply chain manager should be careful, since stable and unstable behavior is possible depending on the choice of  $h$  and  $\tau$ .

Borrowing from Riddalls et al. [2002a], the inventories  $i(t)$  are expressed as  $\frac{di(t)}{dt} = O(t - h) - d(t)$ , where  $d(t)$  is the demand (part of the non-homogeneous terms). Deploying non-homogenous part of the dynamics from the cited work, we present in Figure 8 how inventories (where  $i(0) = 200$ )

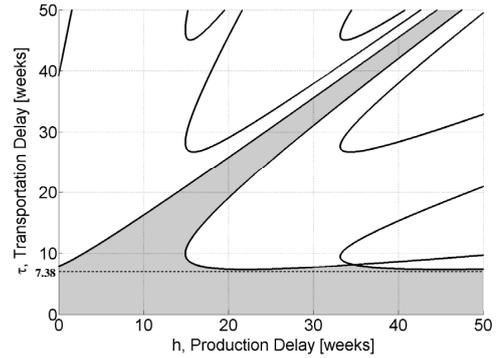


Fig. 7. Stability of the supply network dynamics in the parameter space of the delays.

behave in response to a step change in *demand* (20 units increase) for the two choices (a)  $h = 6$ ,  $\tau = 0$  (the case analyzed by Riddalls) and (b)  $h = 6$ ,  $\tau = 2$  (the new case studied here). The remaining parameters, which have no effect on the stability, remain the same as in Riddalls et al. [2002a].

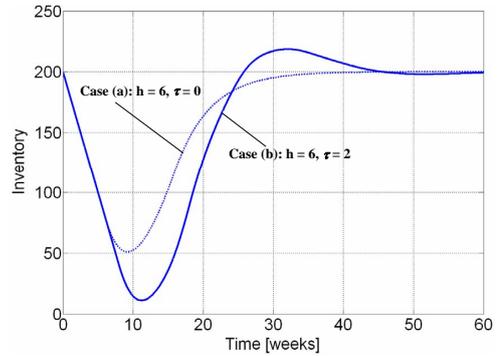


Fig. 8. Time domain simulation of inventories with production delay  $h = 6$  weeks, while transportation delay is either  $\tau = 0$  (dotted curve) or  $\tau = 2$  (solid curve) weeks.

The simulations in Figure 8 indicate that taking into account the effects of transportation delays may make the inventories more prone to oscillations and even towards their depletion.

## 6. CONCLUSION

Supply network dynamics, an open research field in Business and Operations research, particularly from the perspective of arising delays due to production lead times, material deliveries, information and decision lags and transportation times is discussed. Based on the physics laws giving rise to such delays, interpretations and corresponding mathematical descriptions (modeling) of delays are formulated. Furthermore, along with existing results in the literature, delay models are incorporated within the differential equations governing the supply networks dynamics, in which the inventories, demands and supply delivery rates are the main parameters. The connection of the problem with System Dynamics and especially with Time Delay Systems is motivated and an illustrative case study in this context is presented. This work is intended to establish a step towards our comprehension of supply

network dynamics via such connections. It is foreseen that this effort will motivate multi-disciplinary research and put light on some parts of the field unrevealed so far.

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