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## Regular polyhedra of index 2

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# REGULAR POLYHEDRA OF INDEX 2

A dissertation presented  
by

Anthony Cutler

to  
The Department of Mathematics

In partial fulfillment of the requirements for the degree of

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in the field of

Mathematics

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by

Anthony Cutler

## ABSTRACT OF DISSERTATION

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in the Graduate School of Arts and Sciences of  
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We classify all finite regular polyhedra of index 2, as defined in Section 2 herein. The definition requires the polyhedra to be combinatorially flag transitive, but does not require them to have planar or convex faces or vertex-figures, and neither does it require the polyhedra to be orientable. We find there are 10 combinatorially regular polyhedra of index 2 with vertices on one orbit, and 22 infinite families of combinatorially regular polyhedra of index 2 with vertices on two orbits, where polyhedra in the same family differ only in the relative diameters of their vertex orbits. For each such polyhedron, or family of polyhedra, we provide the underlying map, as well as a geometric diagram showing a representative face for each face orbit, and a verification of the polyhedron's combinatorial regularity. A self-contained completeness proof is given. Exactly five of the polyhedra have planar faces, which is consistent with a previously known result. We conclude by describing a non-Petrie duality relation among regular polyhedra of index 2, and suggest how it can be extended to other combinatorially regular polyhedra.

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# **Introduction**

## **Background and Context**

The definition of a finite regular polyhedron in Euclidean 3-space has changed over the centuries, and as a result so has the number of such polyhedra, as Coxeter [6] and Grünbaum [17], among other authors, have pointed out. Kepler appears to have been the first to realize that some star polyhedra, built by stellation from the dodecahedron or icosahedron, are regular if the convexity requirement is dropped. Poincaré later rediscovered Kepler's two polyhedra, and used star vertex-figures to construct the remaining two regular star polyhedra. Thus as the notion of a face (or vertex-figure) was expanded to include self-intersecting boundary edges, the four Kepler-Poincaré polyhedra joined the five classical Platonic solids as regular polyhedra.

In the 1920's and 1930's, Coxeter and Petrie discovered that there are additional regular polyhedra if planarity is removed as a requirement of vertex-figures; these polyhedra, known as the Petrie-Coxeter polyhedra, are infinite and have convex faces but skew vertex-figures. Petrie also was the first to observe that the family of "Petrie polygons" of a regular map (cell-decomposition) on a closed surface (real 2-manifold) usually forms another regular map on another closed surface, called the "Petrie dual" of the original map; here, a Petrie polygon is a path along edges such that every two, but no three, consecutive edges lie in the same face. In the 1970's, Grünbaum [19] restored the symmetry in the definition by admitting the possibility that a polyhedron has skew faces.

Thus a polyhedron became a geometric graph in space equipped with a distinguished (connected) family of simple edge-cycles, called faces, independent of whether or not topological discs (membranes) could be spanned into these cycles to result in a surface free of self-intersections. As the generally accepted definition of regularity changed, the Petrie duals of the known regular polyhedra were recognized as regular polyhedra, bringing the number of finite regular polyhedra to eighteen (see [21, §7E]).

These gradual changes in perception of a polyhedron emphasized the combinatorial aspects of the structure over geometric ones, and went hand-in-hand with the gradual change in defining the term ‘regular’, eventually leading to the modern definition in terms of a flag-transitive symmetry group. The combinatorial structure of a finite polyhedron can be represented by a finite map, which is a cell-decomposition of a closed compact surface into topological polygons. Coxeter [6] and Coxeter-Moser [8] listed all regular maps of genus less than 3. This was extended by Sherk [27] and Garbe [13], who classified the orientable maps of genus 3, and 4, 5, and 6, respectively. Subsequently, Conder and Dobesanyi [4] using computer search expanded this list to regular maps of genus up to 15, and more recently Conder [5] has listed all regular maps of genus up to 100 in the orientable case, and 200 in the nonorientable case.

Less is known about the geometric realizations of these combinatoric structures. Not every regular map has a polyhedral realization, although most do. Conversely, two or more geometrically dissimilar polyhedra may be realized from the same regular map. Each map in Appendix 1 provides an example of this. Every geometric polyhedron

realizing a regular map has associated with it the (geometric) symmetry group, which can be viewed as a subgroup of the (combinatorial) automorphism group of the underlying regular map. When the index of this subgroup is finite, it is called the index of the polyhedron. We shall only consider finite polyhedra of small index, usually of index 2. A polyhedron has index 1 if and only if it is geometrically regular. The 18 polyhedra mentioned above are the only finite regular polyhedra of index 1. Wills [29] showed there are precisely five orientable finite regular polyhedra of index 2 with planar faces. For photographs of these, see Richter [23].

## Synopsis

The main result of this paper is to classify all finite regular polyhedra of index 2, regardless of orientability or planarity of faces or vertex-figures. Section 2 sets out definitions and basic properties of maps and combinatorially regular polyhedra. In Section 3, covering results applicable throughout the paper, we show that the symmetry group of a combinatorially regular polyhedron of index 2 can not be the rotation group of a Platonic solid. We also introduce a notation describing the shape of faces. This allows the development of powerful insights into the symmetry constraints on faces of combinatorially regular polyhedra. It is normal for such classification proofs, particularly those that principally proceed by examining combinatorial properties such as the polyhedral type, to contain extensive structure-by-structure analysis; in this case the additional emphasis on geometric properties of the polyhedra, especially the face-shape and relative position of the vertices, provides general results, so that the amount of specific analysis is significantly reduced. This enables the completeness proof to be a

‘stand alone’ one, without recourse to computer generated lists of maps, or leaving much verification to the reader.

Sections 4 and 5 cover the polyhedra with two and one vertex orbits, respectively. The main results here are Theorems 4.1 and 5.4, which set out all possibilities for finite regular polyhedra of index 2 with those vertex orbits. That each of the structures described in those theorems is indeed a regular index 2 polyhedron is shown in Appendices 1 and 2, which contain maps, geometric diagrams of representative faces, and verifications of combinatorial regularity. Appendix 3 contains verification of some results concerning polyhedra with edges of different lengths that were suggested, but not proved, in Section 5.

In practice, the regular index 2 polyhedra were first found using a spreadsheet program, and many of the constructs of the proof given here, in particular the face shape and the calculation of the face stabilizer under the rotation group, originated in the techniques developed for that search. However, the classification proof presented here is independent of the computer search.

## Section 2: Basic Notions

### Maps

Coxeter-Moser [8] defines a map  $M$  as a decomposition of a closed 2-manifold into non-overlapping, simply-connected regions by means of arcs. The regions are called the faces of  $M$ , the arcs are called the edges of  $M$ , and the intersections of the arcs are called the vertices of  $M$ . Thus each edge is incident with precisely two vertices, one at each end, and subtends precisely two faces. In this way, the vertices, edges, and faces of a map form a partially ordered set, which usually is identified with the underlying map. In most cases,  $M$  will be finite, so that the underlying surface is compact.

The cyclically ordered set of edges bordering each face is called the face boundary (or sometimes the edge boundary) of that face. The  $f$ -vector of  $M$  is the triplet  $(f_0, f_1, f_2)$ , where  $f_i$  is the number of vertices, edges, and faces, respectively. A map is finite if each of these is finite. A flag of  $M$  is a set consisting of one vertex, one edge incident with this vertex, and one face containing this edge. The map is said to be of type  $\{p, q\}$  if each of its faces is a topological  $p$ -gon (that is, has a face boundary consisting of  $p$  edges), and if  $q$  edges meet at each vertex. We always assume that  $p$  and  $q$  are finite.

The combinatorial automorphism group of  $M$  is the group of incidence preserving bijections of the partially ordered set of vertices, edges, and faces of  $M$ , and is denoted by

$\Gamma(M)$ . Following McMullen and Schulte [21], we say that a map  $M$  is (combinatorially) regular if it is flag-transitive. Thus a regular map is reflexible in the sense of [8].

The automorphism group  $\Gamma(M)$  of a regular map  $M$  is generated by three involutions,  $\rho_0$ ,  $\rho_1$ , and  $\rho_2$ . For a fixed flag, consisting of a vertex  $F_0$ , an edge  $F_1$ , and a face  $F_2$ , these generators can be defined by

$$(I) \quad \rho_i(F_j) = F_j \text{ if and only if } j \neq i, \text{ for } i, j = 0, 1, 2.$$

These generators satisfy the standard relations

$$(II) \quad \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = 1,$$

but in general satisfy other independent relations also.

The automorphism  $\rho_0\rho_1$  cyclically permutes the successive edges on the edge boundary of the face  $F_2$ , and the automorphism  $\rho_1\rho_2$  cyclically permutes the successive edges meeting at the vertex  $F_0$  of this face. Following McMullen and Schulte [21], for a regular map, we define the automorphisms,  $\sigma_1$  and  $\sigma_2$ , by  $\sigma_1 := \rho_0\rho_1$  and  $\sigma_2 := \rho_1\rho_2$ , with the convention that  $\rho_i\rho_j(x) := \rho_i(\rho_j(x))$ . The elements  $\sigma_1$  and  $\sigma_2$  generate the combinatorial rotation subgroup  $\Gamma^+(M)$  of  $\Gamma(M)$ , consisting of the automorphisms that can be expressed as products of an even number of generators from  $\rho_0$ ,  $\rho_1$ , and  $\rho_2$ .

A fundamental region for the group  $\Gamma(M)$  of a regular map  $M$  is given by any triangle of the barycentric subdivision of  $M$  on the surface of  $M$ . For the triangle determined by the flag  $\{F_0, F_1, F_2\}$  (in other words, the triangle whose vertices are  $F_0$ , the mid-point of  $F_1$ ,

and the center of  $F_2$ ), the involutions  $\rho_i$  can be thought of as reflections in the sides of the triangle, each  $\rho_i$  reflecting in the side of the triangle opposite the vertex of the triangle lying in  $F_i$ , as specified in (I).

Every map  $M$ , with  $f$ -vector  $(f_0, f_1, f_2)$ , has associated with it a dual map  $M^*$  with  $f$ -vector  $(f_2, f_1, f_0)$ , with one vertex of  $M^*$  lying in each face of  $M$ , and vice versa, and with each edge of  $M^*$  crossing one edge of  $M$ . Thus if  $M$  is of type  $\{p, q\}$  then  $M^*$  is of type  $\{q, p\}$ .

Additionally, if  $M$  is regular then it also has associated with it a polygon called the Petrie polygon. Given  $M$ , a Petrie polygon is a 'zigzag' along the edges of  $M$  such that every two but no three successive edges of the polygon are edges of a single face of  $M$ . Because of the flag-transitivity of  $\Gamma(M)$ , the Petrie polygons of  $M$  are all combinatorially equivalent under  $\Gamma(M)$ , so we may talk about the Petrie polygon of  $M$ .

Identifying those pairs of vertices of the regular tessellation  $\{p, q\}$  on the 2-sphere or in the Euclidean or hyperbolic plane which are separated by  $r$  steps along a Petrie polygon of the tessellation gives, for suitable values of  $r$ , a finite regular map  $M$  denoted by  $\{p, q\}_r$ . The surface of this map is orientable if and only if  $r$  is even. The value of  $r$  gives us one further relation to complement those in (II).

$$(III) \quad (\rho_0\rho_1\rho_2)^r = 1.$$

The map  $\{p, q\}_r$  then has Petrie polygons of length  $r$ .

More generally, if a regular map  $M$  of type  $\{p, q\}$  has Petrie polygons of length  $r$ , then its group  $\Gamma(M)$  has a presentation consisting of the standard relations (II), the Petrie relation (III), and in general some additional independent relations. Thus the map  $M$  is obtained from the corresponding (finite or infinite) map  $\{p, q\}_r$  by making suitable identifications; that is,  $M$  is a quotient map of  $\{p, q\}_r$ . In other words, the regular maps  $\{p, q\}_r$  are universal among all regular maps of type  $\{p, q\}$  with Petrie polygons of length  $r$ .

We say that a regular map  $M$  is of type  $\{p, q\}_r$  if it is of type  $\{p, q\}$  and has Petrie polygons of length  $r$ . Thus  $\{p, q\}_r$  is the ‘largest’ regular map of type  $\{p, q\}_r$ . Note that most of the universal maps  $\{p, q\}_r$  are infinite, even though there frequently exist maps of type  $\{p, q\}_r$  which are finite. Most regular polyhedra of index 2 provide examples of this kind.

If we leave the vertices and edges of the regular map  $M$  of type  $\{p, q\}_r$  alone, and replace the faces by the Petrie polygons, we get a new map of type  $\{r, q\}_p$  with the same automorphism group, but not necessarily on the same surface. Thus there is one common automorphism group for six related regular maps of type  $\{p, q\}_r$ ,  $\{p, r\}_q$ ,  $\{q, p\}_r$ ,  $\{q, r\}_p$ ,  $\{r, p\}_q$ , and  $\{r, q\}_p$ . (see [8], [20], [21], [29]).

For a map on a closed surface of genus  $g$ , denote by  $\chi$  the Euler characteristic,  $\chi := f_0 - f_1 + f_2$ . The well-known Euler-Poincaré relations then give  $\chi = 2 - 2g$  if the underlying surface is orientable, and  $\chi = 2 - g$  if the surface is non-orientable.

For a regular map, we also have the combinatorial identities  $qf_0 = 2f_1$ , and  $pf_2 = 2f_1$ , which can be obtained by counting edge ‘ends’ and edge ‘sides’, respectively.

## Polyhedra

A polyhedron,  $P$ , is a faithful geometric realization in Euclidean 3-space,  $\mathbf{E}^3$ , of a map,  $M$  (see [21, Ch. 5]). A face,  $F$ , of  $P$  is determined by its edge boundary, which is a closed simple sequence of edges of  $P$ . Thus the vertices, edges, and faces of  $P$  are faithfully represented respectively by points, line segments, and closed simple sequences of edges; that is, there exist bijections between the sets of vertices, edges, and faces of  $M$  and the sets of vertices, edges, and faces of  $P$ , respectively. If  $M$  is a regular map, then  $P$  is called a combinatorially regular polyhedron. We note that faces of  $P$  can be specified by the cyclically ordered vertices on the edge ‘boundary’ of the face. An edge boundary of a face of a combinatorially regular polyhedron need not be planar or convex.

We note that a consequence of the definition given above of a combinatorially regular polyhedron,  $P$ , as a faithful realization of a regular map,  $M$ , through bijections between  $M$  and  $P$ , is that no face of  $P$  has a vertex occurring more than once in its edge boundary.

We refer to this as the “non-repetitive property” of faces.

According to our definition, a polyhedron  $P$  is a geometric graph in space with a distinguished family of edge cycles, the faces of  $P$ . These faces may be planar or non-

planar (skew) polygons, but a priori their edge boundary does not actually bound a topological disc that is made part of the definition of the polyhedron. Still, we can often span a topological disc (usually piecewise linear) into the faces, thereby obtaining a geometric model for the underlying surface of  $M$  embedded into space. Typically, this model will have self-intersections and will have features not representative of the original polyhedron.

The polyhedral realizations that we consider do not include realizations of regular maps where more than one edge joins two vertices or subtends two faces. Thus we do not consider the degenerate dihedra  $\{p, 2\}$  or hosohedra  $\{2, q\}$ . Specifically, we require that  $p \geq 3$  and  $q \geq 3$ , for maps of type  $\{p, q\}$ , in order to achieve non-degenerate geometric realization in  $\mathbf{E}^3$ . A consequence of this constraint is that it is not the case for any combinatorially regular polyhedron that two faces share two consecutive edges in their edge boundaries; if it were so it would require that  $q = 2$ .

The geometric symmetry group,  $G(P)$ , of  $P$  consists of the isometries of  $\mathbf{E}^3$  that map  $P$  to itself. This is the group of rotations and (point or plane) reflections that preserve  $P$ , and can be viewed as a subgroup of  $\Gamma(M) = \Gamma(P)$ , when we have identified  $M$  and  $P$ . The index of  $P$  is defined to be the index of this subgroup in  $\Gamma(P)$ .

If the geometric symmetry group of a polyhedron  $P$  is flag-transitive,  $P$  is said to be a (geometrically) regular polyhedron. A combinatorially regular polyhedron of index  $n$  is also referred to as a regular polyhedron of index  $n$ . Thus a "regular polyhedron of index

$2^n$  is a combinatorially regular, geometric polyhedron whose geometric symmetry group,  $G(P)$ , has index 2 in the combinatorial automorphism group,  $\Gamma(M)$ . An immediate consequence of this is that a regular polyhedron of index 2 has precisely two flag orbits under  $G(P)$ .

An important subgroup of  $G(P)$  is  $G^+(P)$ , the group of rotations that preserve  $P$ . If  $P$  is non-orientable, the rotation group,  $G^+(P)$ , is the same as the symmetry group,  $G(P)$ ; but if  $P$  is orientable, then  $G^+(P)$  is a subgroup of  $G(P)$  of index 2.

It follows from the enumeration of the finite subgroups of  $O(3)$  (see [14]), that if  $P$  is a regular polyhedron of index 2, then its geometrical symmetry group,  $G(P)$ , is either  $G(Q)$  or  $G^+(Q)$ , where  $Q$  is a Platonic solid, or is isomorphic to a cyclic or dihedral group (in which case  $G(P)$  is reducible). Wills [29, Lemma 1] showed that there is no combinatorially regular polyhedron of index 2 with a cyclic or dihedral group as symmetry group. Although he used a more restrictive definition of a regular polyhedron of index 2, the proof carries through for the definition we have adopted.

Lemmas 3.4 and 3.5 in the next section show that when  $P$  is a regular index 2 polyhedron there are three possibilities for its face orbits: either  $P$  has one face orbit under both  $G(P)$  and  $G^+(P)$ ; or  $P$  has two face orbits under both  $G(P)$  and  $G^+(P)$ ; or  $P$  has one face orbit under  $G(P)$  and two face orbits under  $G^+(P)$ . The following lemmas detail the possibilities for vertex orbits under  $G(P)$ .

**Lemma 2.1:** Let  $Q$  be a Platonic solid  $\{p, q\}$ , let  $\{v, e, f\}$  be a flag of  $Q$ , and let  $T$  be the triangle whose vertices are  $v$ , the center of  $e$ , and the center of  $f$ . Then

- (a)  $T$  is a fundamental region for the action of  $G(P)$  on the boundary of  $Q$ . In particular, the orbit of every point on the boundary of  $Q$  under  $G(P)$  meets  $T$  in exactly one point, and no two points of  $T$  are equivalent under  $G(P)$ ;
- (b) There are seven types of orbits of boundary points of  $Q$  under  $G(P)$ , the type depending on whether its representative point in  $T$  lies at one of the three vertices of  $T$ , in the relative interior of one of the edges of  $T$ , or in the relative interior of  $T$ . Accordingly, the size of the orbit is  $f_0$ ,  $f_1$ ,  $f_2$ , or  $pf_2$ ,  $2f_1$ ,  $pf_2$ , or  $2pf_2 = |G(Q)|$ , respectively, where  $f_0$ ,  $f_1$ ,  $f_2$  are the number of vertices, edges, and faces of  $Q$ .

**Proof:** Part (a) is well known, and part (b) follows immediately by inspection of the possible types of orbits.

■

**Lemma 2.2:** If  $P$  is a regular polyhedron of index 2 with  $G(P) = G(Q)$ , where  $Q$  is a Platonic solid, then each vertex orbit of  $P$  under  $G(P)$  is of one of the seven types described in Lemma 2.1(b), and the vertex set of  $P$  is the disjoint union of such orbits.

**Proof:** This follows from Lemma 2.1 if we regard  $T$  as an infinite simplicial cone emanating from the center of  $Q$ , rather than a triangle. This we may do as  $P$  and  $Q$  are geometric realizations. ■

**Lemma 2.3:** If  $P$  is a regular polyhedron of index 2, then, under action of  $G(P)$ ,  $P$  has at most two vertex orbits, at most two edge orbits, and at most two face orbits.

**Proof:** Since  $P$  is a regular polyhedron of index 2, it has exactly two flag orbits under  $G(P)$ . Select two flags of  $P$  on different orbits to represent the two flag orbits. Then if  $v$  is a vertex of  $P$ , choose a flag that contains  $v$  and map this flag to the flag representing its orbit. It follows that  $v$  is in the same vertex orbit as the vertex from the representing flag. Thus there can be at most two vertex orbits of  $P$  under  $G(P)$ . The argument for edge orbits and face orbits is similar.

■

The Petrie dual (sometimes called the Petrial) of a combinatorially regular polyhedron  $P$  is the structure sharing the same vertices and edges as  $P$ , but with Petrie polygons of  $P$  as faces. Thus if the underlying map of  $P$  has type  $\{p, q\}_r$ , then the underlying map of the Petrie dual of  $P$  has type  $\{r, q\}_p$ . Since Petrie polygons of  $P$  are all combinatorially equivalent, the Petrie dual of  $P$  is well defined. We have that the Petrie dual of the Petrie dual is the original polyhedron. For reference, we list the following result, due to McMullen and Schulte [21, Lemma 7B3], as a lemma.

**Lemma 2.4:** The Petrie dual of a combinatorially regular polyhedron is a combinatorial polyhedron (and thus combinatorially regular) if and only if a Petrie polygon of the

original polyhedron visits any given vertex at most once (i.e. if and only if a Petrie polygon of the original polyhedron satisfies the non-repetitive property).

## Regular Compounds

A regular polyhedral compound is a structure, composed of several combinatorially regular polyhedra, which is vertex-transitive, edge-transitive, and face-transitive. There are five regular compounds (see [6]), and we shall refer to three of them. The compounds are:

i) the compound of two tetrahedra, sometimes called the stella octangula. The convex hull of the two tetrahedra is a cube, where the diagonals of the faces of the cube coincide with the edges of the tetrahedra and the edges of the cube connect vertices on different tetrahedra. This observation shows that the edge graph of the cube is bipartite. The intersection of the two tetrahedra is an octahedron, which shares the same face-planes as the compound, so that the compound is a stellation of the octahedron.

ii) the compound of five cubes. The convex hull of the five cubes is a dodecahedron. Each edge of any given cube is a diagonal of one of the twelve faces of the dodecahedron, and the five diagonals of any given face of the dodecahedron coincide with an edge of each of the five cubes. The intersection of the vertex sets of any two of

the cubes is a single pair of opposite vertices of the dodecahedron, and each of the ten pairs of opposite vertices of the dodecahedron lie on a (different) pair of cubes.

iii) the compound of five octahedra. The convex hull of the five octahedra is an icosidodecahedron, and the intersection of them is an icosahedron, so that the compound is also a stellation of the icosahedron. This compound is dual to the compound of five cubes, in the sense of duality between faces and vertices. The intersection of the vertex sets of any two of the octahedra is empty.

iv) and v) The other two regular compounds, which are not referred to in this paper, are the compound of ten tetrahedra, formed from the compound of five cubes by making each of the cubes a compound of two tetrahedra; and the “chiral” compound of five tetrahedra, which is formed from the compound of ten tetrahedra by removing five tetrahedra which are isomorphic under rotational icosahedral symmetry.

## Section 3: General Results

### Symmetry Group of P

**Theorem 3.1:** There is no regular index 2 polyhedron  $P$  whose symmetry group  $G(P)$  is the rotational group of a Platonic solid.

Before proving this, we establish three lemmas.

Throughout this subsection let  $P$  be a regular index 2 polyhedron whose symmetry group  $G(P)$  is the rotation group of a Platonic solid  $S$ , that is  $G(P) = G^+(S)$ . Let the  $f$ -vector of  $P$  be  $(f_0, f_1, f_2)$ .

**Lemma 3.1:** The vertices of  $P$  lie on the vertices of  $S$  or its (polar) dual.

**Proof:** We have that  $2f_1/f_0 = q \geq 3$ . Since  $G(P) = G^+(S)$  and  $P$  has index 2,  $P$  has the same number,  $f_1$ , of edges as  $S$ .

Up to scaling, every vertex of  $P$  is equivalent, under  $G^+(S)$ , to a vertex in the fundamental triangle  $T$  for  $G^+(S)$  on the boundary of  $S$ . We may take  $T$  to consist of two adjacent fundamental triangles for  $G(S)$ , such that  $T$  has two of its vertices at vertices of  $S$  and the third vertex at a face center of  $S$ . Thus, each vertex of  $P$  has a representative

vertex (modulo  $G^+(S)$ ) lying in one of the seven (non-empty) faces of  $T$  (at the vertices, on the edges, or in the interior). For four of these faces (the edges and the interior),  $f_0$  would be at least as great as the number of edges of  $S$ , which is  $f_1$ , but this contradicts  $f_0 \leq 2 f_1/3$ . In the other three cases, the vertices of  $P$  coincide, up to scaling, with the vertices of  $S$  or the centers of the faces of  $S$ . But the centers of the faces of  $S$  are just the vertices of the polar dual of  $S$ . Note here that the vertices of  $P$  must lie in one orbit under  $G^+(S)$ , for if not,  $f_0$  would be such that either  $q < 3$  or  $q$  does not divide  $|G^+(S)|$ .

■

**Corollary:** No edge of  $P$  joins opposite vertices of  $P$  (when  $S$  is centrally symmetric).

**Proof:** If so, then by the rotational symmetry of  $S$  such an edge would lie on at least three faces of  $P$ .

■

From now on we assume that the vertex set of  $P$  coincides with the vertex set of  $S$ . This is justified by Lemma 3.1. Then, since  $\Gamma(P)$  and  $G(S)$  have the same order, we also know that the Schläfli symbols of  $P$  and  $S$  have the same second entry,  $q$ , and hence  $q = 3, 4, 5$ .

**Lemma 3.2:** All edges of  $P$  have the same length.

**Proof:** If  $q$  is odd, then the edges emanating from a vertex are equivalent under the  $q$ -fold rotation about the vertex; hence all edges are equivalent under  $G^+(S) = G(P)$ . Thus if  $P$

has edges of two different lengths, then  $q$  is even. The only Platonic solid with even  $q$  is the octahedron. But if  $S$  is the octahedron and  $P$  has edges of different length, the longer edges must be between opposite vertices of  $S$ . However, no edge of  $P$  can join opposite vertices of  $P$ , by the Corollary above.

■

**Lemma 3.3:** Let  $F$  be any face of  $P$  such that  $F$  has a plane of symmetry,  $H$ . Then  $H$  is not a reflection plane of  $S$ .

**Proof:** Suppose that  $H$  is a reflection plane of  $S$ , and  $r_H$  is the reflection in  $H$ , so that  $r_H \in G(S)$  and hence the reflection  $r_H$  in  $H$  maps the vertex set  $V$  of  $P$  onto itself.

If  $q$  is even then  $q = 4$ , and  $S$  is the octahedron. The edge length of  $P$  must be the same as that of  $S$ , by the Corollary to Lemma 3.1. The only such polyhedra are the octahedron, and the Petrie dual of the octahedron, both of which are index 1, not 2.

So we may suppose that  $q$  is odd, so that  $q = 3$  or  $5$ . Then  $\sigma_2 \in G(P) = G^+(S)$ , since  $\sigma_2^2 \in G(P)$ , so  $G(P)$  contains the  $q$ -fold rotational symmetry of  $P$  about the base vertex of  $P$ . Hence  $\sigma_2$  cyclically permutes the  $q$  faces of  $P$  around the base vertex and, since  $G(P)$  is vertex transitive, there are similar such symmetries at every vertex of  $P$ . It follows that  $G(P)$  is face transitive; in fact, since  $P$  is connected we can move every face of  $P$  to every other face of  $P$  by successively applying symmetries that cyclically rotate about vertices.

Let  $F'$  be any face of  $P$ . Since  $G(P)$  is face transitive and  $G(P) = G^+(S)$ , we have  $F' = s'(F)$  for some  $s' \in G^+(S)$ . Since  $r_H \in G(S)$  and  $G^+(S)$  is normal in  $G(S)$ , we have  $r_H s' r_H = s''$  for some  $s'' \in G^+(S)$ . Hence, since  $r_H(F) = F$ , we have  $r_H(F') = r_H s'(F) = s'' r_H(F) = s''(F)$ .

But  $s'' \in G(P) = G^+(S)$ , so  $s''(F)$  is a face of  $P$ . Thus  $r_H(F')$  is a face of  $P$ , and so  $r_H$  maps every face  $F'$  of  $P$  to a face of  $P$ . Thus  $r_H$  is a symmetry of  $P$ , a contradiction to our assumption that  $G(P) = G^+(S)$ .

■

We can now prove Theorem 3.1.

**Proof of Theorem 3.1:** Suppose there is a regular index 2 polyhedron  $P$  whose symmetry group  $G(P)$  is the rotation group of a Platonic solid  $S$ . As before, we assume that the vertex set  $V$  of  $P$  coincides with the vertex set of  $S$ .

Now let  $F$  be any face of  $P$ , and let  $v_1, v_2, v_3$  be three consecutive vertices of  $F$ . Since  $P$  is a regular index 2 polyhedron,  $\sigma_1^2$  must be in  $G(P)$  and must be either a rotation or a rotary reflection. But  $G(P)$  is the rotation group of  $S$ , so  $\sigma_1^2$  can not be a rotary reflection. Since  $\sigma_1^2$  is a rotation and all edges of  $P$  have the same length, the perpendicular bisector  $H$  of  $v_1$  and  $v_3$  contains the vertex  $v_2$  of  $F$  and hence the face  $F$  of  $P$  has mirror symmetry about  $H$ . However, since  $G(P)$  is the rotation group of  $S$ ,  $H$  can not be a plane of symmetry of the entire polyhedron  $P$ .

Let  $k$  be the length of the minimal path from  $v_1$  to  $v_3$  along edges of  $S$ , so that  $k = 1$  if  $S$  is  $\{3,3\}$ ;  $k \leq 2$  if  $S$  is  $\{3,4\}$ ;  $k \leq 3$  if  $S$  is  $\{4,3\}$  or  $\{3,5\}$ ; and  $k \leq 5$  if  $S$  is  $\{5,3\}$ . We consider the various possibilities for  $k$ . In each case we shall arrive at a contradiction.

If  $k = 1$ , or if  $k = 2$  and  $S$  is the dodecahedron,  $H$  will bisect an edge of  $S$ . Otherwise, if  $k$  is even, an edge of  $S$  will lie in  $H$ . In all these cases,  $H$  actually is a reflection plane of  $S$ , in contradiction of Lemma 3.3.

If  $v_1$  and  $v_3$  are opposite vertices of the cube, the icosahedron, or the dodecahedron (and hence  $k = 3, 3$ , or  $5$ , respectively), then  $H$  is not a reflection plane of  $S$ , but in each of these cases  $V \cap r_H(V) = \{v_1, v_3\}$ , so, in particular,  $r_H$  does not map  $F$  onto itself. However, this contradicts the fact that  $F$  has mirror symmetry with respect to  $H$ .

The only remaining case occurs when  $S$  is the dodecahedron, and  $k = 3$ , as shown in the diagram, with  $A, B, C, D$  labeling vertices of  $S$  and the lines indicating edges of  $S$ .

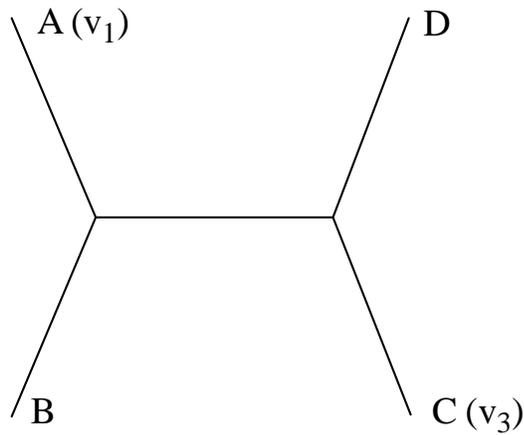


Figure 3.1

Since  $ABCD$  is a planar rectangular rhombus, it is a square. Together with the four opposite vertices,  $A', B', C', D'$  it forms a cube inscribed in the dodecahedron. So the reflection  $r$  in the plane containing  $BDB'D'$  will transpose  $A$  and  $C$ , and  $A'$  and  $C'$ . None of the other 12 vertices of  $S$  are mapped by  $r$  onto a vertex of  $S$ , so all vertices of  $F$  must lie on this cube; note for the latter that  $r$  is just the reflection  $r_H$ , and that  $F$  is invariant under  $r_H$ . This is the compound of five cubes described in Section 2. Each edge of the cube is a diagonal of one of the twelve faces of  $S$ . Since each face of  $S$  has five diagonals, there are five such cubes inscribed in  $S$ , and  $G(P) = G^+(S)$  acts as a permutation group on this set of cubes. The intersection of the vertex sets of any two of these cubes is a single pair of opposite vertices of  $S$ . Since all vertices of  $F$  lie on a single cube, and since no edge of  $P$  can join opposite vertices, no edge of  $F$  connects a pair of vertices common to two cubes.

Thus  $G(P)$  acts on  $F$  and its images to produce a compound of five (pairwise

disconnected) objects, rather than a polyhedron. (Note here that  $q = 3$ , so  $\sigma_2 \in G(P)$  and hence  $G(P)$  would have to act transitively on the faces of  $P$ .)

■

## Face Stabilizers

The classification of regular polyhedra of index 2 makes continual use of the stabilizer of a face of a polyhedron, so for completeness we review the results for face stabilizers that are needed. We adopt the following notation, repeating the first four for convenience:

$\Gamma(P)$  := combinatorial automorphism group of  $P$

$\Gamma^+(P)$  := combinatorial rotation subgroup of  $P$

$G(P)$  := geometric symmetry group of  $P$

$G^+(P)$  := rotation subgroup of  $G(P)$

$\Gamma_F(P)$  := stabilizer of face  $F$  in  $\Gamma(P)$

$G_F(P)$  := stabilizer of face  $F$  in  $G(P)$

$\Gamma_F^+(P) := (\Gamma_F(P))^+ :=$  stabilizer of face  $F$  in  $\Gamma^+(P)$

$G_F^+(P) := (G_F(P))^+ :=$  stabilizer of face  $F$  in  $G^+(P)$

The following facts are well known:

1.  $\Gamma(P) = \langle \rho_0, \rho_1, \rho_2 \rangle$ , because  $P$  is combinatorially regular. Here,  $\rho_0, \rho_1, \rho_2$  are the distinguished generators of  $\Gamma(P)$  given by a fixed base flag.
2.  $\Gamma^+(P) = \langle \sigma_1, \sigma_2 \rangle$ , with  $\sigma_1 := \rho_0\rho_1, \sigma_2 := \rho_1\rho_2$ , and  $\rho_0, \rho_1, \rho_2$  as in (1).
3.  $\Gamma_F(P) = \langle \rho_0(F), \rho_1(F) \rangle \approx D_p$ , where  $\rho_0(F)$  is a (combinatorial) reflection symmetry of  $F$  where the “plane of reflection” intersects an edge  $\{u,v\}$  of  $F$ , and  $\rho_1(F)$  is a combinatorial reflection symmetry of  $F$  where the vertex  $v$  of  $F$  lies on the “plane of reflection”. Note that  $\rho_0(F), \rho_1(F)$  are conjugates of  $\rho_0, \rho_1$  under a mapping that takes the base flag to the flag given by  $v, \{u,v\}$ , and  $F$ .
4.  $|\Gamma(P) : \Gamma_F(P)|, |G(P) : G_F(P)|, |G^+(P) : G_F^+(P)|$  is the length (number of elements) of the orbit of  $F$  under the action of  $\Gamma(P), G(P),$  or  $G^+(P)$  respectively, on the faces of  $P$ .
5.  $f_2 = |\Gamma(P) : \Gamma_F(P)| = |\Gamma(P)| / |\Gamma_F(P)| = 2|G(P)| / |\Gamma_F(P)| = |G(P)| / p$ .

The two following lemmas describe the actions of  $G(P)$  and  $G^+(P)$ :

**Lemma 3.4:** Let  $F$  be a face of  $P$ . Then

- a)  $G_F(P)$  is a subgroup of  $\Gamma_F(P)$  of index 1 or 2.
- b)  $G(P)$  has at most two orbits on the faces of  $P$ .

- c)  $\Gamma_F(P) = G_F(P)$  (that is, the index is 1) if and only if there are two orbits of faces under  $G(P)$ .
- d)  $G_F(P)$  is of index 2 in  $\Gamma_F(P)$  if and only if  $G(P)$  acts face transitively on  $P$  (that is, there is just one face orbit under  $G(P)$ ).

Thus there are two orbits of  $F$  under  $G(P)$  if and only if  $|G_F(P)| = |\Gamma_F(P)| = 2p$ , and there is one orbit of  $F$  under  $G(P)$  if and only if  $|G_F(P)| = \frac{1}{2}|\Gamma_F(P)| = p$ .

**Proof:** Let  $i := |\Gamma_F(P) : G_F(P)|$ . Then

- a) By facts 4 and 5 above,  $f_2 = |\Gamma(P) : \Gamma_F(P)| = |\Gamma(P)| / |\Gamma_F(P)| = 2|G(P)| / i|G_F(P)| = 2|G(P) : G_F(P)| / i \leq 2f_2 / i$ . Hence  $i \leq 2$ .
- b) From part (a) we obtain  $2|G(P) : G_F(P)| = if_2$ , so  $|G(P) : G_F(P)| = f_2/2$  or  $f_2$ , according as  $i = 1$  or  $2$ . Hence there are at most two orbits of  $G(P)$  on the faces.
- c) If  $\Gamma_F(P) = G_F(P)$ , then  $f_2 = 2|G(P) : G_F(P)|$ , so  $|G(P) : G_F(P)| = f_2 / 2$ ; that is, there are two face orbits under  $G(P)$ , each of size  $f_2 / 2$ . Conversely, if there are two orbits under  $G(P)$ , then  $|G(P) : G_F(P)| = f_2 / 2$ , so we must have  $i = 1$ , that is,  $\Gamma_F(P) = G_F(P)$ .
- d) Part (d) is a restatement of part (c). We can prove it directly, as follows. If  $G_F(P)$  is of index 2 in  $\Gamma_F(P)$ , then  $i = 2$  and  $f_2 = |G(P) : G_F(P)|$ ; hence there is just one face orbit under  $G(P)$ . Conversely, if there is just one face orbit under  $G(P)$ , then  $f_2 = |G(P) : G_F(P)|$  and hence  $i = 2$ .

■

It follows from Lemma 3.4 that:

- 1) If there are two orbits of faces under  $G(P)$ , then

$G_F(P) = \Gamma_F(P) = \langle \rho_0(F), \rho_1(F) \rangle \approx D_p$ . That is, each combinatorial symmetry of  $P$  that preserves  $F$  can be realized by a geometric symmetry of  $P$  that preserves  $F$ . In particular,  $\sigma_1(F) \in G_F(P)$ , where  $\sigma_1(F) := \rho_0(F) \cdot \rho_1(F)$ .

- 2) If  $G_F(P)$  is of index 2 in  $\Gamma_F(P)$ , then there are two possibilities: either

- a)  $G_F(P) \approx C_p$ ,  $p$  even or odd, whence  $\sigma_1(F) \in G_F(P)$ ; or  
b)  $G_F(P) \approx D_{p/2}$ ,  $p$  even. Then  $\sigma_1(F)$  is not in  $G_F(P)$ , but  $\sigma_1^2(F) \in G_F(P)$ .

Here  $G_F(P) = \langle \rho_0(F), \sigma_1(F)\rho_0(F)\sigma_1^{-1}(F) \rangle$  or  $\langle \rho_1(F), \sigma_1(F)\rho_1(F)\sigma_1^{-1}(F) \rangle$ .

**Lemma 3.5:** Let  $F$  be a face of  $P$ . Then

- a)  $G_F^+(P)$  is a subgroup of  $\Gamma_F(P)$  of index 2 or 4.  
b)  $G_F^+(P)$  is of index 2 in  $\Gamma_F(P)$  if and only if there are two orbits of faces under  $G^+(P)$ .  
c)  $G_F^+(P)$  is of index 4 in  $\Gamma_F(P)$  if and only if  $G^+(P)$  acts face transitively on  $P$  (that is, there is just one face orbit under  $G^+(P)$ ).

**Proof:** Let  $j := |\Gamma_F(P) : G_F^+(P)|$ . Then

- a) By facts 4 and 5 above,  $f_2 = |\Gamma(P) : \Gamma_F(P)| = |\Gamma(P)| / |\Gamma_F(P)| = 4|G^+(P)| / j|G_F^+(P)|$   
 $= 4|G^+(P) : G_F^+(P)| / j \leq 4f_2 / j$ . Hence  $j \leq 4$ , so  $j = 1, 2$ , or  $4$ , since it is clear

that  $j|4$ . (Note that  $j = |\Gamma_F(P) : G_F^+(P)| = |\Gamma_F(P) : G_F(P)| \cdot |G_F(P) : G_F^+(P)|$ , and both factors on the right are 1 or 2).

If  $j = 1$ , then  $\Gamma_F(P) = G_F^+(P)$ , so  $\rho_0(F), \rho_1(F) \in G_F^+(P)$ . Thus  $\rho_0(F)$  and  $\rho_1(F)$ , being involutions, are half-turns, and so  $F$  must be planar and have the center,  $O$ , of  $P$  as its center. It follows that any edge of  $F$  determines  $F$ , and so every edge of  $P$  subtends only one face of  $P$ . Therefore  $P$  is not a polyhedron.

- b) If  $G_F^+(P)$  has index 2 in  $\Gamma_F(P)$ , then  $j = 2$  and  $f_2 = (4/j)|G^+(P) : G_F^+(P)|$ , so the orbit of  $F$  under  $G^+(P)$  contains  $f_2/2$  faces. Conversely, if the orbit of  $F$  under  $G^+(P)$  contains  $f_2/2$  faces then necessarily  $j = 2$ , that is, the index of  $G_F^+(P)$  in  $\Gamma_F(P)$  is 2.

Clearly, if the orbit of  $F$  under  $G^+(P)$  contains  $f_2/2$  faces there are at least two face orbits, and by part a) they are each of size at least  $f_2/2$  (because the case  $j = 1$  can not occur). Hence there are exactly two face orbits if the orbit of  $F$  under  $G^+(P)$  contains  $f_2/2$  faces.

- c) If  $G_F^+(P)$  has index 4 in  $\Gamma_F(P)$ , then  $j = 4$  and

$f_2 = (4/j)|G^+(P) : G_F^+(P)| = |G^+(P) : G_F^+(P)|$ . Hence there is just one face orbit under  $G^+(P)$ , so  $G^+(P)$  acts face-transitively on  $P$ . Conversely, if there is just one face orbit under  $G^+(P)$ , then  $f_2 = |G^+(P) : G_F^+(P)|$  and hence  $j = 4$ .

■

It follows from Lemma 3.5, that:

1) If  $G_F^+(P)$  is of index 2 in  $\Gamma_F(P) \approx D_p$ , there are two possibilities:

a.  $G_F^+(P) \approx C_p$ ,  $p$  even or odd. Then  $\sigma_1(F) \in G_F^+(P)$ .

b.  $G_F^+(P) \approx D_{p/2}$ ,  $p$  even. Then  $\sigma_1(F)$  is not in  $G_F^+(P)$ , but  $\sigma_1^2(F) \in G_F^+(P)$ .

Now  $G_F^+(P) = \langle \rho_0(F), \sigma_1(F)\rho_0(F)\sigma_1^{-1}(F) \rangle$  or  $\langle \rho_1(F), \sigma_1(F)\rho_1(F)\sigma_1^{-1}(F) \rangle$ .

2) If  $G_F^+(P)$  has index 4 in  $\Gamma_F(P) \approx D_p$ , then  $p$  is even, and the possibilities are:

a)  $G_F^+(P) \approx C_{p/2}$ . Then  $\sigma_1(F)$  is not in  $G_F^+(P)$ , but  $\sigma_1^2(F) \in G_F^+(P)$ .

b)  $G_F^+(P) \approx D_{p/4}$ ,  $p \equiv 0 \pmod{4}$ . Then  $\sigma_1^2(F)$  is not in  $G_F^+(P)$ , but  $\sigma_1^4(F) \in$

$G_F^+(P)$ . Now  $G_F^+(P)$  is one of the four subgroups

$\langle \rho_0(F), \sigma_1^2(F)\rho_0(F)\sigma_1^{-2}(F) \rangle$ ,  $\langle \sigma_1(F)\rho_0(F)\sigma_1^{-1}(F), \sigma_1^3(F)\rho_0(F)\sigma_1^{-3}(F) \rangle$ ,

$\langle \rho_1(F), \sigma_1^2(F)\rho_1(F)\sigma_1^{-2}(F) \rangle$ , or  $\langle \sigma_1(F)\rho_1(F)\sigma_1^{-1}(F), \sigma_1^3(F)\rho_1(F)\sigma_1^{-3}(F) \rangle$ .

Although case 2b above is a theoretical possibility, it does not actually occur, as we see when we find all regular polyhedra of index 2.

## Face Shapes

We suppose, as before, that  $P$  is a finite regular polyhedron of index 2, with vertices coincident with the vertices of  $S$ , where  $S$  is a Platonic solid or a cuboctahedron or an icosidodecahedron (see Section 2). We noted earlier that if  $S$  is a Platonic solid then no

edge of  $P$  can join two opposite vertices of  $S$ , for if so then that edge would subtend at least three faces under the rotational symmetries of  $S$ . The same is more generally true, as shown in the next lemma.

**Lemma 3.6:** No edge of any regular polyhedron,  $P$ , of index 2 joins vertices that are collinear with the center,  $O$ , of  $P$ .

**Proof:** Suppose that  $P$  does have an edge,  $e_0$ , joining two vertices collinear with  $O$ , and let  $F$  and  $F'$  be the two faces of  $P$  that include  $e_0$  in their edge boundaries. Then  $e_0$  must lie on a 2-fold rotational axis of  $P$  (as is the case, for example, if the vertices of  $P$  coincide with the vertices of a cuboctahedron or an icosidodecahedron), and the corresponding half-turn maps  $F$  to  $F'$ . Since  $P$  has reflexive symmetry by Theorem 3.1,  $F$  and  $F'$  lie in a reflection plane,  $H$ , of  $P$  that contains  $e_0$ , for otherwise  $e_0$  would subtend four copies of  $F$  under rotation and reflection. If  $P$  had two face orbits under  $G(P)$ , then faces in different orbits would alternate around each vertex, and so  $F$  and  $F'$ , being neighboring faces, would lie on different orbits; however this is not the case. Therefore  $P$  has one face orbit under  $G(P)$ , so each face must lie in a plane through  $O$ . Thus every face of  $P$  adjacent to  $F$  must also lie in  $H$ . Since  $P$  is connected, it follows, by repetition of the argument, that  $P$  must be planar. This is a contradiction since  $P$  is finite.

■

Suppose  $\{v_1, v_2\}$  is an edge of  $P$ , where  $v_1$  and  $v_2$  are vertices of  $P$ . It is useful to consider the projection of that edge onto the sphere that circumscribes the set,  $S$ , of vertices of  $P$

that lie in a specified vertex orbit (recall that  $P$  has one or two vertex orbits). The projection is the (shorter) arc of the great circle, which is the intersection of the sphere with the plane spanned by  $v_1$ ,  $v_2$ , and  $O$  (the center of  $S$ ). Since no projected edge of  $P$  can join opposite vertices of the sphere, all projected edges are less than half a circumference in length. Note also that the projected images of vertices of  $P$  are vertices of  $S$ , except when  $S$  is a tetrahedron and the vertices of the other vertex orbit of  $P$  lie on a tetrahedron opposed to  $S$ . Since the projection of edges is confined to the surface of a sphere, the angle between two adjoining projected edges is a 2-dimensional ‘measure’.

We use this observation to introduce a notation that describes the shape of faces of  $P$ , and which exploits the existence of certain rotational symmetries of  $P$ . Each face,  $F$ , of  $P$  is defined by the cyclically ordered vertices in its edge boundary,  $\{v_1, v_2, v_3, \dots\}$ . If the edge lengths of  $F$  are known, as well as the vertex orbits that  $v_1, v_2, v_3, \dots$  lie on, and the position of vertices of  $P$ , then  $F$  can also be specified by  $[v_1, v_2; a, b, c, d]$  where  $\{v_1, v_2\}$  is a directed edge of the face boundary of  $F$ , called the starting edge, and  $a, b, c,$  and  $d$  are symbols such as ‘right’ or ‘left’ that specify the change of direction from one edge to the next along the edge boundary of  $F$  (traversed in the direction  $\{v_1, v_2, v_3, \dots\}$ ) when it is projected onto the sphere circumscribing the vertices in a specified vertex orbit of  $P$ .

Thus ‘ $a$ ’ represents the change of direction from  $\{v_1, v_2\}$  to  $\{v_2, v_3\}$  (as well as that from  $\{v_5, v_6\}$  to  $\{v_6, v_7\}$ , etc), while ‘ $b$ ’ represents the change of direction from  $\{v_2, v_3\}$  to  $\{v_3, v_4\}$ , etc. We explain below why four symbols –  $a, b, c, d$  – suffice to describe the face.

We require that the starting edges for face shapes of faces of  $P$  all lie in the same edge orbit of directed edges under  $G^+(P)$ . This stipulation is necessary to properly compare faces of  $P$ ; an example of such a comparison is given below.

The number of different directions in which an edge boundary path may continue is dependent on  $P$ . If there is an odd number then the change of direction represented by the middle path will always be designated by 'f'. Unless  $S$  is a dodecahedron and the edge length of  $P$  is 2 or 3, this change of direction will be 'straight forward', meaning that the two adjoining edges at that vertex lie in the same great circle. We shall see that, for all regular polyhedra of index 2, we will have  $q \leq 6$  if the edges have equal length and  $q \leq 10$  otherwise, since the polyhedron must have tetrahedral, octahedral, or icosahedral symmetry. It will follow from this that there are at most five possible directions in which an edge path can continue at any vertex, and so the symbols needed to denote these changes of direction have at most five possible values. If three symbols suffice for a specific  $P$ , then we use 'r', 'l', and (if needed) 'f', to represent turns to the right, left, or forward. Note that 'r' and 'l' may each represent different angles (on the sphere of projection) for different  $P$ , but that within any specific  $P$  the angles represented do not change. If  $P$  needs symbols with more than three values, we use 'hr', 'sr', 'sl', and 'hl' (and possibly also 'f'), representing 'hard right', 'soft right', 'soft left', and 'hard left'.

Since  $P$  is an index 2 polyhedron, the combinatorial automorphism  $\sigma_1 = \sigma_1(F)$  satisfies  $\sigma_1^2 \in G(P) = G(S)$  and  $\sigma_1^4 \in G^+(P) = G^+(S)$ . So modulo  $G^+(S)$ , four directions (the a, b, c, d above) are sufficient to specify  $F$ , irrespective of the number of vertices of  $F$  or the

number of vertex orbits of  $P$ . If the starting edge of  $F$  is understood, we say  $F$  is of shape  $[a,b,c,d]$ . In general this is not a unique designation;  $F$  may also be any or all of  $[b,c,d,a]$ ,  $[c,d,a,b]$ , or  $[d,a,b,c]$ , depending on the number of directed edge orbits of  $P$  under  $G^+(P)$ . In addition, if we traverse the boundary of  $F$  in the opposite direction we see that the four shapes above are the same as  $[d',c',b',a']$ ,  $[a',d',c',b']$ ,  $[b',a',d',c']$ , and  $[c',b',a',d']$ , where  $x'$  represents the direction  $x$  when traversing the face boundary of  $F$  in the opposite direction. For example, if  $x$  is 'right' or 'left', then  $x'$  is 'left' or 'right', respectively.

Note that while only the above eight face shapes are possible shapes for  $F$ , it is not necessarily the case that each of the eight represents the face shape of  $F$ . In particular, each of the eight shapes has a different starting edge (recall that the starting edge is a directed edge), and two face shapes can represent the same face only if their starting edges are equivalent under  $G^+(P)$ .

We can illustrate this by applying it to the hypothetical face in Figure 3.2.

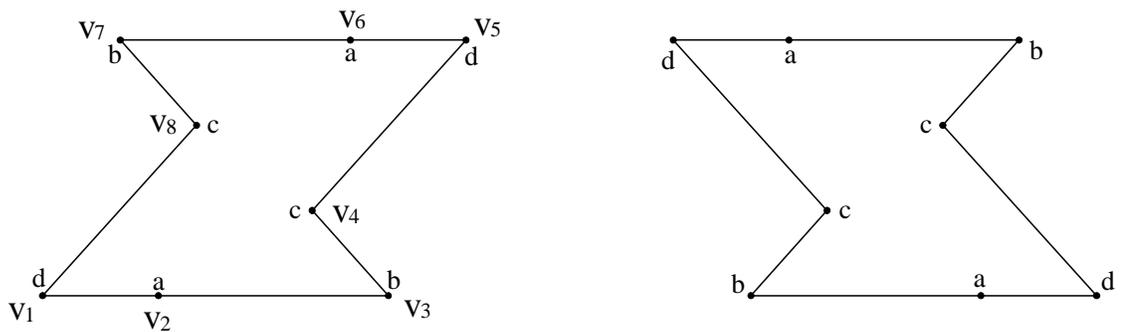


Figure 3.2: A hypothetical face, with  $p = 8$ , and its reflection image with symbols denoting change of direction at each vertex.

In this particular example,  $a = f$  and  $b = d$ ; nevertheless, the following statements apply generally. If we take  $\{v_1, v_2\}$  as the starting edge, then the shape of the face above is  $[a,b,c,d]$ . However, if we have not specified the starting edge, then the shape could be any of the eight cases given in the previous paragraph. Similarly, the shape of the reflection image of the face is a cyclic permutation of either  $[d,c,b,a]$  or  $[a',b',c',d']$ . In practice, as explained above, the (directed) starting edge is confined to a specified edge transitivity class under  $G^+(P)$ . For example, if the face in Figure 3.2 has edges that can be reversed under  $G^+(P)$ , and we are using the convention that the starting edge is the shorter edge (if there are two edge lengths in  $P$  and the edges of a face alternate in length), then each of the four shapes  $[a,b,c,d]$ ,  $[c,d,a,b]$ ,  $[d',c',b',a']$ , and  $[b',a',d',c']$ , and only those shapes, can denote the shape of the face, while its reflection image will be given by one of  $[d,c,b,a]$ ,  $[b,a,d,c]$ ,  $[a',b',c',d']$ , or  $[c',d',a',b']$ .

As further examples, we apply this notation to some regular polyhedra of index 1. Since for such polyhedra  $\sigma_1^2 \in G^+(P) = G^+(S)$ , where again  $\sigma_1 = \rho_0 \rho_1$ , modulo  $G^+(S)$  it suffices to describe a face shape by  $[a,b]$ . Thus, in our earlier notation,  $[a,b]$  stands for  $[a,b,a,b]$ . If  $P$  is a cube, then each face of  $P$  has shape  $[r,r]$ , where 'r' represents a turn to the right. If we traverse the edge boundary in the opposite direction, then the same face has shape  $[l,l]$ , where 'l' represents a turn to the left. Because this is the same turn in the opposite direction, we have  $l' = r$ , and  $r' = l$ . If  $P$  is the Petrie dual of the cube, then each of its faces has shape  $[r,l]$ . If we traverse in the opposite direction, or choose a different starting edge, the face shape is  $[l,r]$ . Since the edge lengths are the same and the vertices are on one orbit, this represents the same face shape.

As another example, if  $P$  is the small stellated dodecahedron with Schläfli symbol  $\{5/2, 5\}$ , the vertices of  $P$  coincide with those of a regular icosahedron,  $S$ , and  $P$  has 12 pentagram faces. Since the edge length of  $P$  differs from that of  $S$ , the projected image on  $S$  of a face of  $P$  is not confined to the edges of  $S$ . At any vertex,  $v_2$ , on a face,  $F$ , of  $P$  there are four possible directions in which to continue, relative to the edge  $\{v_1, v_2\}$  of  $F$ , corresponding to the four vertices (other than  $v_1$ ) which are the same distance from  $v_2$  as  $v_1$  is. If we call these directions in counterclockwise order  $hr$ ,  $sr$ ,  $sl$ , and  $hl$  (for hard right, soft right, soft left, and hard left), we have that  $[hr,hr]$  and  $[hl,hl]$  represent the same shape, which is the pentagram shape of each face of  $P$ . For the same  $S$ , and edge length, the shapes  $[sl,sl]$  and  $[sr,sr]$  represent the faces of a great icosahedron,  $\{3, 5/2\}$ .

In addition to being a convenient notation, we can also on occasion compare two face shapes to obtain very useful information. Specifically, if two faces of  $P$ , say  $F_1$  and  $F_2$ , lie in the same face orbit under  $G^+(P)$ , then they will have the same face shape. Thus if we know the face shapes symbols for each of  $F_1$  and  $F_2$ , then their common face shape will satisfy both formulations. We can find constraints on this common face shape in the following manner: Suppose it is known that the face shape of  $F_1$  is  $[a,b,c,d]$ , and that of  $F_2$  is  $[e,f,g,h]$ . Then the shape of  $F_2$  can be represented by up to 8 derivations of this face shape, namely the cyclic permutations of  $[e,f,g,h]$ , as well as those of  $[h',g',f',e']$ . No other quartet of symbols can possibly represent the face shape of  $F_2$ . Therefore, since  $F_1$  has the same face shape as  $F_2$ , one of these eight derivations must be  $[a,b,c,d]$ . However, we further have that the (directed) starting edge of  $[a,b,c,d]$  must be equivalent under

$G^+(P)$  to the directed starting edge of the face shape formulation for  $F_2$  that exactly matches it. This consideration may eliminate some of the eight possible face shape notations for  $F_2$  that could exactly match  $[a,b,c,d]$ . Of those that remain, one of them must be  $[a,b,c,d]$ , with symbol by symbol equality. This technique can be used to give key results, as, for example, in Lemmas 5.6 and 5.9.

### Petrie Polygons

We defined Petrie polygons in the previous Section, and described how to use them to obtain a new map, of type  $\{r, q\}_p$ , with the same automorphism group as the original map. This led to the concept, also introduced in Section 2, of the Petrie dual of a combinatorially regular polyhedron.

**Lemma 3.7:** Let  $P$  be a regular polyhedron of index 2 such that any of its Petrie polygons visits any given vertex at most once. Then the Petrie dual,  $Q$ , of  $P$  is also a regular polyhedron of index 2, and  $G(Q) = G(P)$ .

**Proof:** First consider the underlying combinatorially regular polyhedron, again denoted by  $P$ . By Lemma 2.4, the Petrie dual of a combinatorially regular polyhedron is a combinatorial polyhedron (and thus combinatorially regular) if and only if a Petrie polygon of the original polyhedron does not revisit a vertex. (By the regularity assumption it suffices to check only a single Petrie polygon.) By our assumption on  $P$ , the

condition on its Petrie polygons is satisfied, so the Petrie dual is again a combinatorial polyhedron. Since it retains the vertices and edges of  $P$ , the Petrie dual  $Q$  is also a geometric polyhedron.

Since  $P$  is regular of index 2, we have  $|\Gamma(P) : G(P)| = 2$ . By definition, Petrie polygons are characterized as those edge paths of  $P$  such that any two, but no three, consecutive edges belong to a face of  $P$ . This property is invariant under all combinatorial symmetries of  $P$ , including in particular those in  $G(P)$ . It follows that  $G(P)$  maps faces of  $Q$  to faces of  $Q$ . Hence  $G(P)$  preserves  $Q$ , so that  $G(P) \subset G(Q) \subset \Gamma(Q) \approx \Gamma(P)$ . In fact, since the Petrie dual of the Petrie dual is the original polyhedron, we must have  $G(Q) = G(P)$ , which is of index 2 in  $\Gamma(Q)$ .

■

## Section 4: Vertices on Two Orbits

Throughout this section, we assume that the vertices of  $P$  lie on two orbits.

### Location of the vertices of $P$

Given  $P$ , let  $S$  and  $S^*$  be the same Platonic solid, cuboctahedron or icosidodecahedron with different diameters, but with the same center and axes of rotation, such that the vertices of  $P$  coincide with the vertices of  $S$  and  $S^*$ . Although the asterisk notation is often used to denote duality, that is not the meaning here.  $S^*$  is a homothetic copy of  $S$  or  $-S$ .

$S$  can not be a cuboctahedron or icosidodecahedron, for then the number of vertices,  $f_0$ , of  $P$  would be 24 or 60, respectively, which is the same as the number of edges,  $f_1$ , of  $P$ . This would require that  $q_P = 2$ . So  $S$  and  $S^*$  are the same Platonic solid.

Similarly, since  $f_1 = qf_0/2 \geq 3f_0/2$ , in order to have a sufficient number of edges, the symmetry group of  $P$  must be  $G(S)$ , the full symmetry group of  $S$ , and not  $G^+(S)$ , the rotation group of  $S$ . We also know this, of course, by Theorem 3.1.

## Geometric nature of P

Without loss of generality, we take the edge length of  $S$  to be 1, and the edge length of  $S^*$  to be  $r$ , where  $r \neq 1$ . Note that, by definition,  $r > 0$ .  $P$  thus represents an infinite family of polyhedra, each corresponding to a different value of  $r$ , and one for each  $r$ . Polyhedra in the same family exhibit the same geometric features (and are isomorphic). They differ only in the value of  $r$ , including when  $r < 1$  and when  $r > 1$ . If there is no ambiguity, we use  $P$  to represent both a family of polyhedra and a member of that family, relying on context to make clear which we mean.

If  $S$  is a tetrahedron, then  $S^*$  can be inverted relative to  $S$ , so that it is homothetic to  $-S$ . If two polyhedra differ in this regard then they belong to different families, even though they are combinatorially isomorphic.

## Location of the edges of P

Any edge of  $P$  must either connect two vertices in  $S$ , or connect two vertices in  $S^*$ , or join a vertex of  $S$  with a vertex of  $S^*$ . The first two types of edges belong to flags that are from different flag orbits under  $G(P)$ , and the last belongs to two flags from different flag orbits. If  $P$  contained edges of all three types, there would be at least three flag-orbits and so  $P$  would be of index  $\geq 3$ . So  $P$  has edges of the last type (since  $P$  is connected), and at most one other type. However,  $S$  and  $S^*$  have the same number of vertices, each of which

is an endpoint for  $q_P$  edges. Therefore the number of edges between two vertices of  $S^*$  is the same as the number of edges between two vertices of  $S$ , and so this number must be zero if  $P$  is of index 2. Thus every edge of  $P$  joins a vertex of  $S$  with a vertex of  $S^*$ . It follows that  $p$ , the number of edges in any face boundary of  $P$ , must be even if the vertices of  $P$  lie in two orbits.

For a vertex  $v$  of  $S$ , let  $v^*$  denote the corresponding vertex of  $S^*$ . So  $v$  and  $v^*$  are collinear with  $O$ , the center of  $S$ , and are on the same side of  $O$  if  $S$  is centrally symmetric. Then the observation in the previous paragraph is that every edge of  $P$  is of the form  $\{v_1, v_2^*\}$ , where  $v_1$  and  $v_2$  are separate vertices of  $S$ .

### Length of the edges of $P$

Rather than use the Pythagorean length of each edge of  $P$ , it is more convenient to use a metric for the length, which for simplicity we also call length. There is no ambiguity in using the same term, as the meaning defined below is always used in relation to edge length.

Definition: Suppose  $P$  is such that  $S^*$  is a positively homothetic copy of  $S$  (that is,  $S^* = \lambda S$  with  $\lambda > 0$ ). Let  $\{v_1, v_2^*\}$  be any edge of  $P$ , where  $v_1$  is a vertex of  $S$ , and  $v_2^*$  is a vertex of  $S^*$ . Let  $v_2$  be the vertex of  $S$  which corresponds to  $v_2^*$ . The length of the edge  $\{v_1, v_2^*\}$  is taken to be the combinatorial length of the shortest path from  $v_1$  to  $v_2$  along

edges of  $S$ . The length of any edge of  $P$  is thus an integer. Note that the combinatorial length of an edge path on  $S$  is at least as great as the geometric length of the edge path, since the edges of  $S$  have unit length.

If  $S^*$  is not a positively homothetic copy of  $S$ , then it is a positively homothetic copy of  $-S$ , which occurs only if  $S$  is a tetrahedron and  $S^*$  is inverted relative to  $S$ . The length of any edge  $\{v_1, v_2^*\}$  of  $P$  is then taken to be the combinatorial length of the shortest path from  $v_1$  to  $v_2^*$  along edges of the cube whose vertices coincide with those of  $S$  and  $-S$ . Note that in this case this is strictly greater than the geometric length, since the edges of the cube have less than unit length. Note also that in this case the length of any edge of  $P$  is an odd integer, since each edge of the cube joins a vertex of  $S$  with a vertex of  $S^*$  (and in fact all edge-lengths must be 1, since no edge of  $P$  joins polar opposite vertices).

**Lemma 4.1:** If  $P$  is a regular polyhedron of index 2 with vertices on two orbits then all edges of  $P$  have the same length. In fact,  $G(P)$  is edge-transitive. Moreover,  $q_P = q_S = 3, 4, \text{ or } 5$ .

**Proof:**  $P$  has twice as many vertices and twice as many edges as  $S$ , so  $P$  and  $S$  have the same value of  $q$ , since  $q = 2f_1 / f_0$ . Hence, necessarily,  $q_P = 3, 4, \text{ or } 5$ . Every  $q$ -fold rotational symmetry about a vertex of  $S$  determines  $q$  edges of  $P$  of the same length emanating from this vertex. Hence all edges at a vertex of  $P$  have the same length. Moreover,  $G(P)$  is edge-transitive.

■

This result allows us to talk about the edge length of  $P$  without ambiguity.

### Allowable Edge Configurations of $P$

We now examine allowable edge lengths of  $P$  for each of the five possibilities for  $S$ .

If  $S$  is a cube, then every vertex of  $S$  can be colored red or blue such that each edge of  $S$  goes between vertices of different colors. Since edges of  $P$  go between  $S$  and  $S^*$  and are of equal length, we have that any face of  $P$  contains only red vertices of  $S$  or only blue vertices of  $S$ . It follows that the dual edge graph of  $P$  (that is, the edge graph of the dual of  $P$ ) is not connected, since adjacent faces of  $P$  must necessarily have their vertices on  $S$  colored the same. Thus if  $S$  is a cube,  $P$  can be a compound, but is not a polyhedron.

Since no edge of  $P$  goes between a vertex of  $S$  and the corresponding opposite vertex of  $S^*$  (by Lemma 3.6), the possible edge lengths of  $P$  for different  $S$  are: 1 if  $S$  is a tetrahedron or an octahedron; 1 or 2 if  $S$  is an icosahedron; and 1, 2, 3, or 4 if  $S$  is a dodecahedron. Note that if  $S$  is a tetrahedron and  $S^*$  is inverted relative to  $S$ , the only possible edge length of  $P$  is 1.

**Lemma 4.2:** If  $S$  is a dodecahedron,  $P$  does not have edge length of 2 or 3.

**Proof:** Suppose  $S$  is a dodecahedron and  $P$  has edge length of either 2 or 3. Choose any edge,  $e_0$ , of  $P$  and let one of its endpoints be  $v_0$ . There are 6 possible vertices that can be the other endpoint of  $e_0$ , and the action of the stabilizer of  $v_0$  in  $G(P) = G(S)$  takes  $e_0$  to all of them. This contradicts  $q_P = q_S = 3$ .

■

We summarize these results in the following lemma.

**Lemma 4.3:**  $S$  is not a cube. If  $S$  is a tetrahedron or an octahedron, then the edge length of  $P$  is 1. If  $S$  is an icosahedron, then the edge length of  $P$  is either 1 or 2. If  $S$  is a dodecahedron, then the edge length of  $P$  is either 1 or 4.

**Corollary:** If the vertices of  $P$  are on two orbits,  $G^+(P)$  is edge-transitive.

**Proof:** For  $S$  a Platonic solid, the only ordered pair  $\{u,v\}$  of vertices of  $S$  that is not taken under the action of  $G^+(S)$  to every other ordered pair of vertices of  $S$  the same distance apart is when  $S$  is a dodecahedron and the length of  $\{u,v\}$  is 2 or 3, in which case  $\{u,v\}$  is not an edge of  $P$  by Lemma 4.2.

■

## Generation of Index 2 Polyhedra from Index 1 Polyhedra

**Lemma 4.4:** Each of the 18 finite regular polyhedra (of index 1) described in Section 2 generates a regular polyhedron of index 2, with its vertices on two orbits, and with non-planar faces.

**Proof:** Let  $Q$  be a finite regular polyhedron. The corresponding regular index 2 polyhedron,  $P$ , is constructed from  $Q$  in the following manner. If  $Q$  is the cube or its Petrie dual, then the edge-graph of  $Q$  is bipartite, and the vertices of  $Q$  coincide with the vertices of two regular tetrahedra,  $S$  and  $S^*$ . By changing the diameter of the sphere of  $S^*$ ,  $Q$  is changed into an index 2 polyhedron,  $P$ , which is combinatorially regular because it is combinatorially identical to the original  $Q$ . Neither of these polyhedra have planar faces: the one derived from the cube because  $S$  and  $S^*$  have different diameters, and the one derived from the Petrie dual of the cube because each face consists of three pairs of opposite vertices.

If  $Q$  is not the cube or its Petrie dual, then the edge-graph of  $Q$  has cycles of length 3 or 5, and so is not bipartite. To construct  $P$ , take a concentric copy,  $Q^*$ , of  $Q$ , homothetic to  $Q$  but with a different diameter, and change each pair of edges between  $v_0$  and  $v_1$  in  $Q$  and the corresponding two vertices,  $v_0^*$  and  $v_1^*$ , in  $Q^*$  to two edges in  $P$  from  $v_0$  to  $v_1^*$  and from  $v_0^*$  to  $v_1$ . Both of these edges in  $P$  subtend a pair of faces, each corresponding to the pair of faces of  $Q$  which are subtended by  $\{v_0, v_1\}$ ; if  $q_Q$  is odd, then the two pairs of faces are the same.

The resulting structure is a connected polyhedron (because the edge-graph of  $Q$  is not bipartite) with  $q_P = q_Q$ , but with twice the number of vertices as  $Q$ , since all vertices of  $Q$  and  $Q^*$  occur as vertices of  $P$ , and twice the number of edges as  $Q$ . If  $p_Q$  is odd, then every face boundary of  $P$  tracks the corresponding face boundary of  $Q$  twice, so that  $p_P = 2p_Q$ ; in this case,  $P$  and  $Q$  have the same number of faces, and every pair of faces of  $P$  that are subtended by an edge of  $P$  are also subtended by the opposite edge on the face boundary. If  $p_Q$  is even, then  $p_P = p_Q$ , and  $P$  has twice as many faces as  $Q$ , with each face of  $Q$  corresponding to two faces of  $P$ .

None of these 16 families of polyhedra have planar faces. If  $p_Q$  is odd any face of  $P$  with edge  $\{v_0, v_1^*\}$  also contains the edge  $\{v_0^*, v_1\}$ , and these edges lie in an equatorial plane. If  $p_Q$  is even, then  $Q$  is the Petrie dual of a regular polyhedron with planar faces (other than the cube), and so every face of  $Q$  consists of pairs of opposite edges; in particular, the convex hull of the vertices of a face of  $P$  must contain the center of  $Q$ . Hence, in both cases, if the faces of  $Q$  were planar, they would all have to lie in planes through the center of  $Q$ ; however, this is not possible.

■

Maps and geometric diagrams of these 18 (families of) polyhedra are included in Appendix 1, together with 4 additional index 2 polyhedra with vertices on two orbits. As we shall see, this comprises the complete list of index 2 polyhedra with vertices on two orbits.

## Faces of P

**Lemma 4.5:** If P is a regular polyhedron of index 2, with its vertices on two orbits, then  $G^+(P) = G^+(S)$  acts transitively on the faces of P. Moreover,  $G_F^+(P)$  has order  $p/2$ , for any face, F, of P.

**Proof:** When we rotate F, a face of P, under application of  $G^+(S)$  we get  $|G^+(S)|$  copies, including duplications. Alternate vertices of F are on different orbits, so rotating F along its edge boundary produces at most  $p/2$  duplications. In addition, a rotation that fixes F can not fix a vertex of F, since it would necessarily have to be a half-turn about this vertex; this, in turn, would require S to be an octahedron and the vertices of F to be coplanar with the center of S, which is impossible by the connectedness of P. Thus the orbit of F under  $G^+(S)$  contains at least  $|G^+(S)| / (p/2)$  faces. Recall here that p is even. But  $f_2 = 2f_1 / p = |G(S)| / p = |G^+(S)| / (p/2)$ , since  $G(P) = G(S)$ . So the orbit of F under  $G^+(S)$  includes all faces of P. Thus  $G^+(S)$  acts transitively on the faces of P, and the stabilizer of any face under  $G^+(S)$  has order  $p/2$ . This, of course, is consistent with Lemma 3.4(c).

■

**Corollary 1:**  $G_F^+(P) = \langle \sigma_1^2 \rangle \approx C_{p/2}$ , where  $\sigma_1 = \sigma_1(F) = \rho_0(F)\rho_1(F)$  for any face, F, of P.

**Proof:** We know from Lemma 4.5 that the stabilizer  $G_F^+(P)$  has order  $p/2$ . Hence  $G_F^+(P)$  is of index 4 in  $\Gamma_F(P) \approx D_p$ . As mentioned after Lemma 3.4 in Section 3, this leaves the two possibilities  $G_F^+(P) \approx C_{p/2}$  or  $G_F^+(P) \approx D_{p/4}$  (if  $p \equiv 0 \pmod{4}$ ). In the former case,  $G_F^+(P) = \langle \sigma_1^2 \rangle$ , and we are done. Hence, in order to prove the corollary, we need to eliminate the second possibility. This can be done on geometric grounds as follows, exploiting the fact that the vertices of  $P$  are on two orbits.

We show that no ‘combinatorial reflection’ in  $\Gamma_F(P)$  can be realized by an element in  $G_F^+(P)$ ; this forces  $G_F^+(P)$  to be a subgroup of  $\langle \sigma_1 \rangle$ , and hence  $G_F^+(P) = \langle \sigma_1^2 \rangle$ . A combinatorial reflection in  $\Gamma_F(P)$  is conjugate in  $\Gamma_F(P)$  to  $\rho_0 = \rho_0(F)$  or  $\rho_1 = \rho_1(F)$ , and hence either interchanges the vertices of an edge of  $F$ , or fixes a vertex of  $F$ . The two vertices of an edge belong to different vertex orbits, and vertices in different vertex orbits can not be interchanged by elements in  $G(P)$ . Hence, no conjugate of  $\rho_0$  can be realized by any symmetry of  $P$ . Moreover, if an element of  $G^+(P) = G^+(S)$  realizes a conjugate of  $\rho_1$ , then this, being an involution, must be a half-turn with a rotation axis through the center of  $S$ . Since this half-turn fixes a vertex  $v$  (say) of  $F$ , the two vertices of  $F$  adjacent to  $v$ , which necessarily belong to the vertex orbit not containing  $v$ , must lie in a plane containing the rotation axis. Now a vertex of  $F$  adjacent to either of these vertices, but distinct from  $v$ , must lie in the same vertex orbit as  $v$ . This is geometrically impossible since  $G^+(P)$  is edge transitive, and so all edges have the same length. Therefore, no conjugate of  $\rho_1$  can be realized by an element of  $G^+(P)$ .

■

**Corollary 2:** Every face,  $F$ , of  $P$  has  $p/2$  planes of reflection, which each pass through opposite vertices, and are planes of reflection of  $P$ . Moreover,  $G_F^+(P) \approx D_{p/2}$ .

**Proof:**  $G(S)$  is the symmetry group of  $P$ , and  $G^+(S)$  acts transitively on the faces of  $P$ . Hence  $G(S)$  also acts transitively on the faces, and by Lemma 3.3(d) of Section 3,  $G_F(P)$  is of index 2 in  $\Gamma_F(P)$ , for every face  $F$  of  $P$ . Note that  $\sigma_1(F)$  can not be realized by an element in  $G(S)$ , since it cyclically permutes the vertices of  $F$ , but alternate vertices lie on different vertex orbits. Hence, the additional symmetries in the stabilizer must be plane reflections, and the mirrors can not pass through the mid-point of edges since alternate vertices on any face boundary lie on different orbits.

■

Corollary 1 of Lemma 4.5 allows us to talk about faces of  $P$  having face shape  $[a,b]$ , meaning that every face of  $P$  is of shape  $[a,b,a,b]$ .

To make this specific, we apply it to the allowable edge configurations for  $P$ , set out in Lemma 4.3, using the convention that the starting edge goes from a vertex on  $S$  to a vertex on  $S^*$ .

If  $S$  is a tetrahedron (and the edge length is 1) or a dodecahedron (and the edge length is 1 or 4), then  $q = 3$  and there are only two directions in which any face boundary of  $P$ , when projected on  $S$ , can continue at any vertex. Call these directions  $r$  and  $l$  (right and left, in the standard orientation on  $S$ ). If we traverse  $F$  in the opposite direction, then  $r$  and  $l$

change, so that  $r' = l$  and  $l' = r$ . The projected image of the face boundary on  $S$  goes along edges of  $S$  if the edge length of  $P$  is 1, but if the edge length is 4 when  $S$  is a dodecahedron the projected face boundary goes across faces of  $S$ , connecting opposite vertices of adjacent pentagonal faces.

If  $S$  is an octahedron (and the edge length is 1),  $q = 4$  and we have three possible directions that a face boundary can continue at any vertex when projected on  $S$ :  $r$ ,  $l$ , and  $f$  (where  $f$  stands for 'forward'). Here  $r' = l$ ,  $l' = r$ , and  $f' = f$ .

Finally, if  $S$  is an icosahedron (and the edge length is 1 or 2),  $q = 5$  and there are four possible directions in which a face boundary may continue at any vertex when projected on  $S$ . Calling these directions  $hr$ ,  $sr$ ,  $sl$ , and  $hl$  (for hard right, soft right, soft left, and hard left), we have that  $hr' = hl$ ,  $hl' = hr$ ,  $sr' = sl$ , and  $sl' = sr$ . As before, the projected image of a face of  $P$  onto  $S$  goes along edges of  $S$  if the edge length of  $P$  is 1, but across faces of  $S$  if the edge length is 2.

## **Enumeration of Families of P**

We now have sufficient restrictions on  $P$  to be able to identify all regular polyhedra of index 2 with vertices on two orbits, which we do by examining all possible face configurations for each of the allowable combinations of  $S$  and edge length of  $P$ .

If  $S$  is a tetrahedron or a dodecahedron,  $q = 3$  and so there are only two directions in which any face boundary of  $P$  can continue at any vertex. So  $P$  must have shape  $[r,r]$ ,  $[l,l]$ ,  $[r,l]$ , or  $[l,r]$ .  $[l,l]$  represents the same face as  $[r,r]$ , traversed in the opposite direction. Similarly,  $[l,r]$  is the same face as  $[r,l]$ , going in the reverse direction. Thus there are only two possible configurations for  $P$ , namely  $[r,r]$  and  $[r,l]$  respectively, and these correspond to the index 2 polyhedra generated as in Lemma 4.4 from

- (i) the tetrahedron (Schläfli symbol  $\{3, 3\}$ ) and the Petrie dual of the tetrahedron  $(\{4, 3\})$ , when  $S$  is a tetrahedron and  $S^*$  is aligned with  $S$ ;
- (ii) the cube  $(\{4, 3\})$  and the Petrie dual of the cube  $(\{6, 3\})$ , when  $S$  is a tetrahedron and  $S^*$  is opposed to  $S$ ;
- (iii) the dodecahedron  $(\{5, 3\})$  and the Petrie dual of the dodecahedron  $(\{10, 3\})$ , when  $S$  is a dodecahedron and the edge length of  $P$  is 1; and
- (iv) the great stellated dodecahedron  $(\{5/2, 3\})$  and the Petrie dual of the great stellated dodecahedron  $(\{10/3, 3\})$ , when  $S$  is a dodecahedron and the edge length of  $P$  is 4.

If  $S$  is a tetrahedron and  $S^*$  is opposed to  $S$ , the projected image of a face of  $P$  on  $S$  takes, as its vertices, certain vertices and face centers of  $S$ . The face shapes  $[r,r]$  and  $[l,l]$  are taken relative to this system of points. This results in the polyhedra mentioned in (ii), which are isomorphic to the cube or the Petrie dual of the cube.

If  $S$  is an octahedron,  $q = 4$  and there are 9 possible configurations for  $P$ , represented by  $[r,r]$ ,  $[r,l]$ ,  $[r,f]$ ,  $[l,r]$ ,  $[l,l]$ ,  $[l,f]$ ,  $[f,r]$ ,  $[f,l]$ , and  $[f,f]$ . However,  $[f,f]$  does not produce a

(connected) polyhedron because all vertices in any face boundary are coplanar with the center of  $S$ . Eliminating this, as well as duplicates caused by traversing faces in the reverse direction, we have that the only possible face configurations for  $P$  are  $[r,r]$ ,  $[r,l]$ ,  $[r,f]$ , and  $[f,r]$ . Examining these, we have that  $[r,r]$  produces the index 2 polyhedron generated, as in Lemma 4.4, from the octahedron, and  $[r,l]$  produces the polyhedron generated from the Petrie dual of the octahedron. The configuration  $[r,f]$ , which represents a face with  $p = 4$ , does not produce a polyhedron because every face boundary is of the cyclically ordered form  $\{u, v, w, v^\circ\}$ , where  $v^\circ$  is the vertex opposite  $v$ . In fact, for every face with cyclically ordered vertices  $\{u, v, w, v^\circ\}$ , there is also a face with cyclically ordered vertices  $\{u^\circ, v, w, v^\circ\}$ , where  $u^\circ$  is the vertex opposite  $u$ ; these two faces both contain the adjoining edges  $\{v,w\}$  and  $\{w,v^\circ\}$  in their face boundaries. Thus these two faces completely determine the neighborhood of  $w$  (topologically speaking), and there can be no other face containing  $w$ . This forces  $q = 2$ , in contradiction to Lemma 4.1, so the object produced by  $[r,f]$  can not be a polyhedron. Finally, the configuration  $[f,r]$  is combinatorially equivalent to  $[r,f]$  (and geometrically equivalent if we interchange  $S$  and  $S^*$  and rescale so that the edge length of  $S$  is 1), and so does not produce a polyhedron either. Thus there are exactly two polyhedra of index 2 when  $S$  is an octahedron, which are the index 2 polyhedra generated, as in Lemma 4.4, from the octahedron and the Petrie dual of the octahedron.

If  $S$  is an icosahedron,  $q = 5$  and there are now 16 possible configurations for  $P$ , corresponding to the 4 possible directions in which a face boundary may continue at any vertex. As before, these include traversing the face boundary in the opposite direction.

Eliminating these duplications leaves the 10 configurations [hr,hr], [hr,hl], [hl,hr], [sr,sr], [sr,sl], [sl,sr], [hr,sr], [sr,hr], [hr,sl], and [sl,hr]. In addition, [hr,hl] and [hl,hr] are geometrically equivalent, as are each of the three pairs [sr,sl], [sl,sr] and [hr,sr], [sr,hr] and [hr,sl], [sl,hr], all four by interchanging S and S\* and rescaling. Thus there are 6 possible configurations, for families of geometrically equivalent polyhedra, namely [hr,hr], [hr,hl], [sr,sr], [sr,sl], [hr,sr], and [hr,sl], for both edge length of 1 and edge length of 2. It is the case that all 12 of these configurations produce index 2 regular polyhedra. The proof of this is in Appendix 1, which contains maps of these polyhedra, together with their geometric diagrams and verification of their combinatorial regularity.

Eight of these polyhedra are generated, as in Lemma 4.4, from regular index 1 polyhedra. The [hr,hr] configuration produces the polyhedron generated from the icosahedron (Schläfli symbol  $\{3, 5\}$ ) for edge length 1, and from the small stellated dodecahedron ( $\{5/2, 5\}$ ) for edge length 2; and the [hr,hl] configuration produces the polyhedra generated from the Petrie duals of  $\{3, 5\}$  and  $\{5/2, 5\}$  for edge lengths 1 and 2, respectively. The [sr,sr] configuration produces the polyhedron generated from the great dodecahedron ( $\{5, 5/2\}$ ) for edge length 1, and from the great icosahedron ( $\{3, 5/2\}$ ) for edge length 2; and the [sr,sl] configuration produces the polyhedra generated from the Petrie duals of  $\{5, 5/2\}$  and  $\{3, 5/2\}$  for edge lengths 1 and 2, respectively.

The [hr,sr] configuration with edge length 1 and the [hr,sl] configuration with edge length 2 produce two combinatorially equivalent oriented polyhedra of type  $\{4, 5\}_6$ , and the [hr,sl] configuration with edge length 1 and the [hr,sr] configuration with edge length 2

produce two combinatorially equivalent oriented polyhedra of type  $\{6, 5\}_4$ . These are not generated from any regular index 1 polyhedra by means of Lemma 4.4. In fact, if the vertices are forced into one orbit, by making  $r = 1$ , the vertices and edges, respectively, of the two orbits coincide, halving in number, but the faces do not; consequently, each edge would subtend four faces, and the resulting object would not be a regular polyhedron of any index. Additionally, for these four families of polyhedra we have that the convex hull of the vertices of any face of the  $\{4, 5\}$  polyhedron contains the center of  $P$  if the edge length is 2 (in fact, one diagonal of the face passes through the center), and does not if the edge length is 1. The vertices of any face of either of the two  $\{6, 5\}$  polyhedra lie on two parallel planes, one for each orbit, which are separated by the center of  $P$  when the edge length is 2, but not if the edge length is 1. Thus, polyhedra of either family with edge length 2 do not have planar faces, but if the edge length is 1 then for exactly one value of  $r$  will the polyhedron have planar faces. This occurs when  $r = \phi$  for the  $\{4, 5\}$  polyhedron, and  $r = 2\phi + 1$  for the  $\{6, 5\}$  polyhedron, where  $\phi = (1+\sqrt{5})/2$  is the golden ratio. (Recall that  $r$  is the edge length of  $S^*$ , and so is the ratio of the diameters of  $S^*$  and  $S$ .) Note also that ‘swapping’ the orbits  $S$  and  $S^*$  of  $P$ , by changing  $r$  to the reciprocal of its original value, will result in a geometrically similar polyhedron if and only if  $P$  has been generated, as specified in Lemma 4.4, from a regular (index 1) polyhedron; polyhedra belonging to any of the four families of polyhedra just described do not maintain geometric similarity if the vertex orbits are swapped.

Thus we have established the following result:

**Theorem 4.1:** There are precisely 22 infinite families of regular polyhedra of index 2 with vertices on two orbits, where two polyhedra belong to the same family if they differ only in the relative size of the spheres containing their vertex orbits. In particular, 18 of these are related to regular polyhedra (of index 1) as described in Lemma 4.4. Of these 22 families, four have tetrahedral symmetry, two have octahedral symmetry, and sixteen have icosahedral symmetry. Precisely two of these polyhedra (not families of polyhedra) have planar faces.

Maps of these polyhedra, together with their geometric diagrams and verification of their combinatorial regularity, are in Appendix 1. A summary of these (families of) polyhedra is given below. All of the polyhedra are orientable, and transitive on the faces. The first column gives the type  $\{p, q\}_r$  of a polyhedron, but note that this notation does not imply that  $P$  is isomorphic to the universal regular map  $\{p, q\}_r$ .

| $\{p, q\}_r$     | $(f_0, f_1, f_2)$ | Edge length | Generated by Lemma 4.4 from...         | Map classification (Conder [5]) (Note 2) |
|------------------|-------------------|-------------|--|--|
| $\{4, 3\}_6$     | (8, 12, 6)        | 1           | Petrie of Tetrahedron                  | (Note 3)                                 |
| $\{4, 3\}_6$     | (8, 12, 6)        | 1           | Cube                                   | (Note 3)                                 |
| $\{6, 3\}_4$     | (8, 12, 4)        | 1           | Tetrahedron                            | (Note 3)                                 |
| $\{6, 3\}_4$     | (8, 12, 4)        | 1           | Petrie of Cube                         | (Note 3)                                 |
| $\{6, 4\}_6$     | (12, 24, 8)       | 1           | Octahedron                             | dual of R3.4                             |
| $\{6, 4\}_6$     | (12, 24, 8)       | 1           | Petrie of Octahedron                   | dual of R3.4                             |
| $\{10, 3\}_{10}$ | (40, 60, 12)      | 1           | Dodecahedron                           | dual of R5.2                             |
| $\{10, 3\}_{10}$ | (40, 60, 12)      | 4           | Petrie of Great Stellated Dodecahedron | dual of R5.2                             |
| $\{10, 3\}_{10}$ | (40, 60, 12)      | 1           | Petrie of Dodecahedron                 | dual of R5.2                             |
| $\{10, 3\}_{10}$ | (40, 60, 12)      | 4           | Great Stellated Dodecahedron           | dual of R5.2                             |
| $\{4, 5\}_6$     | (24, 60, 30)      | 1           | (Note 1)                               | R4.2                                     |
| $\{4, 5\}_6$     | (24, 60, 30)      | 2           | (Note 1)                               | R4.2                                     |
| $\{6, 5\}_4$     | (24, 60, 20)      | 1           | (Note 1)                               | dual of R9.16                            |
| $\{6, 5\}_4$     | (24, 60, 20)      | 2           | (Note 1)                               | dual of R9.16                            |
| $\{6, 5\}_{10}$  | (24, 60, 20)      | 1           | Icosahedron                            | dual of R9.15                            |
| $\{6, 5\}_{10}$  | (24, 60, 20)      | 2           | Petrie of Small Stellated Dodecahedron | dual of R9.15                            |
| $\{6, 5\}_{10}$  | (24, 60, 20)      | 1           | Petrie of Great Dodecahedron           | dual of R9.15                            |
| $\{6, 5\}_{10}$  | (24, 60, 20)      | 2           | Great Icosahedron                      | dual of R9.15                            |
| $\{10, 5\}_6$    | (24, 60, 12)      | 1           | Petrie of Icosahedron                  | dual of R13.8                            |
| $\{10, 5\}_6$    | (24, 60, 12)      | 2           | Small Stellated Dodecahedron           | dual of R13.8                            |
| $\{10, 5\}_6$    | (24, 60, 12)      | 1           | Great Dodecahedron                     | dual of R13.8                            |
| $\{10, 5\}_6$    | (24, 60, 12)      | 2           | Petrie of Great Icosahedron            | dual of R13.8                            |

**Table 4.1:** Complete list of finite regular polyhedra of index 2 with vertices on two orbits

Note 1: As noted earlier, these four polyhedra are only polyhedra if  $r \neq 1$ , and are not generated from any polyhedra in the manner specified by Lemma 4.4. We shall see later, in Table 5.4, that the two maps for these four families of ‘sporadic’ polyhedra are dual to the maps of two regular index 2 polyhedra with vertices on one orbit.

Note 2: In the Conder, Dobesanyi [4] and Conder [5] classification of regular maps, ‘R’ indicates an orientable, non-chiral (i.e., reflexible) map. The number before the period is the genus of the map. Thus R4.2 is the 2<sup>nd</sup> orientable map of genus 4 in the listing. Note that the listing only contains maps of type  $\{p, q\}_r$  where  $p \leq q$ ; this is sufficient since the dual of such a map has type  $\{q, p\}_r$ .

Note 3: The Conder listing of regular maps does not include maps that have genus  $\leq 1$ . However, these are the 7<sup>th</sup> and 6<sup>th</sup> maps, respectively, listed in Table 8 of Coxeter-Moser [8].

It is interesting to note that Table 4.1 provides three separate instances of four regular, but geometrically dissimilar, polyhedra having the same regular map. Moreover, as we can see from the maps as set out in Appendix 1, there is a duality between pairs of polyhedra in Theorem 4.1, which is different from the Petrie duality or the vertex/face duality. This duality is described more fully at the end of this paper, since it can be adapted to apply to all regular polyhedra of index 2.

## **Section 5: Vertices on One Orbit**

We turn now to examining polyhedra with vertices on one orbit. Accordingly, throughout this section, we assume that  $P$  is a regular index 2 polyhedron whose vertices lie on one orbit, under  $G(P)$ .

For a regular polyhedron,  $P$ , of index 2 and having its vertices in one orbit, Lemmas 2.1 and 2.2 constrain the vertices of  $P$  to lie at the vertices, midpoint of edges, or centers of faces of a Platonic solid. In other words, the vertices of  $P$  will coincide with the vertices of  $S$ , where now  $S$  is either a Platonic solid; or a truncated Platonic solid, that is, an octahedron occurring as a truncated tetrahedron; or a cuboctahedron; or an icosidodecahedron. Up to similarity,  $P$  is metrically unique. Without loss of generality, we fix the size of  $P$  by taking the edge length of  $S$  to be 1.

Recall that we are working under the assumption that  $G(P)$  is the full symmetry group of a Platonic solid, justified by Theorem 3.1. In the present context it can occur that  $G(P)$  is a proper subgroup of  $G(S)$ , namely exactly when  $G(P)$  is the full tetrahedral group and  $S$  is the octahedron, viewed as a truncated tetrahedron. However, we eliminate this case in Lemma 5.1.

## Calculation of $q_P$

Let  $q_P$  be the number of edges of  $P$  that end at any specified vertex of  $P$ , and let  $q_S$  be the number of edges of  $S$  that end at any specified vertex of  $S$ .

**Lemma 5.1:** If  $P$  has tetrahedral symmetry then the vertices of  $P$  do not lie on the midpoints of the edges of a tetrahedron.

**Proof:** Suppose that  $P$  has tetrahedral symmetry and the vertices of  $P$  are on the midpoints of the edges of a tetrahedron. In this case  $S$  is an octahedron;  $P$  has 12 edges, since  $|G(P)| = 24$ ; and  $q_P = 2f_1 / f_0 = 4$ . Consequently, since at least three of the four edges at a vertex must be edges of  $S$ , the symmetry also forces the fourth edge to be an edge of  $S$ . Thus all edges of  $P$  must coincide with those of  $S$ . Since  $P$  is a polyhedron, adjacent edges of any face of  $P$  can not also be adjacent edges of any other face of  $P$ . It follows that in this case, adjacent edges of any face of  $P$  must be adjacent edges of a face of  $S$ , allowing only two possibilities for the type of faces of  $P$ , namely triangles or skew hexagons. Thus  $P$  can only be a regular octahedron or the Petrie dual of a regular octahedron, and so is not an index 2 polyhedron with tetrahedral symmetry.

■

Further, if  $S$  is a cube then  $P$  can not have tetrahedral symmetry as that would require two vertex orbits, not one. Thus we have that if  $S$  is a tetrahedron,  $P$  has tetrahedral symmetry; if  $S$  is an octahedron or a cube,  $P$  has octahedral symmetry; and in all other

cases,  $P$  has icosahedral symmetry. Consequently, it follows from Theorem 3.1 that the symmetry group,  $G(P)$ , of  $P$  is the full symmetry group,  $G(S)$ , of  $S$ .

**Lemma 5.2:** If  $S$  is a Platonic solid,  $q_P = 2q_S$ . If  $S$  is a cuboctahedron or an icosidodecahedron,  $q_P = 4$ .

**Proof:** Since the symmetry group of  $P$  is  $G(S)$ , it follows that  $f_1$ , the number of edges of  $P$ , is 12 if  $P$  has tetrahedral symmetry, 24 if  $P$  has octahedral symmetry, and 60 if  $P$  has icosahedral symmetry. Since  $P$  has the same number of vertices as  $S$ , the result follows from  $q_P = 2f_1 / f_0$ .

■

**Corollary:**  $S$  is not a tetrahedron or an octahedron.

**Proof:** For any polyhedron,  $P$ , we have  $f_0 > q_P$ . But if  $S$  is a tetrahedron or an octahedron,  $f_0 = q_P - 2$  by Lemma 5.2.

■

## Edges of P

As in the previous section, we define the length of any edge  $\{v_1, v_2\}$  of P to be the length of the shortest path from  $v_1$  to  $v_2$  along edges of S. The length of any edge of P is thus an integer. The concept of length will be modified later when S is a cuboctahedron or icosidodecahedron, but for now we adopt the above definition.

**Lemma 5.3:** If S is a Platonic solid, then all edges of P are the same length if and only if S is a dodecahedron and the edge lengths of P are either 2 or 3.

**Proof:** If S is an icosahedron,  $q_P = 10$ , but for given edge length there are at most 5 vertices equidistant from any specified vertex of P. If S is a cube or dodecahedron,  $q_P = 6$ , and there are at most 3 vertices equidistant from any specified vertex of P, unless S is a dodecahedron, and the edges of P have length 2 or 3, when there are 6 vertices.

■

**Lemma 5.4:** If S is a cuboctahedron or an icosidodecahedron, then all edges of P are the same length.

**Proof:** Let S be a cuboctahedron or an icosidodecahedron, so that  $q_P = 4$  by Lemma 5.2. Let  $u$  be a specified vertex of P, and let  $u^\circ$  be the vertex opposite  $u$ . Since the stabilizer of  $u$  in  $G(S)$  has order 4, the orbit of a vertex  $v$  (distinct from  $u, u^\circ$ ) under this stabilizer contains 4 vertices, except when  $v$  lies on the perpendicular bisector of  $u$  and  $u^\circ$ , when it

contains only 2 vertices. Since  $q_P = 4$  and there are only two vertices of the exceptional kind,  $u$  must be connected by an edge of  $P$  to a vertex  $v$  not on the perpendicular bisector of  $u$  and  $u^\circ$ ; but then there must be 4 edges at  $u$  equivalent to  $\{u,v\}$ . Hence, if there were two edges of different lengths,  $u$  would have to lie in 6 or 8 edges, contradicting  $q_P = 4$ .

■

Since (by Lemma 3.6) no edge of  $P$  goes between opposite vertices of  $S$ , the longest possible edge lengths of an index 2 polyhedron for different  $S$  are: 2 if  $S$  is a cube, an icosahedron or a cuboctahedron; and 4 if  $S$  is a dodecahedron, or an icosidodecahedron.

We summarize these results in the following lemma:

**Lemma 5.5:** Let  $P$  be a regular index 2 polyhedron with only one vertex orbit, whose vertices coincide with the vertices of  $S$ . If  $P$  has edges of different lengths, there are only three possible combinations of  $S$  and edge lengths of  $P$ . These are: a)  $S$  is a cube and  $P$  has edges of length 1 and 2; b)  $S$  is a dodecahedron and  $P$  has edges of length 1 and 4; or c)  $S$  is an icosahedron and  $P$  has edges of length 1 and 2. If all edges of  $P$  are the same length, then either d)  $S$  is a dodecahedron and  $P$  has edges of length 2 or 3; or e)  $S$  is a cuboctahedron or an icosidodecahedron.

## Constraints on Face Orbits

Let  $F$  be a face of  $P$ .  $P$  has  $2|G^+(S)|/p$  faces, and either all of them or half of them are in the same orbit as  $F$  under  $G^+(S)$  and thus have the same shape as  $F$ . So the stabilizer,  $G_F^+(P)$ , of  $F$  in  $G^+(S)$ , which is the subgroup of rotational symmetries of  $F$  contained in  $G^+(S)$  has order either  $p/2$  (if all faces of  $P$  are in one orbit under  $G^+(S)$ , in which case  $p$  is even) or  $p$  (if  $P$  has two orbits of faces under  $G^+(S)$ ), as stated in Lemma 3.5.

Since  $G_F^+(P)$  is a subgroup of  $\Gamma_F(P)$ , and  $\Gamma_F(P)$  is a dihedral group generated by  $\rho_0(F)$  and  $\rho_1(F)$ , it follows that each symmetry in  $G_F^+(P)$  is a combinatorial automorphism in  $\Gamma_F(P)$  of the form  $\sigma_1^j(F)$ , where  $\sigma_1(F) := \rho_0(F)\rho_1(F)$  and  $j = 0, \dots, p-1$ , or  $\rho_v(F)$  or  $\rho_e(F)$ , where  $\rho_v(F)$  flips about a vertex  $v$  of  $F$  (while preserving  $F$ ) and  $\rho_e(F)$  flips the ends of an edge  $e$  of  $F$  (while preserving  $F$ ). The face shape of  $F$  depends on  $G_F^+(P)$ , as described in the following lemma.

### **Lemma 5.6:**

- (a) If  $G_F^+(P)$  contains  $\sigma_1(F)$ , the face shape of  $F$  is of the form  $[a,a,a,a]$ .
- (b) If  $G_F^+(P)$  contains  $\sigma_1^2(F)$ , the face shape of  $F$  is of the form  $[a,b,a,b]$ .
- (c) If  $G_F^+(P)$  contains an automorphism of the form  $\rho_v(F)$ , then  $S$  is a cuboctahedron or an icosidodecahedron, and the face shape of  $F$  is of the form  $[a,f,a',f]$  or  $[f,a,f,a']$ .
- (d) If  $G_F^+(P)$  contains an automorphism of the form  $\rho_e(F)$ , the face shape of  $F$  is of the form  $[a,a',b,b']$  or  $[a,b,b',a']$ .

**Proof:**

- (a) Clearly, if  $\sigma_1(F)$  belongs to  $G_F^+(P)$ , then all four entries in the face shape must be the same, and so  $F$  has face shape  $[a,a,a,a]$ .
- (b) Similarly, if  $\sigma_1^2(F)$  belongs to  $G_F^+(P)$ , then  $a$  and  $c$ , and  $b$  and  $d$ , must be the same, and so  $F$  must have face shape  $[a,b,a,b]$ .
- (c) Suppose  $G_F^+(P)$  contains an automorphism  $\rho_v(F)$ . Then  $\rho_v(F)$ , being an involution, must be a half-turn about an axis passing through a vertex,  $v$ , of  $P$ .  $S$  can not be an octahedron, by the Corollary to Lemma 5.2, and the other Platonic solids do not admit half-turns about a vertex; so  $S$  must be a cuboctahedron or an icosidodecahedron. The edges of  $F$  that contain  $v$  must span a plane that contains the rotation axis of  $\rho_v(F)$ . Hence,  $F$  does not change the direction at  $v$ , indicated by an 'f' at the corresponding position in the face shape of  $F$ . Moreover, since  $\sigma_1^4(F) = \rho_v(F)\rho_{v'}(F)$ , for a suitable vertex  $v'$  of  $F$  two steps away from  $v$  along the edge boundary of  $F$ , we can apply the same analysis to  $\rho_{v'}(F)$ ; note here that  $\rho_{v'}(F)$  lies in  $G_F^+(P)$ , since both  $\sigma_1^4(F)$  and  $\rho_v(F)$  lie in  $G_F^+(P)$ . It follows that the face shape also has an 'f' at the position corresponding to  $v'$ . Finally, the two other entries in the face shape are of the form  $a$  and  $a'$ , since they represent the vertices of  $F$ , distinct from  $v$ , that are the endpoints of the edges of  $F$  at  $v$ , and these are mapped into each other by the half-turn  $\rho_v(F)$ . Thus  $F$  has face shape  $[a,f,a',f]$  or  $[f,a,f,a']$ .
- (d) Suppose  $G_F^+(P)$  contains an automorphism  $\rho_e(F)$ . Then, since  $\sigma_1^4(F) = \rho_e(F)\rho_{e'}(F)$ , for a suitable edge  $e'$  of  $F$  two steps away from  $e$  along the edge boundary of  $F$ ,

we also have  $\rho_{e'}(F)$  in  $G_F^+(P)$ . Both  $\rho_e(F)$  and  $\rho_{e'}(F)$  must be half-turns. Since  $\rho_e(F)$  is a half-turn that flips the two vertices of  $F$  in  $e$ , it is immediately clear that the entries of the face shape corresponding to those vertices are adjacent and of the form  $a, a'$ . We can argue similarly for  $\rho_{e'}(F)$  and the two vertices of  $e'$ , giving entries  $b, b'$  in the face shape. Thus the face shape of  $F$  is of the form  $[a, a', b, b']$  or  $[a, b, b', a']$ .

■

Lemma 5.6 shows that the occurrence of these automorphisms in  $G_F^+(P)$  depends on the metrical shape of  $F$ .

As stated earlier,  $G_F^+(P)$  has order  $p$  or  $p/2$ , and  $G_F^+(P)$  then has two or one orbits respectively on the faces of  $P$ .

If  $G_F^+(P)$  has order  $p/2$ , then all faces of  $P$  are equivalent under  $G^+(P)$ , and thus under  $G(P)$ ; hence Lemma 3.4 implies that  $G_F(P)$  has order  $p$ . It follows in this case that  $G_F(P)$  contains improper isometries that preserve  $F$ . If  $G_F(P)$  is not isomorphic to  $C_p$ , then  $G_F(P)$  contains involutions and hence contains plane reflections. If  $G_F(P) \approx C_p$ , then  $\sigma_1(F)$  is a rotary reflection generating  $G_F(P)$ . In this case  $p$  must be even and  $F$  does not have mirror symmetry under a plane reflection preserving  $P$ .

If  $G_F^+(P)$  has order  $p$ , then  $G^+(P)$  has two orbits on the faces, so  $G(P)$  has two or one orbits on the faces. In this case, by Lemma 3.4,  $G_F(P) = \Gamma_F(P)$  or  $G_F(P) = G_F^+(P)$ ,

respectively. Only in the former case does  $F$  have mirror symmetry under a plane reflection in  $G(P)$ .

In either case, observe that  $G_F^+(P)$  can not contain involutions of the form  $\rho_e(F)$  as well as of the form  $\rho_v(F)$ . In fact, as already exploited in the proof of Lemma 5.6, if  $G_F^+(P)$  contains  $\rho_e(F)$  (or  $\rho_v(F)$ , resp.), then it must also contain  $\rho_{e'}(F)$  (or  $\rho_{v'}(F)$ , resp.) for alternating edges  $e'$  (vertices  $v'$ ) along  $F$ , including  $e$  ( $v$ , resp.). Hence, if  $G_F^+(P)$  contains both kinds of automorphisms, then it also contains a pair  $\rho_e(F)$ ,  $\rho_v(F)$ , where  $v$  is a vertex of  $e$ . In this case  $\sigma_1(F) = \rho_e(F)\rho_v(F)$  belongs to  $G_F^+(P)$  and, together with  $\rho_e(F)$ , generates a group of  $2p$ . This contradicts the fact that  $G_F^+(P)$  has order  $p/2$  or  $p$ .

These considerations impose constraints on the shape of  $F$ , which are set out below.

| $G_F^+(P)$ contains automorphisms of the form...                         | and $F$ has shape...                       | with edge length pattern...             |
|--|--|---|
| $\sigma_1^2(F)$ ,<br>but not $\sigma_1(F)$ or $\rho_e(F)$ or $\rho_v(F)$ | $[a,b,a,b]$                                | $\cdot h \cdot g \cdot h \cdot g \cdot$ |
| $\rho_e(F)$ ,<br>but not $\sigma_1^2(F)$ or $\rho_v(F)$                  | $[a,a',b,b']$ or $[a,b,b',a']$<br>(Note 1) | $\cdot h \cdot g \cdot h \cdot g \cdot$ |
| $\rho_v(F)$ ,<br>but not $\sigma_1^2(F)$ or $\rho_e(F)$                  | $[a,f,a',f]$ or $[f,a,f,a']$<br>(Note 2)   | $\cdot h \cdot g \cdot g \cdot h \cdot$ |

**Table 5.1a:** Possible face shapes when  $G_F^+(P)$  has order  $p/2$

| $G_F^+(P)$ contains automorphisms of the form...                          | and $F$ has shape...        | with edge length pattern...             |
|---|-----------------------------|---|
| $\sigma_1(F)$ ,<br>but not $\rho_e(F)$ or $\rho_v(F)$                     | $[a,a,a,a]$                 | $\cdot h \cdot h \cdot h \cdot h \cdot$ |
| $\sigma_1^2(F)$ and $\rho_e(F)$ ,<br>but not $\sigma_1(F)$ or $\rho_v(F)$ | $[a,a'a,a']$<br>(Note 1)    | $\cdot h \cdot g \cdot h \cdot g \cdot$ |
| $\sigma_1^2(F)$ and $\rho_v(F)$ ,<br>but not $\sigma_1(F)$ or $\rho_e(F)$ | $[f,f,f,f]$<br>(Notes 2, 3) | $\cdot h \cdot h \cdot h \cdot h \cdot$ |

**Table 5.1b:** Possible face shapes when  $G_F^+(P)$  has order  $p$

The symbols  $a$  and  $b$  in the 2<sup>nd</sup> column represent the change of direction between consecutive edges of  $F$ . Here  $a$  and  $b$  may or may not be different from each other, or from 'f'. Note that  $f$  in the face shape indicates straight forward (i.e. there are an odd number of changes of direction, and  $f$  is the middle one) at that vertex in the edge boundary. As previously,  $x'$  represents the change of direction represented by  $x$  when the edge boundary of  $F$  is traversed in the opposite direction.

Edge lengths  $h$  and  $g$  may or may not be different.

Note 1:  $\rho_e(F)$  represents a half-turn about the midpoint of an edge of  $F$ . Thus vertices of  $F$  equivalent under  $\rho_e(F)$  are represented by  $x$  and  $x'$ . See Lemma 5.6(d).

Note 2:  $\rho_v(F)$  represents a half-turn about a vertex of  $F$ . Here,  $S$  must be the cuboctahedron or an icosidodecahedron. Vertices of  $F$  that are equivalent under  $\rho_v(F)$  are represented by  $x$  and  $x'$ . There is no change of direction at the point of rotation, as represented by  $f$ . See Lemma 5.6(c).

Note 3: Note that in this case ( $G_F^+(P)$  containing  $\sigma_1^2(F)$  and automorphisms of the form  $\rho_v(F)$ ) the group  $G_F^+(P)$  does not contain  $\sigma_1(F)$ , even though the face shape is  $[f,f,f,f]$ . Thus for this case to occur, there must be some property (such as edges of different lengths) which does not allow  $G_F^+(P)$  to contain  $\sigma_1(F)$ .

## Configurations when P has Different Edge Lengths

If P has edges of different lengths, there are only three possible combinations of S and edge lengths of P, by Lemma 5.5. These are: a) S is a cube and P has edges of length 1 and 2; b) S is a dodecahedron and P has edges of length 1 and 4; or c) S is an icosahedron and P has edges of length 1 and 2. In particular, S is a Platonic solid.

The following preliminary lemma applies to all regular polyhedra of index 2, not just those with different edge lengths.

**Lemma 5.7:** Suppose P is a regular polyhedron of index 2, with its vertices coincident with the vertices of S, and that P has one face orbit under  $G(P)$ . Suppose further that if S is a dodecahedron, then the edges of P are of two different lengths. Then no face of P has face shape  $[f,f,f,f]$ .

**Proof:** Let F be a face of P, and assume that F has face shape  $[f,f,f,f]$ . It is easy to verify that for all combinations of S and edge length of P this assumption implies that F is planar, and is coplanar with O, the center of P. If  $\{u,v\}$  is an edge of F, then u and v are not collinear with O, by Lemma 3.6. Thus O and any edge of F define the plane in which F lies. Since P has one face orbit under  $G(P)$ , each face of P must also have face shape  $[f,f,f,f]$ , and by the same argument must lie in the plane defined by O and any edge of that face. Thus every face of P adjacent to F must also lie in the plane of F. Since P is

connected, it follows, by repetition of the argument, that  $P$  must be planar. This is a contradiction since  $G^+(P)$  is the rotation group of a Platonic solid, and so  $P$  is not planar.

■

We do not use the following generalization of Lemma 5.7, but include it for completeness.

**Lemma 5.7(generalized):** If  $P$ , a regular polyhedron of index 2, has one face orbit under  $G(P)$ , then no face of  $P$  has face shape  $[f,f,f,f]$ .

**Proof:** By Lemma 5.7, it is sufficient to assume that  $S$  is a dodecahedron and the edge lengths of  $P$  are all the same (and so must be 2 or 3). Let  $F$  be a face of  $P$ , with face shape  $[f,f,f,f]$ . Note here that  $F$  is not planar. If  $P$  has one face orbit under  $G^+(P)$ , then each face of  $P$  also has shape  $[f,f,f,f]$ . If  $P$  has two face orbits under  $G^+(P)$ , then each face in the same face orbit as  $F$  has shape  $[f,f,f,f]$ , and each face in the other face orbit is equivalent under  $G^+(P)$  to a planar reflection of  $F$  (since  $P$  has one face orbit under  $G(P)$ ), and so has shape  $[f',f',f',f']$ . It is straightforward to verify that if  $S$  is a dodecahedron and the edge lengths of  $P$  are either 2 or 3, then  $f' = f$ , so that  $[f',f',f',f']$  is the same shape as  $[f,f,f,f]$ . Thus, in both cases, all faces of  $P$  have shape  $[f,f,f,f]$ . Let  $F'$  be a face of  $P$  adjacent to  $F$ , and let  $\{u,v\}$  be their common edge. Both  $F$  and  $F'$  have a change of direction of  $f$  at  $v$ , and so both share another common edge  $\{v,w\}$ . But this is not possible, as  $q \geq 3$ .

■

**Lemma 5.8:** If  $P$  has edges of different lengths, then  $G_F^+(P)$  does not contain automorphisms of the form  $\rho_v(F)$  for any face,  $F$ , of  $P$ .

**Proof:** If  $G_F^+(P)$  contains automorphisms of the form  $\rho_v(F)$ , then  $S$  must be the cuboctahedron or an icosidodecahedron, by Lemma 5.6(c), and it then follows by Lemma 5.4 that all edges of  $P$  have the same length.

■

We introduce a further adaptation of the face shape notation. If the vertex positions and edge lengths of  $P$  are clear, as well as the convention for identifying the starting edge of the face shape, then  $P$  can be characterized in terms of the shape of its faces. The notation used here is that if  $P$  has one face orbit under  $G^+(P)$  with the faces of  $P$  having shape  $[a,b,c,d]$  then  $P$  is said to have shape  $[a,b,c,d]$ ; whereas if  $P$  has two face orbits under  $G^+(P)$  with the faces in one face orbit having shape  $[a,b,c,d]$ , and the faces in the other face orbit having shape  $[e,f,g,h]$ , then  $P$  is said to have shape  $[a,b,c,d] \& [e,f,g,h]$ , or equivalently,  $[e,f,g,h] \& [a,b,c,d]$ .

**Lemma 5.9:** If  $P$  has edges of different lengths, then  $P$  has one of the four shapes  $[a,a,a,a]$  or  $[a,a,a',a']$  or  $[a,a',a',a]$  or  $[a,a',a,a'] \& [a',a,a',a]$ . In each case  $a \neq a'$ . Moreover, in all cases  $G(P)$  acts face-transitively.

**Proof:** Since  $P$  has edges of different lengths,  $G_F^+(P)$  can not contain  $\sigma_1(F)$  (and hence  $p$  is even), and by Lemma 5.8, neither can it contain automorphisms of the form  $\rho_v(F)$ .

Therefore  $G_F^+(P)$  can only contain  $\sigma_1^2(F)$ , or an automorphism of the form  $\rho_e(F)$ , or both. It follows, as set out in Tables 5.1a and 5.1b, that the edge length pattern of  $F$  is  $\cdot h \cdot g \cdot h \cdot g \cdot$ , where  $h$  and  $g$  represent different lengths.

Let  $F'$  be the image of  $F$  under any plane reflection of  $P$ , so that  $F'$  and  $F$  are in the same face orbit under  $G(P)$ . Then both  $F$  and  $F'$  have alternating edge lengths. If the face shape of  $F$  is  $[a,b,c,d]$  then the face shape of  $F'$  is one of  $[d,c,b,a]$ ,  $[b,a,d,c]$ ,  $[a',b',c',d']$ , or  $[c',d',a',b']$ , because the starting edge of the face shape of  $F'$  must be the same length as that for the face shape of  $F$ , as they lie in the same directed edge orbit under  $G^+(P)$  (Refer to Section 2 and Figure 3.2).

If  $G_F^+(P)$  contains both  $\sigma_1^2(F)$  and an automorphism of the form  $\rho_e(F)$ , then  $G_F^+(P)$  has order  $p$ , so  $P$  has two face orbits under  $G^+(P)$ , and the face shape of  $F$  is  $[a,a',a,a']$  (see Table 5.1b). Since  $P$  has two different edge lengths,  $\sigma_1(F)$  can not belong to  $G_F(P)$ , so  $G_F(P) = G_F^+(P)$  and  $P$  has only one face orbit under  $G(P)$ . Consequently, the face shape of  $F'$  is  $[a',a,a',a]$ , (where the starting edges for the face shape of each face are in the same edge orbit under  $G^+(P)$ , and so, in particular, are the same length). We can eliminate the possibility that  $a' = a$ , since then the face shape of both  $F$  and  $F'$  is  $[f,f,f,f]$ , which is excluded by Lemma 5.7 (since  $P$  has just one face shape under  $G(P)$ ). Thus  $F$  has face shape  $[a,a',a,a']$  with  $a \neq f$ . We claim that  $F'$  is not in the same face orbit as  $F$  under  $G^+(P)$ . In fact, if  $F' = \rho(F)$ , where  $\rho$  is a plane reflection of  $P$ , then  $F$  and  $F'$  lie in the same orbit under  $G^+(P)$  if and only if there exists an element  $\tau \in G^+(P)$  such that  $F = \tau(F') = \tau\rho(F)$ ; here  $\tau\rho$  would belong to  $G_F(P)$ , contradicting  $G_F(P) = G_F^+(P)$  since  $\rho$  is not

contained in  $G^+(P)$ . It follows that the second face orbit under  $G^+(P)$  is represented by  $F'$ , while the first is represented by  $F$ . Hence  $P$  has shape  $[a,a',a,a']$  &  $[a',a,a',a]$  with  $a \neq f$ , and has only one face orbit under  $G(P)$ .

If  $G_F^+(P)$  contains  $\sigma_1^2(F)$ , but no automorphisms of the form  $\rho_e(F)$ , then  $G_F^+(P)$  has order  $p/2$ , so  $P$  has one face orbit under  $G^+(P)$  and  $G(P)$ , and the face shape of  $F$  is  $[a,b,a,b]$ . Consequently, the face shape of  $F'$  is  $[b,a,b,a]$  or  $[a',b',a',b']$  (again, refer to Figure 3.2). Since  $F$  and  $F'$  are in the same face orbit under  $G^+(P)$ , they have the same face shape. By equating the shapes of  $F$  and  $F'$ , as described in Section 3, this gives just two possibilities: either  $a = b$ ; or  $a = a'$  and  $b = b'$  (and hence  $a = b = f$ ). The latter gives the face shape  $[f,f,f,f]$  for  $F$ , which is excluded by Lemma 5.7. Hence  $F$  must have face shape  $[a,a,a,a]$ , with  $a \neq f$ .

If  $G_F^+(P)$  contains an automorphisms  $\rho_e(F)$ , but not  $\sigma_1^2(F)$ , then  $G_F^+(P)$  has order  $p/2$ , so  $P$  again has one face orbit under both  $G^+(P)$  and  $G(P)$ , and the face shape of  $F$  is either  $[a,a',b,b']$ , or  $[a,b,b',a']$ , depending on whether the edge flipped by  $\rho_e(F)$  is an even or odd number of steps along the face boundary of  $F$  away from the starting edge (in other words, whether or not the two edges have the same length). It follows that the face shape of  $F'$  is either  $[b',b,a',a]$  or  $[a',a,b',b]$  in the first instance, and either  $[a',b',b,a]$  or  $[b,a,a',b']$  in the second instance. Since  $F$  and  $F'$  are in the same face orbit under  $G^+(P)$ , they have the same face shape, which we find by equating the shapes of  $F$  and  $F'$ . In the first instance this gives either  $b = a'$ , for a face shape of  $[a,a',a',a]$ ; or  $a = a'$  and  $b = b'$ , for a face shape of  $[f,f,f,f]$ . In the second instance this gives either  $a = a'$  and  $b = b'$ , for a

face shape of  $[f,f,f,f]$ ; or  $b = a$ , for a face shape of  $[a,a',a',a]$ . However, by Lemma 5.7, if  $P$  has one face orbit under  $G(P)$  then the face shape can not be  $[f,f,f,f]$ , and so  $F$  must have face shape  $[a,a',a',a]$ ; and again there is only one face shape modulo  $G^+(P)$ .

■

With this result we obtain all candidates for finite regular index 2 polyhedra with different edge lengths.

If  $S$  is a cube (and  $P$  has edges of length 1 or 2) or a dodecahedron (and  $P$  has edges of length 1 or 4), then  $q = 6$ , by Lemma 5.2, and so there are three possible directions ( $r$ ,  $f$ , and  $l$ ) for a face boundary to continue from any vertex, since edge lengths alternate along a face and around a vertex. By Lemma 5.9, there are seven possible face shapes for  $P$ . Of these possible face shapes, we have that  $[l,l,l,l]$  and  $[l,r,r,l]$  represent the same faces as  $[r,r,r,r]$  and  $[r,l,l,r]$  traversed in the opposite direction, and  $[l,l,r,r]$  represents the same face as  $[r,r,l,l]$  when we choose a starting edge that is two steps further along on the edge boundary of the face (because edge lengths alternate along a face, this new starting edge has the same length). Eliminating these duplications from separate consideration reduces the analysis to polyhedra with faces of one of four possible shapes, namely  $[r,r,r,r]$ ,  $[r,l,r,l]$  &  $[l,r,l,r]$ ,  $[r,r,l,l]$ , and  $[r,l,l,r]$ .

If  $S$  is an icosahedron (and  $P$  has edges of length 1 or 2), then  $q = 10$ , by Lemma 5.2, and so there are five possible directions for a face boundary to continue from any vertex. By Lemma 5.9, there are then 14 possible face shapes for  $P$ , and in precisely the same

manner as above, we see that 6 pairs of these represent duplicate faces, so that we are left with 8 possible different structures for P, namely [hr,hr,hr,hr], [hr,hl,hr,hl] & [hl,hr,hl,hr], [hr,hr,hl,hl], and [hr,hl,hl,hr], as well as [sr,sr,sr,sr], [sr,sl,sr,sl] & [sl,sr,sl,sr], [sr,sr,sl,sl], and [sr,sl,sl,sr].

These sixteen structures, four each when S is a cube or a dodecahedron and eight when S is an icosahedron, are thus the only candidates for finite regular polyhedra of index 2 with edges of different length. Since polyhedra, by definition, are vertex faithful, only those structures with vertex faithful faces are actual possibilities.

Figure 5.1, below, shows by direct construction (using the convention that the starting edge in the face shape notation is an edge of length 1) a portion of a face for seven of the sixteen structures. In each case the faces self-intersect, so these structures will be ruled out immediately. Note that three of the diagrams each show a portion of a face for two structures, since in each case the faces of the two structures share the portion shown.

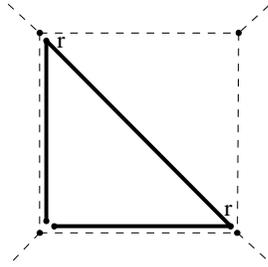


Fig. 5.1a Portion of face  $[r,r,r,r]$  on cube, and also portion of face  $[r,r,l,l]$  on cube.

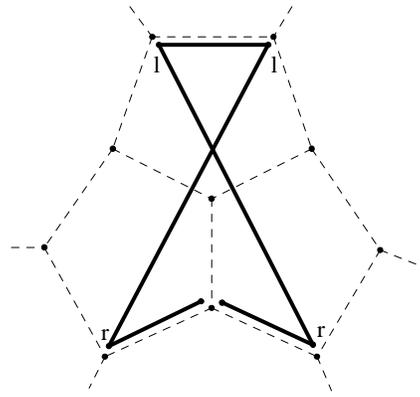


Fig. 5.1b Portion of face  $[r,l,l,r]$  on dodecahedron.

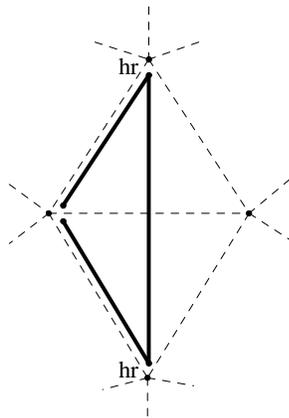


Fig. 5.1c Portion of face  $[hr,hr,hr,hr]$  on icosahedron, and also portion of face  $[hr,hr,hl,hl]$  on icosahedron.

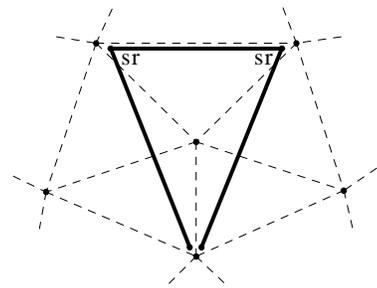


Fig. 5.1d Portion of face  $[sr,sr,sr,sr]$  on icosahedron, and also portion of face  $[sl,sr,sr,sl]$  on icosahedron.

The edges in Figure 5.1 have been shrunk slightly to show the underlying structure of  $S$  more clearly. Note that the face  $[sl,sr,sr,sl]$  in Figure 5.1d is the face  $[sr,sl,sl,sr]$  with the same starting edge, but traversed in the opposite direction.

None of the diagrams in Figure 5.1 represents the entire face, as they each have an odd number of edges, whereas any face with edges of different lengths must have an even number of edges. Thus each of the seven faces that are partially shown in Figure 5.1 self-intersects at a vertex, so does not have the non-repetitive property of faces and therefore is not vertex faithful.

Since every finite regular polyhedron is vertex faithful by definition, this eliminates from consideration 7 of the 16 candidate structures.

We now reduce this list further by considering properties of Petrie dual polyhedra. In this context, some guidance is provided by the following lemma, which holds independent of the requirement that P have edges of two different lengths.

**Lemma 5.10:** Let P be a finite regular polyhedron of index 2 such that the Petrie dual Q of P is a polyhedron. If the shape of P is either  $[a,a',a,a']$  or  $[a,a',a,a'] \& [a',a,a',a]$ , where a represents any specific change of direction, then the shape of Q is either  $[a,a,a,a]$  or  $[a,a,a,a] \& [a',a',a',a']$ . If the shape of P is either  $[a,a,a,a]$  or  $[a,a,a,a] \& [a',a',a',a']$ , then the shape of Q is either  $[a,a',a,a']$  or  $[a,a',a,a'] \& [a',a,a',a]$ .

**Proof:** Our assumption that Q be a polyhedron is equivalent to requiring that the Petrie polygons of P not revisit vertices of P. By Lemma 3.7 it remains to determine the face shape of Q. Now let  $\{u,v\}$  be any edge of P, and let  $F_1$  and  $F_2$  be the two faces of P bordering  $\{u,v\}$ . Let the edge boundary of  $F_1$  be  $\{u,v,w_1,\dots\}$ , and the edge boundary of  $F_2$  be  $\{u,v,w_2,\dots\}$ . The face boundaries of  $F_1$  and  $F_2$  each have a change of direction represented by either a or a' at vertex v. If these two changes of direction were the same, then  $w_1$  and  $w_2$  would coincide since the edges  $\{v,w_1\}$  and  $\{v,w_2\}$  have the same length, and so the face boundaries of  $F_1$  and  $F_2$  would share two consecutive edges, and P could not be a polyhedron. Thus at vertex v, the face boundary of either  $F_1$  or  $F_2$  has a change of direction represented by a, and the face boundary of the other one has a change of

direction represented by  $a'$ . (Note that this implies  $a \neq f$ .) Therefore the neighboring faces along any edge of  $P$  have different changes of direction at an end-point of that edge. It follows from the way the Petrie dual is constructed that if each face of  $P$  has shape either  $[a,a',a,a']$  or  $[a',a,a',a]$ , then each face of  $Q$  has shape either  $[a,a,a,a]$  or  $[a',a',a',a']$ . If  $Q$  has one face orbit under  $G^+(P)$  then these face shapes represent the same face, and  $Q$  has shape  $[a,a,a,a]$ ; if not, then they represent two different faces, and  $Q$  has shape  $[a,a,a,a]$  &  $[a',a',a',a']$ . Similarly, if each face of  $P$  has shape either  $[a,a,a,a]$  or  $[a',a',a',a']$ , then each face of  $Q$  has shape either  $[a,a',a,a']$  or  $[a',a,a',a]$ . If  $Q$  has one face orbit under  $G^+(P)$  then these face shapes represent the same face, and  $Q$  has shape  $[a,a',a,a']$ ; if not, then they represent two different faces, and  $Q$  has shape  $[a,a',a,a']$  &  $[a',a,a',a]$ .

■

As pointed out in the proof of Lemma 5.10, Petrie duality relates structures of shape  $[a,a',a,a']$  or  $[a,a',a,a']$  &  $[a',a,a',a]$  to those of shape  $[a,a,a,a]$  or  $[a,a,a,a]$  &  $[a',a',a',a']$ , while preserving  $S$  at the same time. In particular, the following lemma shows which structures with different edge length are related by Petrie duality for given  $S$ .

**Lemma 5.11:** Let  $P$  and  $Q$  be Petrie dual polyhedra of index 2, with edges of different length, and let the vertices of  $P$  and  $Q$  coincide with those of the Platonic solid  $S$ . If  $S$  is a cube or a dodecahedron, then  $P$  and  $Q$  either have shape  $[r,r,r,r]$  and  $[r,l,r,l]$  &  $[l,r,l,r]$ , or have shape  $[r,r,l,l]$  and  $[r,l,l,r]$ . If  $S$  is an icosahedron, then  $P$  and  $Q$  have shape either  $[hr,hr,hr,hr]$  and  $[hr,hl,hr,hl]$  &  $[hl,hr,hl,hr]$ ; or  $[hr,hr,hl,hl]$  and  $[hr,hl,hl,hr]$ ; or  $[sr,sr,sr,sr]$  and  $[sr,sl,sr,sl]$  &  $[sl,sr,sl,sr]$ ; or  $[sr,sr,sl,sl]$  and  $[sr,sl,sl,sr]$ .

**Proof:** Follows from Lemma 5.9 (and the subsequent remarks) and Lemma 5.10, together with the observation that if all the shape symbols of  $P$  are either  $a$  or  $a'$ , for some specified  $a$ , then all the shape symbols of  $Q$  must also be either  $a$  or  $a'$ .

■

We now apply to the related structures in Lemma 5.11 the information provided by Figure 5.1 concerning vertex faithfulness. This gives that if  $S$  is the cube or the icosahedron, the structures in Lemma 5.11 corresponding to  $S$ , as well as two of the four structures corresponding to the dodecahedron, either are not polyhedra or do not have Petrie duals which are polyhedra. This suggests that none of the seven structures that are Petrie dual to the structures in Figure 5.1 is a polyhedron. Indeed, this can be confirmed directly from the geometry determined by their shape.

It is in fact the case that three of these seven structures, namely  $[r,r,l,l]$  when  $S$  is a dodecahedron, and  $[hr,hl,hl,hr]$  and  $[sr,sr,sl,sl]$  when  $S$  is an icosahedron, do not have the non-repetitive property of faces, and so are not vertex faithful. Diagrams, similar to those in Figure 5.1, demonstrating this are in Appendix 3. It is also the case that for none of the fourteen structures – the seven in Figure 5.1, and the seven that are Petrie dual to them – is the combinatorial automorphism  $\rho_1$  well-defined (that is, the underlying map is not combinatorially regular). Appendix 3 contains diagrams demonstrating this just for the remaining four structures that have not been shown to lack the non-repetitive property of

faces, namely  $[r,l,l,r]$  and  $[r,l,r,l]$  &  $[l,r,l,r]$  when  $S$  is a cube, and  $[hr,hl,hr,hl]$  &  $[hl,hr,hl,hr]$  and  $[sr,sl,sr,sl]$  &  $[sl,sr,sl,sr]$  when  $S$  is an icosahedron

Thus we are left with just one possible Petrie pair for index 2 polyhedra with different edge lengths, namely the two structures with shapes  $[r,r,r,r]$  and  $[r,l,r,l]$  &  $[l,r,l,r]$ , where  $S$  is a dodecahedron. It is the case that these structures are both regular polyhedra of index 2, and their maps and geometric diagrams are set out in Appendix 2. These then are the only two such polyhedra with edges of different lengths.

Each of these two polyhedra have  $p = 6$  and  $q = 6$  (see Lemma 5.2), an  $f$ -vector of  $(20, 60, 20)$ , and edge lengths of 1 and 4. The polyhedron with shape  $[r,r,r,r]$  is orientable, with one face orbit under  $G^+(P)$  (and hence also  $G(P)$ ), and has planar faces; the polyhedron with shape  $[r,l,r,l]$  &  $[l,r,l,r]$  is non-orientable, with non-planar faces, and has two face orbits under  $G^+(P)$ , but only one face orbit under  $G(P)$ , so that faces in different orbits are mirror reflections of each other.

In summary we have established the following.

**Theorem 5.1:** There are precisely two combinatorially regular polyhedra of index 2 with vertices on one orbit and edges of different lengths. They are Petrie duals of each other, are of type  $\{6, 6\}_6$ , have their vertices at the vertices of a dodecahedron, and have the full icosahedral group as their geometric symmetry group. One is orientable, of genus 11, with planar faces; the other is non-orientable, of genus 22, with non-planar faces. The

underlying regular maps for the two polyhedra are the self-dual map R11.5 and the (non-self-dual) map N22.3 (not the dual of N22.3), respectively, of Conder [5].

### **Configurations when the Edges of P all have the Same Length**

For the remainder of Section 5 we assume that all edges of P have the same length. Recall that all vertices of P lie at the vertices of S, and are in one orbit under  $G(P)$ . By Lemma 5.5, the edges of P are the same length if and only if S is a dodecahedron and the edge length is 2 or 3, or S is a cuboctahedron or an icosidodecahedron.

If S is a Platonic solid, any pair of vertices of P at a given distance (defined as the shortest distance along edges of S) is equivalent under  $G(P)$  to any other pair of vertices at this distance. However, when S is a cuboctahedron or an icosidodecahedron this property does not hold, and accordingly we modify the definition of edge length of P in order to maintain it (except in one unimportant case, explained below). The reason that non-equivalent edges of the same length exist is that the cuboctahedron and icosidodecahedron have two different faces (eight triangles and six squares for the cuboctahedron, and twenty triangles and twelve pentagons for the icosidodecahedron). So when S is a cuboctahedron or an icosidodecahedron, we modify the measure of the length of an edge of P by taking the shortest path along diagonals of faces of S (where a diagonal of a triangle is the same as an edge of the triangle). By keeping the side of each triangle to be of length 1, and designating the diagonal of the square in the case of the

cuboctahedron (equal to  $\sqrt{2}$ ), or the pentagon in the case of the icosidodecahedron (equal to the golden ratio,  $(1+\sqrt{5})/2$ ) to be  $d$ , then we can uniquely specify, up to the action of  $G(P)$ , any edge of  $P$  by giving its length as  $m+nd$ , where  $m$  and  $n$  are integers and  $m+nd$  is minimized. Since there are no edges between opposite vertices, this gives possible edge lengths of 1, 2, or  $d$  for the cuboctahedron, and possible edge lengths of 1, 2, 3, 4,  $d$ ,  $2d$ , or  $1+d$  for the icosidodecahedron. Note that for the cuboctahedron  $1 < d < 2$ , and for the icosidodecahedron  $1 < d < 2 < 1+d < 3 < 2d < 4$ .

This modification of the definition of edge length of  $P$  maintains in general the property referred to in the previous paragraph, in that all vertices equidistant from a given vertex,  $v$ , are equivalent under the vertex stabilizer of  $v$ , except when  $S$  is the icosidodecahedron and the edge length is  $1+d$ . This latter case will be excluded on other grounds in Lemma 5.13.

**Lemma 5.12:** If  $S$  is a cuboctahedron,  $P$  can not have edge length of  $d$ .

**Proof:** If so,  $q = 2$ , and every face of  $P$  would be a square, coplanar with the center of  $S$ .

■

**Lemma 5.13:** If  $S$  is an icosidodecahedron,  $P$  can not have edge length of  $1+d$ .

**Proof:** The 30 vertices of the icosidodecahedron are coincident with the vertices of five regular octahedra, any two of which do not share a vertex. This is the compound of five

octahedra described in Section 2. Any edge of  $P$ , of length  $1+d$ , from a specified vertex  $v$ , will have its endpoint coincide with a vertex on the octahedron that contains  $v$ .  $P$  can thus be a compound, but not a polyhedron.

■

### Edge Orientation

From now on, because of Lemmas 5.12 and 5.13, we exclude the edge lengths  $d$  for the cuboctahedron and  $1+d$  for the icosidodecahedron.

We now look at the action of  $G^+(P) = G^+(S)$  on edges of  $P$ . As pointed out earlier, any pair of vertices of  $S$  separated by a given distance (the graph distance when  $S$  is a Platonic solid, or as defined above when  $S$  is the cuboctahedron or icosidodecahedron) is equivalent under  $G(P) = G(S)$  to any other pair of vertices the same distance apart. (Note here that we excluded the distance  $1+d$  for the icosidodecahedron, for which this is not true.) However, this no longer remains true for equivalence under the rotation subgroup  $G^+(P)$ . Indeed, it fails precisely when  $S$  is a dodecahedron and the edge length is 2 or 3, or when  $S$  is a cuboctahedron or an icosidodecahedron (for any distance). These are exactly the cases for which all edges of  $P$  have the same length.

For the remainder of this section we suppose that  $S$  is the dodecahedron and the edge length of  $P$  is 2 or 3, or that  $S$  is the cuboctahedron and the edge length of  $P$  is 1 or 2, or

that  $S$  is the icosidodecahedron and the edge length of  $P$  is 1, 2, 3, 4,  $d$ , or  $2d$ . In these cases,  $G(P)$  will be transitive on the edges of  $P$ . Thus all polyhedra  $P$  with one vertex orbit, and all edges of the same length will have an edge transitive symmetry group  $G(P)$ .

Recall that  $P$  has  $|\Gamma(P)|/4 = |G^+(P)|$  edges. The length of an orbit of an edge,  $e$ , under  $G^+(P)$  equals the index (in  $G^+(P)$ ) of the stabilizer of  $e$  in  $G^+(P)$ . We claim that this stabilizer has order 1 or 2, and hence that the orbit of  $e$  under  $G^+(P)$  consists of either all edges of  $P$  or one-half of them. Indeed, if a nontrivial element  $\varphi \in G(P)$  stabilizes  $e$ , then either  $\varphi$  fixes  $e$  pointwise and must be the reflection in the plane through  $e$  and  $O$  (the center of  $P$ ), or  $\varphi$  interchanges the endpoints of  $e$  and must be either the reflection in the perpendicular bisector of  $e$  or the half-turn about the midpoint of  $e$ . Thus the stabilizer of  $e$  in  $G^+(P)$  has order 1 or 2.

We now have two possible scenarios. Either  $G^+(P)$  acts transitively on the edges of  $P$  (which occurs when the stabilizer of at least one edge is trivial), or there are precisely two edge orbits of the same size under  $G^+(P)$  (which occurs when all edge stabilizers of  $P$  are generated by the half-turn about the midpoint of the edge). Note that if  $G^+(P)$  acts edge transitively, then it acts sharply edge transitive (since the edge stabilizer is trivial).

Suppose  $e$  is any edge of  $P$  with vertices  $u$  and  $v$ . Consider the corresponding ‘directed’ edge, denoted  $\{u,v\}$ , obtained by equipping  $e$  with a ‘direction’ pointing from  $u$  to  $v$ . Given  $e$  there are two directed edges associated with  $e$ , namely  $\{u,v\}$  and  $\{v,u\}$ . Then the stabilizer of  $e$  in  $G^+(P)$  is nontrivial if and only if the two directed edges  $\{u,v\}$  and  $\{v,u\}$

are equivalent under  $G^+(P)$ . In this case the two directed edges associated with any edge of  $P$  are equivalent under  $G^+(P)$ , since the corresponding edge stabilizer in  $G^+(P)$  is nontrivial as well. In other words, if the edge stabilizer of one, and hence of each, edge is trivial, then no edge of  $P$  can be ‘inverted’ modulo  $G^+(P)$ . In this case, if we begin with a directed edge and take its (directed) images under  $G^+(P)$ , we obtain a directed copy of the edge graph of  $P$ . These observations motivate the definition of ‘directed type’, given below.

Two directed edges  $\{u,v\}$  and  $\{u',v'\}$  of  $P$  are said to have the same orientation if they are equivalent under  $G^+(P)$ ; that is, there exists an element  $\phi$  in  $G^+(P)$  such that  $\phi(u) = u'$  and  $\phi(v) = v'$ . Otherwise the two directed edges are said to have opposite orientation.

There always are two transitivity classes of directed edges under  $G^+(P)$ . If  $G^+(P)$  is transitive on the (undirected) edges of  $P$ , then, for each edge of  $P$ , the two corresponding directed edges lie in different transitivity classes under  $G^+(P)$ ; that is, each transitivity class of directed edges contains exactly one of the two directed edges associated with any edge of  $P$ . In this case the two transitivity classes are the orbits, under  $G^+(P)$ , of the pair  $\{u,v\}$  and  $\{v,u\}$  of directed edges associated with any given edge of  $P$ . On the other hand, if  $G^+(P)$  is not transitive on the (undirected) edges of  $P$ , then each of the two transitivity classes of (undirected) edges of  $P$  gives rise to a single transitivity class of directed edges of  $P$  of twice the size. In this case the two directed edges associated with an edge of  $P$  always lie in the same transitivity class of directed edges.

Definition: If for one (and hence every) edge of  $P$  with vertices  $u$  and  $v$ , the corresponding directed edges  $\{u,v\}$  and  $\{v,u\}$  have opposite orientation (i.e. are not equivalent under  $G^+(P)$ ), then  $P$  has directed type. If for one (and hence every) edge of  $P$  with vertices  $u$  and  $v$ , the corresponding directed edges  $\{u,v\}$  and  $\{v,u\}$  have the same orientation (are equivalent under  $G^+(P)$ ), then  $P$  has bicolor type.

$P$  has directed type if  $S$  is a dodecahedron and the edge length is 2; or if  $S$  is a cuboctahedron and the edge length is 1; or if  $S$  is an icosidodecahedron and the edge length is 1, 3, or  $d$ .  $P$  has bicolor type if  $S$  is a dodecahedron and the edge length is 3; or if  $S$  is a cuboctahedron and the edge length is 2; or if  $S$  is an icosidodecahedron and the edge length is 2, 4, or  $2d$ .

It is clear from the definition of edge orientation that every regular index 2 polyhedron with vertices on one orbit and all edges having the same length will be of either directed or bicolor type. We will first find all index 2 polyhedra with directed type.

## Directed Type

As noted above, for directed type we can assume that  $S$  is a dodecahedron and the edge length is 2, or  $S$  is a cuboctahedron and the edge length is 1, or  $S$  is an icosidodecahedron and the edge length is 1, 3, or  $d$ .

If  $P$  has directed type, then  $G_F^+(P)$  does not contain automorphisms of the form  $\rho_e(F)$ , as  $\rho_e(F)$  changes the direction of the edge that is flipped, while preserving the orientation. So  $G_F^+(P)$  can only contain automorphisms of the form  $\sigma_1^j(F)$  or  $\rho_v(F)$ . Note that  $\sigma_1(F)$  and  $\rho_v(F)$  generate a dihedral group  $D_p$ , so can not both lie in  $G_F^+(P)$  by Lemma 3.5. Thus the possibilities for  $F$ , together with allowable orientation of its edges, are summarized in the following table, where  $x$  and  $y$  represent opposite orientations. For the 2<sup>nd</sup> and 4<sup>th</sup> columns, recall Tables 5.1a and 5.1b, and Lemma 3.5. Since  $\sigma_1^4(F)$  belongs to  $G_F(P)$  and hence to  $G_F^+(P)$ , it is sufficient to record the orientation pattern of faces by a string containing four entries.

| If $G_F^+(P)$ contains automorphisms of the form...        | then $F$ has shape...        | and orientation pattern (for the directed face boundary)   | # face orbits under $G^+(P)$ |
|--|------------------------------|--|------------------------------|
| $\sigma_1(F)$ ,<br>but not $\rho_v(F)$                     | $[a,a,a,a]$                  | $\cdot x \cdot x \cdot x \cdot x \cdot$  | 2                            |
| $\sigma_1^2(F)$ ,<br>but not $\sigma_1(F)$ or $\rho_v(F)$  | $[a,b,a,b]$                  | $\cdot x \cdot x \cdot x \cdot x \cdot$ or $\cdot x \cdot y \cdot x \cdot y \cdot$   | 1                            |
| $\rho_v(F)$ , but not $\sigma_1^2(F)$                      | $[a,f,a',f]$ or $[f,a,f,a']$ | $\cdot x \cdot x \cdot y \cdot y \cdot$ or $\cdot x \cdot y \cdot y \cdot x \cdot$ or<br>$\cdot x \cdot y \cdot x \cdot y \cdot$ | 1                            |
| $\sigma_1^2(F)$ and $\rho_v(F)$ ,<br>but not $\sigma_1(F)$ | $[f,f,f,f]$                  | $\cdot x \cdot y \cdot x \cdot y \cdot$  | 2                            |

**Table 5.2:** Possible Face Shapes when  $P$  has Directed Type

We first examine the case that  $S$  is a cuboctahedron or an icosidodecahedron, and  $P$  has directed type. After this is dealt with, we will look at the case that  $S$  is a dodecahedron.

When  $S$  is a cuboctahedron or an icosidodecahedron, there are four vertices at a specified distance from a given vertex, except when the distance is that between antipodal vertices. Thus there are three possible directions in which a face boundary of a polyhedron  $P$  may continue at any vertex. Call these directions (in the standard counterclockwise orientation)  $r$ ,  $f$ , and  $l$ . This is consistent with the fact, from Lemma 5.2, that  $q = 4$ .

From Table 5.2 we see that if  $P$  has two face orbits under  $G^+(P)$ , then all its faces have face shape  $[a,a,a,a]$ , with  $a = r, f, l$ . We now determine the number of vertices,  $p(F)$ , a face  $F$  with face shape  $[r,r,r,r]$ ,  $[f,f,f,f]$ , or  $[l,l,l,l]$  would need to have for a given edge length when  $S$  is a cuboctahedron or an icosidodecahedron. Since all faces of  $P$  must have the same value of  $p(F)$ , certain possibilities can be ruled out. Note that  $[r,r,r,r]$  and  $[l,l,l,l]$  are ‘inverse’ face shapes, obtained by traversing a face boundary in the opposite direction, but that they represent different (and adjacent) faces when they have the same starting edge.

For edge length 1, any face of shape  $[r,r,r,r]$  or  $[l,l,l,l]$  would be a face of  $S$  (with  $[r,r,r,r]$  representing a triangular face), and any face of shape  $[f,f,f,f]$  would be a circumference of  $S$  (parallel to a triangular face when  $S$  is a cuboctahedron, and parallel to a pentagon face when  $S$  is an icosidodecahedron), so  $[r,r,r,r]$ ,  $[l,l,l,l]$ , and  $[f,f,f,f]$  all have different values

of  $p(F)$ , namely 3, 4, 6, respectively, for the cuboctahedron, and 3, 5, 10, respectively, for the icosidodecahedron.

If  $S$  is an icosidodecahedron and the edge length is 3, then the faces with shape  $[r,r,r,r]$  and  $[l,l,l,l]$  are pentagrams centered on pentagon faces of  $S$  and triangles centered on triangle faces of  $S$ , and the faces with shape  $[f,f,f,f]$  go three times round a circumference of  $S$ , parallel to a pentagon face of  $S$ , giving a star-decagon  $\{10/3\}$  inscribed in the circumference. Thus the three face shapes again have different values of  $p(F)$  (5, 3, and 10, respectively).

Finally, when  $S$  is an icosidodecahedron and the edge length is  $d$ , then the faces with shape  $[r,r,r,r]$  and  $[l,l,l,l]$  are pentagrams and pentagons, each centered on a pentagon face of  $S$ , and the faces with shape  $[f,f,f,f]$  are planar hexagons contained in a circumference of  $S$  that is parallel to a triangular face of  $S$  (but does not run along edges of  $S$ ). The two face shapes  $[r,r,r,r]$  and  $[l,l,l,l]$  each have  $p(F) = 5$ , and the face with shape  $[f,f,f,f]$  has  $p(F) = 6$ .

It follows that if  $S$  is a cuboctahedron or an icosidodecahedron, and  $P$  has directed type (but otherwise is as in Table 5.2), then  $P$  does not have a face with shape  $[f,f,f,f]$ , for, if so,  $P$  must have two face orbits under  $G(P)$  by Lemma 5.7, and the faces in the other orbit can not also have face shape  $[f,f,f,f]$  and have the same value of  $p(F)$ . Furthermore, the only candidate structure for such a polyhedron, if  $P$  has two face orbits under  $G^+(P)$ , occurs when  $S$  is an icosidodecahedron and the edge length of  $P$  is  $d$ , and the faces in

each orbit have shape  $[r,r,r,r]$  and  $[1,1,1,1]$ , respectively (bearing in mind that the edge lengths are restricted as mentioned at the beginning of this subsection).

We now use the following result, similar to Lemma 5.10, about the Petrie dual of a polyhedron.

**Lemma 5.14:** If  $P$  is a regular polyhedron of index 2 with all its faces having shape  $[a,f,b,f]$  or  $[f,c,f,d]$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  represent any change of direction (including  $f$ ), then the Petrie dual  $Q$  of  $P$  is a regular polyhedron of index 2 and has a face with shape  $[f,f,f,f]$ . Moreover, if  $P$  is of directed or bicolor type, respectively, so is  $Q$ .

**Proof:** Suppose we knew that the Petrie dual  $Q$  was a polyhedron. Then Lemma 3.7 tells us that  $Q$  is a regular polyhedron of index 2 with  $G(Q) = G(P)$ . Since the edges of  $Q$  are just those of  $P$ , any two directed edges of  $Q$  have the same orientation with respect to  $Q$  if and only if they, as directed edges of  $P$ , have the same orientation with respect to  $P$ . Hence if  $P$  is of directed type, so is  $Q$ ; and if  $P$  is of bicolor type, so is  $Q$ .

Next we establish the existence of a face of  $Q$  of shape  $[f,f,f,f]$ , still ignoring the question of polytopality of  $Q$ . Now let  $\{u,v\}$  be any edge of  $P$ , and let  $F_1$  and  $F_2$  be the two faces of  $P$  bordering  $\{u,v\}$ . The face boundaries of  $F_1$  and  $F_2$  each have a change of direction of  $f$  at either  $u$  or  $v$ . If it were at the same vertex, then the face boundaries of  $F_1$  and  $F_2$  would share two consecutive edges, and  $P$  could not be a polyhedron. So the face boundary of either  $F_1$  or  $F_2$  has an  $f$  at  $u$ , and the face boundary of the other one has an  $f$

at  $v$ . It follows from the way the Petrie dual is constructed that one of the faces of  $Q$  has shape  $[f,f,f,f]$ . This is true regardless of whether or not  $Q$  is polytopal. If  $Q$  is not polytopal, we simply regard  $Q$  as the collection of Petrie polygons of  $P$ , and their vertices and edges.

It remains to prove that the Petrie dual is indeed a polyhedron. As explained in the proof of Lemma 3.7, it suffices to show that a Petrie polygon of  $P$  does not revisit a vertex. Since  $P$  is combinatorially regular, it is enough to verify this condition for a single Petrie polygon, and we will take the polygon which is a face of shape  $[f,f,f,f]$  of  $Q$ . Regardless of the choice of  $S$  (a dodecahedron, cuboctahedron, or icosidodecahedron), inspection of the few possibilities shows that no polygon of shape  $[f,f,f,f]$  and of any given edge length does revisit a vertex. Hence the face of  $Q$  with face shape  $[f,f,f,f]$  does not revisit a vertex either. It follows that  $Q$  is a polyhedron, and the proof is complete.

■

As we saw above, when a polyhedron has directed type and  $S$  is a cuboctahedron or an icosidodecahedron, it does not have a face of shape  $[f,f,f,f]$ , and so Lemma 5.14 implies that for each face  $F$  of  $P$ ,  $G_F^+(P)$  does not contain automorphisms of the form  $\rho_v(F)$ , for, as shown in Table 5.2, that would require  $F$  to have face shape  $[a,f,a',f]$  or  $[f,a,f,a']$ , and the corresponding Petrie dual of  $P$  to have a face of shape  $[f,f,f,f]$ . It follows that either  $G_F^+(P)$  contains  $\sigma_1(F)$  for all  $F$  (we have already dealt with this case since then  $G^+(P)$  has two face orbits), or  $G_F^+(P)$  contains  $\sigma_1^2(F)$ , but not  $\sigma_1(F)$ , for all  $F$ . When  $G_F^+(P)$  is face transitive, there are just one face shape, and so the two possible face shapes,  $[r,l,r,l]$  and

$[1,r,l,r]$ , must represent the same face shape of  $P$  (bear in mind that edges of  $P$  are all the same length). Thus the face shape  $[r,l,r,l]$  is the only face shape for which  $G_F^+(P)$  contains  $\sigma_1^2(F)$  but not  $\sigma_1(F)$ ; if other face shapes do occur,  $G_F^+(P)$  must contain  $\sigma_1(F)$  for all faces  $F$ . Thus, since regular polyhedra come in Petrie pairs, ignoring the question of polytopality for the moment, if any regular polyhedra with directed type exist when  $S$  is a cuboctahedron or an icosidodecahedron, then at least one of the polyhedra has only faces  $F$  for which  $G_F^+(P)$  contains  $\sigma_1(F)$ . Note here that the polyhedra in a Petrie pair can not both be such that they only have faces  $F$  for which  $G_F^+(P)$  contains  $\sigma_1^2(F)$  but not  $\sigma_1(F)$ ; in fact, by Lemma 5.10, a Petrie polygon of a polyhedron with faces of shape  $[r,l,r,l]$  necessarily has shape  $[r,r,r,r]$  or  $[l,l,l,l]$ , which in turn forces  $\sigma_1(F) \in G_F^+(P)$  by Table 5.2.

For a polyhedron of directed type with the property that  $\sigma_1(F) \in G_F^+(P)$  for all faces  $F$ , there are two orbits of faces under  $G^+(P)$  and their face shapes are  $[r,r,r,r]$  and  $[l,l,l,l]$  (here, as before, these shapes represent adjacent faces with a common starting edge). Earlier inspection of the cases with allowable edge lengths shows that there is just one candidate where the two face orbits have same value of  $p(F)$ , obtained when  $S$  is an icosidodecahedron, and the edge length is  $d$ . The Petrie dual of that candidate is a polyhedron, and has faces of shape  $[r,l,r,l]$ . It is the case that both these structures are regular polyhedra of index 2, and the maps and geometric diagrams of these polyhedra, which are described more fully in Theorem 5.2, are in Appendix 2.

We next examine the only remaining case of directed type, which occurs when  $S$  is a dodecahedron, and the edge length of  $P$  is 2. There are exactly 6 vertices at distance 2

from any given vertex of the dodecahedron, so there are five possible directions in which a face boundary may continue at any vertex. This is consistent with the fact, from Lemma 5.2, that  $q = 6$ . Call these directions (in standard orientation)  $hr$ ,  $sr$ ,  $f$ ,  $sl$ , and  $hl$ . Here  $f$  stands for ‘forward’, but unlike in previous uses, a direction of  $f$  does not mean straight forward in the strict geometric sense, meaning that the two edges lie in a common plane through the origin.

The following result applies to polyhedra of both directed and bicolor type when  $S$  is a dodecahedron.

**Lemma 5.15:** When  $S$  is a dodecahedron and the edge length of  $P$  is 2 or 3, then if two consecutive edges on a directed face boundary have a change of direction represented by  $hr$ ,  $f$ , or  $hl$ , the two corresponding directed edges have the same orientation if  $P$  has directed type, and opposite orientation if  $P$  has bicolor type. If the change of direction between the edges is represented by  $sr$  or  $sl$ , the two corresponding directed edges have opposite orientation if  $P$  has directed type, and the same orientation if  $P$  has bicolor type.

**Proof:** Since  $q = 6$ , there are six edges at a vertex  $v$  of  $P$ , namely  $\{v, u_0\}, \dots, \{v, u_5\}$ , listed here in cyclic (counterclockwise) order about  $v$  on  $S$  (which may not be the order in which they occur in  $P$ ). Clearly, the three directed edges  $\{v, u_0\}, \{v, u_2\}, \{v, u_4\}$  are equivalent under  $G^+(P)$  (and hence have the same orientation), as are the three directed edges  $\{v, u_1\}, \{v, u_3\}, \{v, u_5\}$ . However,  $\{v, u_j\}$  and  $\{v, u_{j+1}\}$  ( $j \bmod 6$ ) are only equivalent under  $G(P)$ , not under  $G^+(P)$ . Thus  $\{v, u_i\}$  and  $\{v, u_j\}$  have the same orientation if and only

if  $j \equiv i \pmod{2}$ . If  $\{u_i, v\}$  and  $\{v, u_j\}$  are two consecutive edges of a directed face boundary of  $P$ , then a change of direction represented by  $hr, f, hl$ , or  $sr, sl$ , respectively, translates into the condition  $j \equiv i+1 \pmod{2}$  or  $j \equiv i \pmod{2}$ . Now, by definition,  $\{v, u_i\}$  and  $\{u_i, v\}$  have the same orientation if and only if  $P$  is of bicolor type. Then the lemma is immediate. For example, if the consecutive edges are  $\{u_0, v\}$  and  $\{v, u_1\}$ , then the change of direction is  $hr$ , so  $\{u_0, v\}$  and  $\{v, u_1\}$  have the same orientation if  $P$  is of directed type, and opposite orientation if  $P$  has bicolor type.

■

**Lemma 5.16:** If  $S$  is a dodecahedron and the edge length of  $P$  is 2 or 3, then for each face,  $F$ , of  $P$ ,  $G_F^+(P)$  does not contain automorphisms of the form  $\rho_v(F)$ .

**Proof:** If an element  $\rho_v(F)$  lies in  $G_F^+(P)$  then it must necessarily be a half-turn. However,  $v$  is a vertex of  $S$  and  $G(S)$  does not contain a half-turn about a vertex of  $S$ , so the lemma is immediate.

■

It follows from Lemma 5.16 and Table 5.2 that if  $S$  is a dodecahedron, and the edge length of  $P$  is 2 (so that  $P$  is of directed type), and  $F$  is any face of  $P$ , then  $G_F^+(P)$  either contains  $\sigma_1(F)$ , or  $\sigma_1^2(F)$  but not  $\sigma_1(F)$ . Consequently, by Lemma 5.15 and Table 5.2,  $F$  has shape  $[a, b, a, b]$  (allowing  $a = b$ ), where either the directed face boundary of  $F$  has no change in orientation and  $a, b \in \{hr, f, hl\}$ , or the directed face boundary of  $F$  changes orientation at every vertex and  $a, b \in \{sr, sl\}$ . (A mixed type, such as  $a = sl$  and  $b = hr$ ,

can not occur since at least  $\sigma_1^2(F)$  lies in  $G_F^+(P)$ .) However, the latter case can not occur, for if every change of direction of the edge boundary of  $F$  was either  $sr$  or  $sl$  then the edge boundary of  $F$  would be confined to the edges of a cube inscribed in the dodecahedron that forms part of the regular compound of five cubes whose vertices coincide with the vertices of a regular dodecahedron. Any two of the components of this compound have only two (antipodal) vertices in common, and no edge of  $F$  joins those two vertices. Thus if the shape of  $F$  consists of the symbols  $sr$  and  $sl$ ,  $P$  might be a compound but is not a polyhedron.

Of the remaining nine possible face shapes, we have that  $[hl,hr,hl,hr]$ ,  $[f,hr,f,hr]$ , and  $[f,hl,f,hl]$  are the same faces as  $[hr,hl,hr,hl]$ ,  $[hl,f,hl,f]$ , and  $[hr,f,hr,f]$  traversed in the opposite direction. This then reduces the analysis to polyhedra with faces of one of six possible shapes. Now keep in mind that Petrie duality pairs up polyhedra that have one vertex orbit, have all their edges of the same length, and are of directed type, ignoring for the moment the question of their polytopality. In particular, the face shapes for one polyhedron in a pair of Petrie duals can be derived from those for the other by Lemmas 5.10 and 5.14. Thus there are just three candidates for Petrie pairs (when  $S$  is a dodecahedron and the edges of  $P$  have length 2), which are the structures with shapes  $[hr,hl,hr,hl]$  and  $[hr,hr,hr,hr]$  &  $[hl,hl,hl,hl]$ ;  $[hr,f,hr,f]$  and  $[hr,hr,hr,hr]$  &  $[f,f,f,f]$ ; and  $[hl,f,hl,f]$  and  $[hl,hl,hl,hl]$  &  $[f,f,f,f]$ .

We next eliminate two of these pairs by finding  $p(F)$  for the three kinds of  $\sigma_1$ -invariant faces  $F$  that occur, that is, for the faces of shape  $[hr,hr,hr,hr]$ ,  $[hl,hl,hl,hl]$ , or  $[f,f,f,f]$ .

When  $S$  is a dodecahedron and the edges of  $P$  have length 2, we say that an edge  $\{u,v\}$  of  $P$  has left orientation if the corresponding edge path  $\{u,w,v\}$  of length 2 on  $S$  bends left at  $w$ , and has right orientation if the path bends to the right at  $w$ .

If the starting edge has left orientation, then a face  $F$  of face shape  $[hr,hr,hr,hr]$  is a vertex-figure of  $S$ , with  $p(F) = 3$ ; a face of face shape  $[hl,hl,hl,hl]$  is a pentagram on a face of  $S$ , with  $p(F) = 5$ ; and a face of shape  $[f,f,f,f]$  is a convex pentagon taking alternate vertices of a Petrie polygon of  $S$ , also with  $p(F) = 5$ . These values of  $p(F)$  rule out the first two of the three candidates for Petrie pairs of polyhedra, leaving only the pair of structures with shape  $[hl,f,hl,f]$  and  $[hl,hl,hl,hl]$  &  $[f,f,f,f]$ . It is the case that these structures are indeed polytopal and represent two regular index 2 polyhedra; their geometric diagrams and maps verifying this are in Appendix 2. Thus we have that there are precisely four regular index 2 polyhedra with directed type: two whose vertices coincide with those of a regular icosidodecahedron, and two where the vertices match those of a regular dodecahedron. We describe these polyhedra more fully in the following theorem:

**Theorem 5.2:** There are exactly four finite regular polyhedra of index 2 with directed type (and all edges of the same length). There are two whose vertices coincide with those of a regular icosidodecahedron,  $S$ , and with edge length  $(1+\sqrt{5})/2$  ( $= d$ , being the length of a diagonal of a pentagonal face of  $S$  with sides of length 1). One of these has  $p = 5$ ,  $q = 4$ , with genus = 4, and is orientable with planar faces in two face orbits under  $G^+(S)$ , either convex pentagons of shape  $[r,r,r,r]$  or pentagrams of shape  $[l,l,l,l]$ . Its Petrie dual has  $p =$

6,  $q = 4$ , with genus = 12, and is non-orientable with non-planar hexagonal faces in one face orbit under  $G^+(S)$ , and of face shape  $[r,l,r,l]$ . The underlying regular maps for the two polyhedra are the dual of the map R4.2 and the dual of the map R12.1, respectively, of Conder [5]. The other two polyhedra have vertices coinciding with those of a regular dodecahedron and have edges of length 2. One of these has  $p = 5$ ,  $q = 6$ , with genus = 9, and is orientable with planar faces in two face orbits under  $G^+(S)$ , either pentagrams of shape  $[hl,hl,hl,hl]$  or convex pentagons of shape  $[f,f,f,f]$ . Its Petrie dual has  $p = 4$ ,  $q = 6$ , with genus = 12, and is non-orientable with non-planar square faces in one face orbit under  $G^+(S)$ , and of face shape  $[hl,f,hl,f]$ . The underlying regular maps for the two polyhedra are the map R9.16 and the map N12.1, respectively, of Conder [5].

We now turn our attention to structures with bicolor type, which are the only remaining potential polyhedra.

## Bicolor Type

As noted earlier,  $P$  has bicolor type if (and only if)  $S$  is a dodecahedron and the edge length is 3; or  $S$  is a cuboctahedron and the edge length is 2; or  $S$  is an icosidodecahedron and the edge length is 2, 4, or  $2d$ .

If  $P$  has bicolor type, then the possibilities for  $F$ , together with allowable orientation of its edges, are summarized in the following table, where  $x$  and  $y$  represent opposite orientations. For the entries in the  $2^{\text{nd}}$  and  $3^{\text{rd}}$  columns, recall Tables 5.1a, 5.1b and Lemma 3.5.

| If $G_F^+(P)$ contains automorphisms of the form...                       | then $F$ has shape...             | and orientation pattern (for the directed face boundary)  | # face orbits under $G^+(P)$ |
|---|-----------------------------------|---|------------------------------|
| $\sigma_1(F)$ ,<br>but not $\rho_e(F)$ or $\rho_v(F)$                     | $[a,a,a,a]$                       | $\cdot x \cdot x \cdot x \cdot x \cdot$   | 2                            |
| $\sigma_1^2(F)$ ,<br>but not $\sigma_1(F)$ or $\rho_e(F)$ or $\rho_v(F)$  | $[a,b,a,b]$                       | $\cdot x \cdot x \cdot x \cdot x \cdot$ or<br>$\cdot x \cdot y \cdot x \cdot y \cdot$   | 1                            |
| $\rho_e(F)$ , but not $\sigma_1^2(F)$ or $\rho_v(F)$<br>(Note 1)          | $[a,a',b,b']$ or<br>$[a,b,b',a']$ | $\cdot x \cdot \text{any} \cdot x \cdot \text{any} \cdot$ or<br>$\cdot \text{any} \cdot x \cdot \text{any} \cdot x \cdot$           | 1                            |
| $\rho_v(F)$ , but not $\sigma_1^2(F)$ or $\rho_e(F)$<br>(Note 2)          | $[a,f,a',f]$ or $[f,a,f,a']$      | $\cdot x \cdot x \cdot x \cdot x \cdot$ or<br>$\cdot x \cdot x \cdot y \cdot y \cdot$ or<br>$\cdot x \cdot y \cdot y \cdot x \cdot$ | 1                            |
| $\sigma_1^2(F)$ and $\rho_e(F)$ ,<br>but not $\sigma_1(F)$ or $\rho_v(F)$ | $[a,a'a,a']$                      | $\cdot x \cdot x \cdot x \cdot x \cdot$ or<br>$\cdot x \cdot y \cdot x \cdot y \cdot$   | 2                            |
| $\sigma_1^2(F)$ and $\rho_v(F)$ ,<br>but not $\sigma_1(F)$ or $\rho_e(F)$ | $[f,f,f,f]$                       | $\cdot x \cdot x \cdot x \cdot x \cdot$   | 2                            |

**Table 5.3:** Possible Face Shapes when  $P$  has Bicolor Type

Note 1:  $\rho_e(F)$  represents a half-turn about the midpoint of an edge of  $F$ . Thus, since  $P$  has bicolor type, edges of  $F$  that are ‘equidistant’ from this point of rotation have the same orientation, and vertices of  $F$  that are ‘equidistant’ from the point of rotation have

complementary changes of direction along the face boundary of  $F$ , represented by  $x$  and  $x'$ .

Note 2:  $\rho_v(F)$  represents a half-turn about a vertex of  $F$ . Thus since  $P$  has bicolor type, edges of  $F$  that are 'equidistant' from this point of rotation have the same orientation, and vertices of  $F$  that are 'equidistant' from the point of rotation have complementary changes of direction along the face boundary of  $F$ , represented by  $x$  and  $x'$ . The change of direction at the point of rotation along the face boundary is represented by  $f$ .

Again we look first at the situation when  $S$  is a cuboctahedron or an icosidodecahedron, before examining bicolor polyhedra when  $S$  is a dodecahedron.

When  $S$  is a cuboctahedron or an icosidodecahedron, there are four vertices at a specified distance from a given vertex, except when the distance is that between antipodal vertices. Thus there are three possible directions in which a face boundary of a polyhedron  $P$  may continue at any vertex. Call these directions (in standard orientation)  $r$ ,  $f$ , and  $l$ . This is consistent with the fact, from Lemma 5.2, that  $q = 4$ .

It is useful to have an analogous result to Lemma 5.15. Again this applies to polyhedra of both directed and bicolor type.

**Lemma 5.17:** When  $S$  is a cuboctahedron or an icosidodecahedron, then if two consecutive edges on a directed face boundary have a change of direction represented by

r or l, the two corresponding directed edges have the same orientation if P has directed type, and opposite orientation if P has bicolor type. If the change of direction between the edges is represented by f, the two corresponding directed edges have opposite orientation if P has directed type, and have the same orientation if P has bicolor type.

**Proof:** Since  $q = 4$ , there are four edges at a vertex  $v$  of  $P$ , namely  $\{v, u_0\}, \dots, \{v, u_3\}$ , listed here in cyclic (counterclockwise) order about  $v$  on  $S$  (again, this may not be the order in which they occur in  $P$ ). Clearly, the directed edges  $\{v, u_0\}$  and  $\{v, u_2\}$  are equivalent under  $G^+(P)$  (and hence have the same orientation), as are the directed edges  $\{v, u_1\}$  and  $\{v, u_3\}$ . No other equivalences under  $G^+(P)$  occur. Thus  $\{v, u_i\}$  and  $\{v, u_j\}$  have the same orientation if and only if  $j \equiv i \pmod{2}$ . If  $\{u_i, v\}$  and  $\{v, u_j\}$  are two consecutive edges of a directed face boundary of  $P$ , then a change of direction represented by r, l or by f, respectively, translates into the condition  $j \equiv i+1 \pmod{2}$  or  $j \equiv i \pmod{2}$ . The lemma follows, once we recall that, by definition,  $\{v, u_i\}$  and  $\{u_i, v\}$  have the same orientation (i.e., are equivalent under  $G^+(P)$ ) if and only if  $P$  is of bicolor type.

■

**Corollary:** If  $S$  is a cuboctahedron or an icosidodecahedron, and  $P$  has bicolor type, then the only face,  $F$ , of  $P$  for which  $G_F^+(P)$  contains  $\sigma_1(F)$  is the face with shape  $[f, f, f, f]$ .

**Proof:** Every edge on the directed face boundary of  $F$  must have the same orientation if  $G_F^+(P)$  contains  $\sigma_1(F)$ . By Lemma 5.17, the only such face shape is  $[f, f, f, f]$ .

■

In fact, we can improve on the previous corollary, as follows.

**Lemma 5.18:** If  $P$  has bicolor type and  $F$  is a face of  $P$ , then  $\sigma_1(F)$  can not lie in  $G_F^+(P)$ .

**Proof:** Let  $u, v$  be adjacent vertices of  $F$ . Since  $P$  has bicolor type, the directed edges  $\{u, v\}$  and  $\{v, u\}$  are equivalent under an element  $\tau$  of  $G^+(P)$ . Then  $\tau$  fixes the edge  $\{u, v\}$  of  $P$  but interchanges its vertices. Hence, since  $\tau$  belongs to  $\Gamma(P)$ , we must have either  $\tau = \rho_e(F)$  or  $\tau = \sigma_1(F)\sigma_2(v)$ , the latter being the “combinatorial half-turn” about the midpoint of the edge  $\{u, v\}$  of  $P$ . (Note here that the element  $\sigma_1(F)\sigma_2(v)$  is just the element  $\rho_e(G)$  for a face  $G$  of the Petrie dual of  $P$ .)

Now suppose  $\sigma_1(F)$  belongs to  $G_F^+(P)$ . If  $\tau = \rho_e(F)$ , then  $\sigma_1(F)$  and  $\tau$  generate a dihedral subgroup of order  $2p$  contained in  $G_F^+(P)$ ; but  $G_F^+(P)$  has order at most  $p$ , by Lemma 3.5. If  $\tau = \sigma_1(F)\sigma_2(v)$ , then  $\sigma_2(v)$  lies in  $G^+(P)$  (since both  $\tau$  and  $\sigma_1(F)$  lie in  $G^+(P)$ ) and  $\sigma_1(F)$  and  $\sigma_2(v)$  generate a subgroup of  $G^+(P)$ ; however, this subgroup has index at most 2 in  $\Gamma(P)$ , while  $G^+(P)$  has index 4. In both cases we arrive at a contradiction. Hence,  $\sigma_1(F)$  does not lie in  $G_F^+(P)$ .

■

Next we eliminate the possibility that  $G_F^+(P)$  contains automorphisms of the form  $\rho_e(F)$  if  $P$  has one face orbit under  $G^+(P)$ . We begin with the following general lemma.

For an edge  $e$  of  $P$  we let  $G_e(P)$  and  $G_e^+(P)$  denote the stabilizer of  $e$  in  $G(P)$  and  $G^+(P)$ , respectively. Clearly,  $G_e^+(P)$  is a subgroup of  $G_e(P)$ .

**Lemma 5.19:** Let  $P$  have all its vertices in one orbit under  $G(P)$  and have all its edges of the same length. Then  $G(P)$  is edge transitive and the stabilizer  $G_e(P)$  of an edge  $e$  has order 2. In particular,  $G_e(P)$  is generated by a half-turn and equals  $G_e^+(P)$  if  $P$  is of bicolor type, and  $G_e(P)$  is generated by a plane reflection and  $G_e^+(P)$  is trivial if  $P$  is of directed type.

**Proof:** As pointed out earlier, our assumptions on  $P$  imply that  $G(P)$  is edge transitive. Since  $P$  has exactly  $|G(P)|/2$  edges, the stabilizer in  $G(P)$  of an edge  $e$  must have order 2.

By definition, if  $P$  is of bicolor type, then for every edge  $e$ ,  $G_e^+(P)$  is generated by a half-turn about the midpoint of  $e$ . Since  $G_e(P)$  and  $G_e^+(P)$  have the same order, they must coincide. On the other hand, if  $P$  is of directed type, then  $G_e^+(P)$  is trivial. Since the generating element of  $G_e(P)$  is an involution, it must be a plane reflection, a half-turn, or the central inversion. The last is impossible, since edges of  $P$  can not pass through the center. Neither can it be a half-turn, since then it would lie in  $G_e^+(P)$ . Thus  $G_e(P)$  is generated by the reflection in the perpendicular bisector of  $e$ .

■

**Lemma 5.20:** If  $P$  has bicolor type, and  $P$  has one face orbit under  $G^+(P)$ , then for any face,  $F$ , of  $P$ ,  $G_F^+(P)$  can not contain an automorphism of the form  $\rho_e(F)$ .

**Proof:** By Lemmas 3.4 and 3.5, if  $P$  has one face orbit under  $G_F^+(P)$  and hence under  $G(P)$ , then  $|G_F(P)| = 2|G_F^+(P)| = p$ , so necessarily  $\sigma_1^2(F) \in G_F(P)$ , but  $\sigma_1^2(F)$  does not lie in  $G_F^+(P)$  if  $G_F^+(P)$  contains an element  $\rho_e(F)$ . Now suppose  $G_F^+(P)$  contains an element  $\rho_e(F)$ . Then  $\sigma_1^2(F) = \rho_e(F)\rho_{e'}(F)$ , where  $e'$  is an edge of  $F$  adjacent to  $e$ , and  $\rho_{e'}(F)$  lies in  $G_F(P)$  since both  $\sigma_1^2(F)$  and  $\rho_e(F)$  lie in  $G_F(P)$ , but  $\rho_{e'}(F)$  does not lie in  $G_F^+(P)$  since  $\rho_e(F)$  lies in  $G_F^+(P)$  but  $\sigma_1^2(F)$  does not. This implies that  $\rho_{e'}(F)$  is not a half-turn; it must therefore be a plane reflection. This contradicts Lemma 5.19, since all stabilizers of edges in  $G(P)$  must be generated by half-turns if  $P$  is of bicolor type.

■

From Lemma 5.18 (or the corollary to Lemma 5.17) and from Table 5.3 we have that if  $S$  is a cuboctahedron or an icosidodecahedron, and  $P$  has two face orbits under  $G^+(P)$ , then the only possible face shapes for  $P$  are  $[r,l,r,l]$ ,  $[l,r,l,r]$ , and  $[f,f,f,f]$ . We now calculate  $p(F)$  for these face shapes. We require, of course, that the faces in each orbit have the same value of  $p(F)$ .

In order to compare face shapes, or calculate  $p(F)$ , we need to make the face shape notation unique. That is, we need to identify a starting edge, which we do by distinguishing between the two forms of orientation on  $S$ . If  $S$  is a cuboctahedron and  $P$  has edge length 2, or if  $S$  is an icosidodecahedron and  $P$  has edge length 2 or 4, we assume that the starting edge of  $F$  is a directed edge  $\{u,v\}$  of  $P$  such that in the edge path  $\{u,w,\dots,v\}$  of  $S$  of minimum length connecting  $u$  and  $v$ , the triangle face of  $S$  adjoining

$\{u,w\}$  is to the left of  $\{u,w\}$ . If  $S$  is an icosidodecahedron and  $P$  has edge length  $2d$ , we assume that the starting edge of  $F$  is a directed edge  $\{u,v\}$  of  $P$  such that in the minimum edge path  $\{u,w,v\}$  of  $S$  the centroid of the pentagon face of  $S$  traversed by  $\{u,w\}$  is to the left of  $\{u,w\}$ . With this convention,  $P$  is fully determined by its face shape, vertex position, and edge length, allowing a description of the face  $F$  and a calculation of  $p(F)$ .

If  $S$  is a cuboctahedron and  $P$  has edge length 2, then a face with shape  $[f,f,f,f]$  is a triangle inscribed in a hexagonal circumference of  $S$  parallel to a triangle face, with  $p(F) = 3$ ; a face with shape  $[r,l,r,l]$  is an antiprismatic hexagon with vertices in two opposite triangular faces of  $S$ , and has  $p(F) = 6$ ; and a face with shape  $[l,r,l,r]$  is a nonplanar square with its vertices at opposite square faces of  $S$  and its diagonals given by diagonals of those square faces, and has  $p(F) = 4$ .

If  $S$  is an icosidodecahedron and  $P$  has edge length 2, then a face with shape  $[f,f,f,f]$  is a convex pentagon inscribed in a decagonal circumference of  $S$  parallel to a pentagon face, with  $p(F) = 5$ ; a face with shape  $[r,l,r,l]$  is an antiprismatic decagon positioned along a decagon circumference of  $S$ , and has  $p(F) = 10$ ; and a face with shape  $[l,r,l,r]$  is an antiprismatic hexagon positioned along a circumference of  $S$  parallel to a triangle face of  $S$ , and has  $p(F) = 6$ .

If  $S$  is an icosidodecahedron and  $P$  has edge length 4, then a face with shape  $[f,f,f,f]$  is a pentagram inscribed in a decagonal circumference of  $S$  parallel to a pentagon face of  $S$ , with  $p(F) = 5$ ; a face with shape  $[l,r,l,r]$  is an antiprismatic decagon (over a pentagram)

positioned along a decagonal circumference of  $S$  and with its vertices in the two pentagonal faces of  $S$  parallel to the circumference, and has  $p(F) = 10$ ; and a face with shape  $[r,1,r,1]$  is an antiprismatic hexagon with its vertices in two opposite triangular faces of  $S$ , and has  $p(F) = 6$ .

Finally, when  $S$  is an icosidodecahedron and  $P$  has edge length  $2d$ , then a face with shape  $[f,f,f,f]$  is a triangle inscribed in a circumference of  $S$  parallel to a triangle face of  $S$ , with  $p(F) = 3$ ; a face with shape  $[r,1,r,1]$  is an antiprismatic decagon positioned along a circumference of  $S$  parallel to a pentagon face and with its vertices at opposite pentagon faces of  $S$ , and has  $p(F) = 10$ ; and a face with shape  $[1,r,1,r]$  is an antiprismatic decagon (over a pentagram) positioned along a circumference of  $S$  parallel to a pentagon face, and has  $p(F) = 10$ .

Consequently, if  $S$  is a cuboctahedron or an icosidodecahedron, and  $P$  has bicolor type, then  $P$  does not have a face with shape  $[f,f,f,f]$ , for by Lemma 5.7, if so,  $P$  must have two face orbits under  $G(P)$  (and hence under  $G^+(P)$ ) and the faces in the other orbit can not also have shape  $[f,f,f,f]$  and the same value of  $p(F)$ . Additionally, we have that the only candidate for such a polyhedron with two face orbits under  $G^+(P)$  is the structure with shape  $[r,1,r,1]$  &  $[1,r,1,r]$  when  $S$  is an icosidodecahedron and  $P$  has edge length  $2d$ .

It follows that if  $S$  is a cuboctahedron or an icosidodecahedron, and  $P$  has bicolor type, then  $G_F^+(P)$  does not contain automorphisms of the form  $\rho_v(F)$ , for that would require  $G^+(P)$  to be face transitive and every face  $F$  to have face shape of  $[a,f,a',f]$  or  $[f,a,f,a']$ ,

which by Lemma 5.14 would lead to a Petrie dual of bicolor type with a face of shape  $[f,f,f,f]$ ; but we just saw that this is impossible. Thus if  $P$  has one face orbit under  $G^+(P)$ , it must be that for every face,  $F$ , of  $P$ ,  $G_F^+(P)$  contains  $\sigma_1^2(F)$ ; this follows from Lemma 5.20. Since  $F$  does not have shape  $[a,f,a,f]$  or  $[f,a,f,a]$  (again we appeal to Lemma 5.14), the shape of  $F$  can only contain the symbols  $r$  and  $l$ , and so must be one of the four shapes  $[r,l,r,l]$ ,  $[l,r,l,r]$ ,  $[r,r,r,r]$ , or  $[l,l,l,l]$ . But if  $F$  has face shape  $[r,l,r,l]$  or  $[l,r,l,r]$ , then  $G_F^+(P)$  also contains automorphisms of the form  $\rho_e(F)$ . In fact, if  $e$  is an edge of  $F$ , the half-turn  $\varphi \in G_e^+(P)$  will not interchange  $F$  with the other face of  $P$  at  $e$  but instead map  $F$  to itself, giving  $\varphi = \rho_e(F)$  in  $G_F^+(P)$ ; however this is impossible. So  $F$  must have shape of either  $[r,r,r,r]$  or  $[l,l,l,l]$ . These two shapes have alternating edge orientation by Lemma 5.17, and in fact represent the same face. Since this is the only candidate for such a polyhedron with one face orbit under  $G^+(P)$ , its Petrie dual must have two face orbits (in fact, the element  $\varphi \in G_e^+(P)$  now stabilizes a face of the Petrie dual), and we have already found that there is only one possible polyhedron with two face orbits.

Thus we have that if  $S$  is a cuboctahedron or an icosidodecahedron, and  $P$  has bicolor type, then (modulo the polytopality of the Petrie dual) the only two possible regular polyhedra of index 2 occur when  $S$  is an icosidodecahedron and  $P$  has edge length  $2d$ , and they have shapes  $[r,l,r,l]$  &  $[l,r,l,r]$  and  $[r,r,r,r]$ . The latter shape gives skew hexagonal faces, with three vertices at vertices of triangle faces of  $S$ . It is the case that both these structures, which are more fully described in Theorem 5.3, are regular polyhedra of index 2, and their maps and geometric diagrams are in Appendix 2.

The only remaining case of bicolor type is when  $S$  is a dodecahedron, and the edge length of  $P$  is 3. Here  $q = 6$ , by Lemma 5.2, which is consistent with the fact that there are five possible directions in which a face boundary may continue at any vertex. Call these (in counterclockwise order)  $hr$ ,  $sr$ ,  $f$ ,  $sl$ , and  $hl$ .

By Lemma 5.16, we have that when  $S$  is a dodecahedron and  $P$  has bicolor type,  $G_F^+(P)$  does not contain automorphisms of the form  $\rho_v(F)$  for any face,  $F$ , of  $P$ , and, by Lemma 5.20,  $G_F^+(P)$  does not contain automorphisms of the form  $\rho_e(F)$  if  $P$  has one face orbit under  $G^+(P)$ . Thus for every face of  $P$  we have that  $G_F^+(P)$  contains  $\sigma_1^2(F)$ , but not  $\sigma_1(F)$ , and automorphisms of the form  $\rho_e(F)$  if  $P$  has two face orbits under  $G^+(P)$  (see Lemma 5.18 and Table 5.3).

It follows from Lemma 5.15 that if  $S$  is a dodecahedron, and the edge length of  $P$  is 3, and  $F$  is any face of  $P$ , then  $F$  has shape  $[a,b,a,b]$  (allowing  $b = a'$ ), where either the directed face boundary of  $F$  has no change in orientation and  $a, b \in \{sl, sr\}$ , or the directed face boundary of  $F$  changes orientation at every vertex and  $a, b \in \{hl, f, hr\}$ . (A mixed type such as  $\{sl, hr\}$  can not occur as  $\sigma_1^2(F)$  lies in  $G_F^+(P)$ .) The former case can not occur, for if every change of direction between successive edges of the directed face boundary of  $F$  is represented by either  $sr$  or  $sl$ , then the edge boundary of  $F$  would be confined to the diagonals of the faces of one of the cubes inscribed in  $S$  that form the regular compound of five cubes whose vertices coincide with the vertices of a regular dodecahedron. Any two of the components of this compound have only a pair of antipodal vertices in common, and no edge of  $F$  joins such a pair of antipodal vertices.

Thus if the shape of  $F$  consists of the symbols  $sr$  and  $sl$ ,  $P$  might be a compound but is not a polyhedron.

Of the remaining nine possible face shapes, we have that  $[hl,hl,hl,hl]$ ,  $[f,hr,f,hr]$ , and  $[f,hl,f,hl]$  are the same faces as  $[hr,hr,hr,hr]$ ,  $[hl,f,hl,f]$ , and  $[hr,f,hr,f]$  traversed in the opposite direction, which leaves six distinct face shapes. If  $P$  has two face orbits under  $G^+(P)$ , then, since  $G_F^+(P)$  does not contain  $\sigma_1(F)$  or an element  $\rho_v(F)$ , the possible face shapes are  $[hr,hl,hr,hl]$ ,  $[hl,hr,hl,hr]$ , and  $[f,f,f,f]$  (see Table 5.3). If  $P$  has one face orbit under  $G^+(P)$ , then  $G_F^+(P)$  is generated by  $\sigma_1^2(F)$ , since it can not contain automorphisms of the form  $\rho_v(F)$  or  $\rho_e(F)$ . In this case, if  $e$  is an edge of  $F$ , the half-turn  $\phi$  in  $G_e^+(P)$  must necessarily interchange the faces of  $P$ . Hence, in the Petrie dual  $Q$  of  $P$ ,  $\phi$  occurs as an element  $\rho_e(F')$  for some face  $F'$  (see also the proof of Lemma 5.18), so  $Q$  must have two face orbits under  $G^+(Q) = G^+(P)$ . Thus it suffices to investigate polyhedra with two face orbits, and then derive those with one face orbit by Petrie duality.

As before, we find  $p(F)$  for the cases when  $P$  has two face orbits under  $G^+(P)$ . In order to compare face shapes, or calculate  $p(F)$ , we need to make the face shape notation unique. That is, we need to identify a starting edge, which we do by distinguishing the orientation on  $S$ . When  $S$  is a dodecahedron and the edges of  $P$  have length 3, we say that an edge  $\{u,v\}$  of  $P$  has left orientation if the edge path  $\{u,w,t,v\}$  of  $S$  of minimum length connecting  $u$  and  $v$  bends left at  $w$ , and has right orientation if the path on  $S$  bends to the right at  $w$ . If the starting edge of  $F$  has left orientation, then  $[hr,hl,hr,hl]$  represents an antiprismatic hexagonal polygon with its vertices at the vertices of the vertex-figures of a

pair of antipodal vertices of  $S$ , and has  $p(F) = 6$ ;  $[hl,hr,hl,hr]$  represents an antiprismatic decagon with its vertices at the vertices of antipodal faces of  $S$ , and has  $p(F) = 10$ ; and  $[f,f,f,f]$  represents a star-polygon  $\{10/3\}$  (also an antiprismatic decagon) inscribed in a Petrie polygon of  $S$ , and has  $p(F) = 10$ .

Thus the only candidate for a polyhedron with two face orbits (and with bicolor type and where  $S$  is a dodecahedron) is the structure with shape  $[hl,hr,hl,hr]$  &  $[f,f,f,f]$ . It is the case that this structure is a regular polyhedron of index 2, and its Petrie dual (with one face orbit) is the structure with shape  $[hl,f,hl,f]$ , and is also a regular polyhedron of index 2. The maps and geometric diagrams for these structures, which are more fully described in Theorem 5.3, are set out in Appendix 2.

The following theorem summarizes the results for polyhedra with bicolor type.

**Theorem 5.3:** There are exactly four finite regular polyhedra of index 2 with bicolor type (and all edges of the same length). There are two whose vertices coincide with those of a regular icosidodecahedron,  $S$ , and with edge length of  $1+\sqrt{5}$  ( $=2d$ , being twice the length of the diagonal of a pentagonal face of  $S$  with sides of length 1). One of these polyhedra has  $p = 6$ ,  $q = 4$ , with genus = 6, and is orientable with non-planar hexagonal faces in one face orbit under  $G^+(S)$ , and has shape  $[r,r,r,r]$ . Its Petrie dual has  $p = 10$ ,  $q = 4$ , with genus = 20, and is non-orientable with two kinds of antiprismatic decagons as faces, in two face orbits under  $G^+(S)$ , and has shape  $[r,l,r,l]$  &  $[l,r,l,r]$ . The underlying regular maps for the two polyhedra are the dual of the map R6.2 and the dual of the map N20.1 (not the dual

of N20.2), respectively, of Conder [5]. The other two polyhedra have vertices coinciding with those of a regular dodecahedron and have edges of length 3. One of these has  $p = 4$ ,  $q = 6$ , with genus = 6, and is orientable with non-planar square faces in one face orbit under  $G^+(S)$ , and has shape  $[hl,f,hl,f]$ . Its Petrie dual has  $p = 10$ ,  $q = 6$ , with genus = 30, and is non-orientable with two kinds of antiprismatic decagons as faces, in two face orbits under  $G^+(S)$ , and has shape  $[hl,hr,hl,hr]$  &  $[f,f,f,f]$ . The underlying regular maps for these two polyhedra are the map R6.2 and the dual of the map N30.11 (not the dual of N30.10), respectively, of Conder [5].

Drawing together Theorems 5.1, 5.2, and 5.3, we now have established the following theorem:

**Theorem 5.4:** There are precisely 10 finite regular polyhedra of index 2 with vertices on one orbit under the full symmetry group. These polyhedra all have icosahedral symmetry, and all the polyhedra have faces such that the squares of the combinatorial rotation about the face lies in the geometric rotation group (that is,  $\sigma_1^2(F) \in G^+(P)$ , for all  $F$ ). Precisely three of these polyhedra have planar faces. For each of the polyhedra, the full symmetry group is edge transitive, but the rotation group is not.

Maps of these polyhedra, together with their geometric diagrams and verification of their regularity, are in Appendix 2. A summary of these polyhedra, which are listed in Petrie pairs in the order in which they occur in this Section, is given below. See Section 2 for notation. Recall that a polyhedron of type  $\{p, q\}_r$  has a Petrie dual of type  $\{r, q\}_p$ . We see

that each Petrie pair consists of one orientable polyhedron and one non-orientable polyhedron; one of these has one face orbit under  $G^+(P)$ , and the other has two face orbits under  $G^+(P)$ . There does not appear to be any obvious reason why this should be so.

| $\{p, q\}_r$    | $(f_0, f_1, f_2)$ | Edge length | # face orbits under $G^+(P)$ | Map classification (Conder [5]) (Note 1) | Notes                       |
|-----------------|-------------------|-------------|------------------------------|--|-----------------------------|
| $\{6, 6\}_6$    | (20, 60, 20)      | 1, 4        | 1                            | R11.5                                    | Planar faces; self-dual map |
| $\{6, 6\}_6$    | (20, 60, 20)      | 1, 4        | 2                            | N22.3                                    | One face orbit under $G(P)$ |
| $\{6, 4\}_5$    | (30, 60, 20)      | d           | 1                            | dual of N12.1                            |                             |
| $\{5, 4\}_6$    | (30, 60, 24)      | d           | 2                            | dual of R4.2                             | Planar faces                |
| $\{4, 6\}_5$    | (20, 60, 30)      | 2           | 1                            | N12.1                                    |                             |
| $\{5, 6\}_4$    | (20, 60, 24)      | 2           | 2                            | R9.16                                    | Planar faces                |
| $\{6, 4\}_{10}$ | (30, 60, 20)      | 2d          | 1                            | dual of R6.2                             |                             |
| $\{10, 4\}_6$   | (30, 60, 12)      | 2d          | 2                            | dual of N20.1                            |                             |
| $\{4, 6\}_{10}$ | (20, 60, 30)      | 3           | 1                            | R6.2                                     |                             |
| $\{10, 6\}_4$   | (20, 60, 12)      | 3           | 2                            | dual of N30.11                           |                             |

**Table 5.4:** Complete list of finite regular polyhedra of index 2 with vertices on one orbit

Note 1: In the Conder, Dobesanyi [4] and Conder [5] classification of regular maps, ‘R’ indicates an orientable, non-chiral (i.e., reflexible) map, and ‘N’ indicates a non-orientable map. Additionally, the number before the period is the genus of the map. Thus N30.11 is the 11<sup>th</sup> non-orientable map of genus 30 in the listing. Note that the Conder listing only contains maps of type  $\{p, q\}_r$  where  $p \leq q$ ; this is sufficient since the dual of such a map has type  $\{q, p\}_r$ .

Note that, as shown by the map classification in the 5<sup>th</sup> column, the two polyhedra of type  $\{6, 6\}_6$  are not combinatorially isomorphic, and also not dual to each other: one is orientable, the other non-orientable. However the orientable polyhedron of type  $\{6, 6\}_6$  is

combinatorially self-dual. Moreover, the polyhedra of types  $\{6, 4\}_5$  and  $\{4, 6\}_5$  are combinatorially dual, as are the polyhedra of types  $\{6, 4\}_{10}$  and  $\{4, 6\}_{10}$ . Finally, the two orientable polyhedra with  $p = 5$ , of type  $\{5, 4\}_6$  and  $\{5, 6\}_4$  (which both have planar faces) are combinatorially dual to the orientable polyhedra, of type  $\{4, 5\}_6$  and  $\{6, 5\}_4$  respectively, with vertices on two orbits described in Theorem 4.1.

Taken in conjunction, Theorems 4.1 and 5.4 fully classify all regular finite polyhedra of index 2.

### A Non-Petrie Duality

Looking at the maps in Appendix 1, we see that in each case two different families of polyhedra have the same map. Note that the vertices in the vertex orbit  $S^*$  have been labeled so that the numeric labels of polar opposite vertices have the same sum. (In the case of the tetrahedron, which is not centrally symmetric, we require the union of two copies of  $S^*$ , one homothetic to  $S$ , and one homothetic to  $-S$ .) Thus the effect of re-labeling the vertices of  $S^*$ , as specified in the top right corner of each combinatorial map in Appendix 1, is to ‘invert’ the position of the vertices in one orbit (relative to the center of  $P$ ), while keeping constant the position of the vertices in the other orbit of  $P$ . This operation does not change the type of the polyhedron, since every regular index 2 polyhedron with vertices on two orbits has type  $\{p, q\}$  where  $p$  is even. The position of the vertices is changed only in the tetrahedral case with 4 vertices in each orbit, where the

vertices of  $S^*$  change so as to coincide with the vertices of a tetrahedron inverted to their original position.

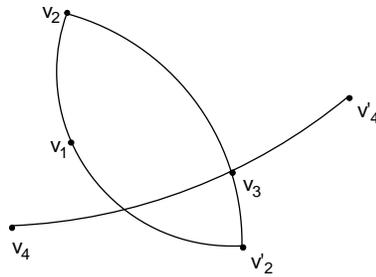
These observations show that there is a correspondence between pairs of regular polyhedra of index 2 with vertices on two orbits, which can be determined in the following manner. Given a regular polyhedron,  $P$ , of index 2 with vertices on two orbits, we let the polyhedron corresponding to  $P$  be  $C(P)$ . Then to obtain  $C(P)$ , we substitute every 2<sup>nd</sup> vertex on each face boundary of  $P$  for its polar opposite vertex, while leaving other vertices of  $P$  in their original position. As stated earlier, for every face  $F$  of  $P$ , we have that  $p(F)$  is even. Note that this construction gives two possibilities for  $C(P)$ , but we do not distinguish between them as they are geometrically equivalent, one being a point reflection of the other about the center of  $P$ . Thus  $C(P)$  is well-defined.

The following properties of  $C(P)$  follow from this operation:

- 1  $C(C(P)) = P$ ; so that  $C(P)$  is a duality. Note that in general it differs from the Petrie duality or the vertex/face duality.
- 2  $C(P)$  is a regular polyhedron of index 2, with the same number of vertex orbits. Thus all edges of  $C(P)$  are the same length.
- 3  $C(P)$  and  $P$  both have the same type,  $\{p,q\}$ , so that  $C(P)$  has the same  $f$ -vector as  $P$ .
- 4 If  $P$  does not have tetrahedral symmetry, then the vertices of  $C(P)$  coincide with those of  $P$ , and the sum of the edge length of  $C(P)$  and the edge length of  $P$  is

equal to the length between polar opposite vertices, which is 2 if  $S$  is an octahedron, 3 if  $S$  is an icosahedron, and 5 if  $S$  is a dodecahedron.

- 5 If  $P$  has tetrahedral symmetry, then the vertices of one of  $P$  and  $C(P)$  coincide with those of two aligned tetrahedra, and the vertices of the other one coincide with the vertices of two opposed tetrahedra (as illustrated in the first two maps of Appendix 1); further,  $P$  and  $C(P)$  have the same edge length of 1.
- 6 If  $P$  has shape  $[a,b]$ , then  $C(P)$  has shape  $[a,b']$ . For suppose the face  $F$  of  $P$  has edge boundary  $\{v_1, v_2, v_3, v_4, v_5, \dots\}$ , so that the corresponding face of  $C(P)$ ,  $F_C$ , has edge boundary  $\{v_1, v'_2, v_3, v'_4, v_5, \dots\}$ , where  $v'$  is the vertex opposite  $v$ . Then, as shown in the diagram below, the three pairs of consecutive projected edges  $\{u, w, u'\}$  are half circumferences of the sphere of projection, so that the change of direction along  $\{v_1, v'_2, v_3\}$  is opposite to that along  $\{v_1, v_2, v_3\}$ , whereas the change of direction along  $\{v'_2, v_3, v'_4\}$  is the same as that along  $\{v_2, v_3, v_4\}$ .



We now extend this duality to all regular polyhedra of index 2, which we do in an analogous way, but allowing for the possibility that  $p(F)$  may be odd. So suppose now that  $P$  is a regular polyhedron of index 2 with just one vertex orbit, of type  $\{p, q\}$ . We consider the following three cases:

- a)  $p$  is even, and no face of  $P$  has an edge boundary containing both  $v$  and  $v'$  (the vertex opposite  $v$ ).
- b)  $p$  is even, and the edge boundary of a face of  $P$  contains both  $v$  and  $v'$ .
- c)  $p$  is odd.

In case (a), the duality is exactly analogous to the situation with polyhedra with two vertex orbits. We have that if  $P$  is in case (a), then so is  $C(P)$ . There are six polyhedra satisfying the conditions of case (a), which are the two with  $p = 4$ , and the four with  $p = 6$ . Since  $P$  and  $C(P)$  are necessarily different, but with the same type  $\{p, q\}$ , this determines the case (a) dualities.

In case (b) it follows that every face of  $P$  has edge boundary of the form  $\{v_1, v_2, \dots, v_{p/2}, v'_1, v'_2, \dots, v'_{p/2}\}$ , and in case (c) it follows that no face of  $P$  has an edge boundary containing both  $v$  and  $v'$ . We have that if  $P$  is in case (b), then  $C(P)$  is in case (c), and vice versa. The case (b) polyhedra are the two with  $p = 10$  (of bicolor type), and the two case (c) polyhedra are the two with  $p = 5$  (of directed type).  $P$  and  $C(P)$  have the same value of  $q$  in all cases, so the dualities of these index 2 can be determined.

Comparing the properties of  $C(P)$  with those given above, we have that while the duality preserves the value of  $q$ , the values of  $p$  for  $C(P)$  can be half or double (or equal to) the value of  $p$  for  $P$ .  $C(P)$  thus has the same number of vertices and edges as  $P$ , but the number of faces can differ by a factor of two. Further the sum of lengths of

corresponding edges of  $C(P)$  and  $P$  is the distance between polar vertices. It is instructive to compare this to the construction in Lemma 4.4 showing how index 2 polyhedra can be generated from regular polyhedra. Finally, if  $P$  has shape  $[a,b,c,d]$ , then  $C(P)$  has shape  $[a,b',c,d']$ .

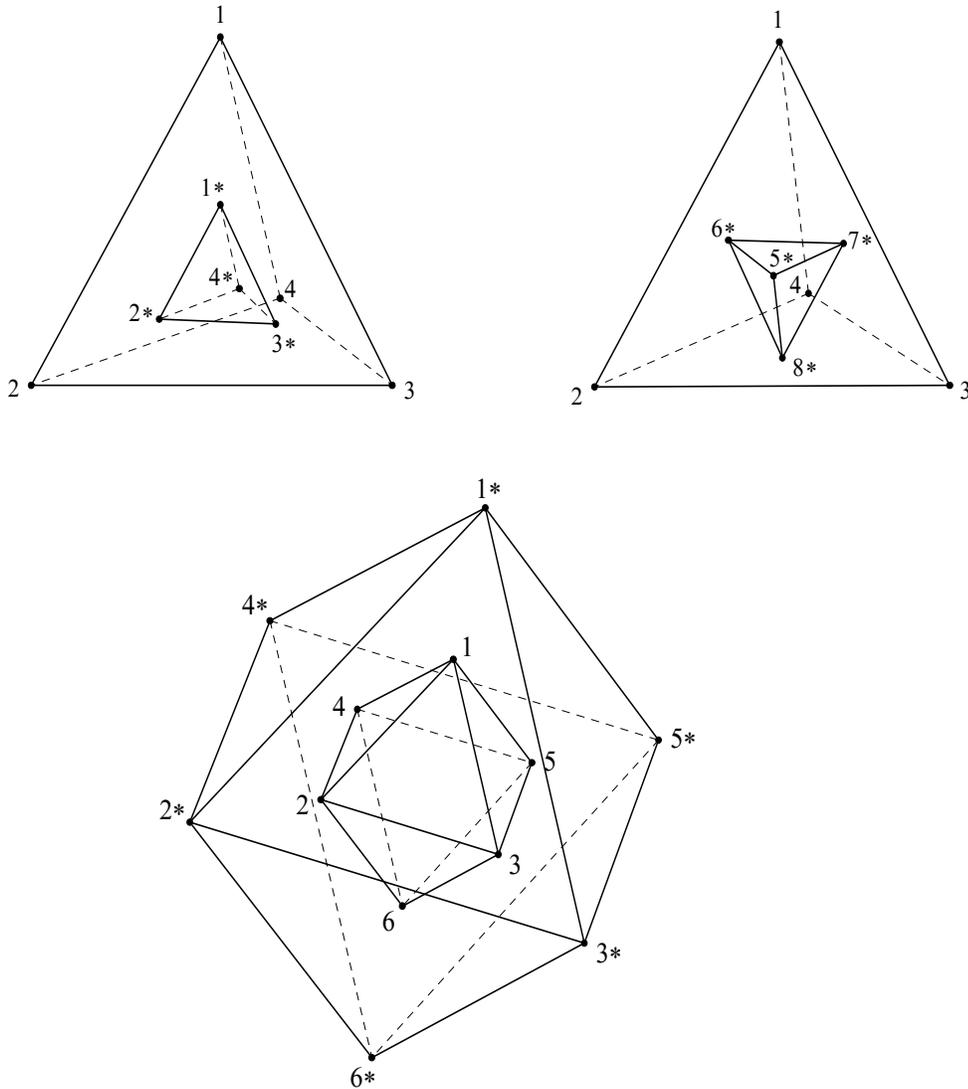
We note that this duality can be extended in a natural way to regular polyhedra. Doing so, we obtain that two regular polyhedra  $P$  and  $Q$  correspond under this duality the regular index 2 polyhedra generated, as in Lemma 4.4, from  $P$  and  $Q$  also correspond under this duality. We conjecture that this duality applies to all combinatorially regular of any index. This requires showing, inter alia, that  $C(P)$  is a polyhedron, not a compound.

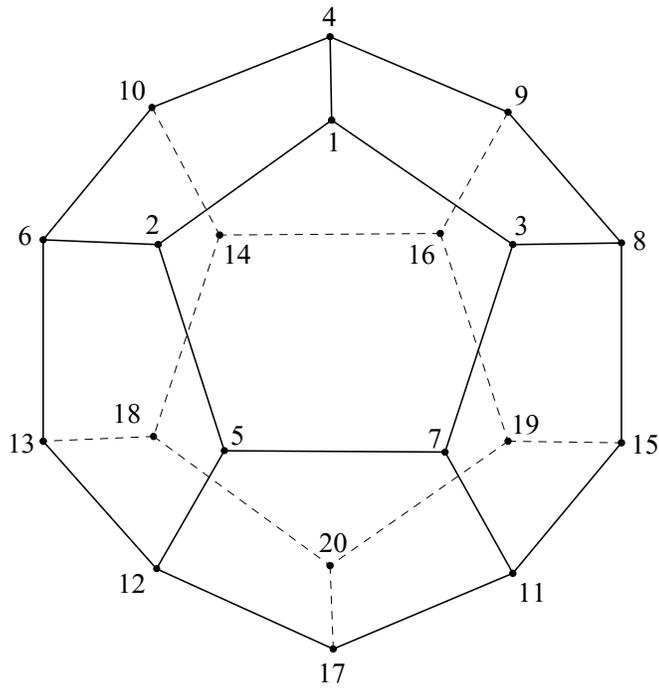
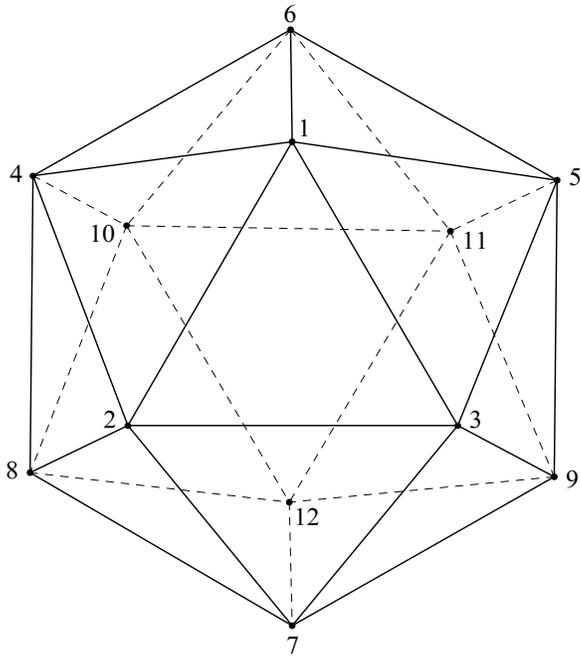
## Appendix 1: Diagrams and maps for Polyhedra with vertices on two orbits

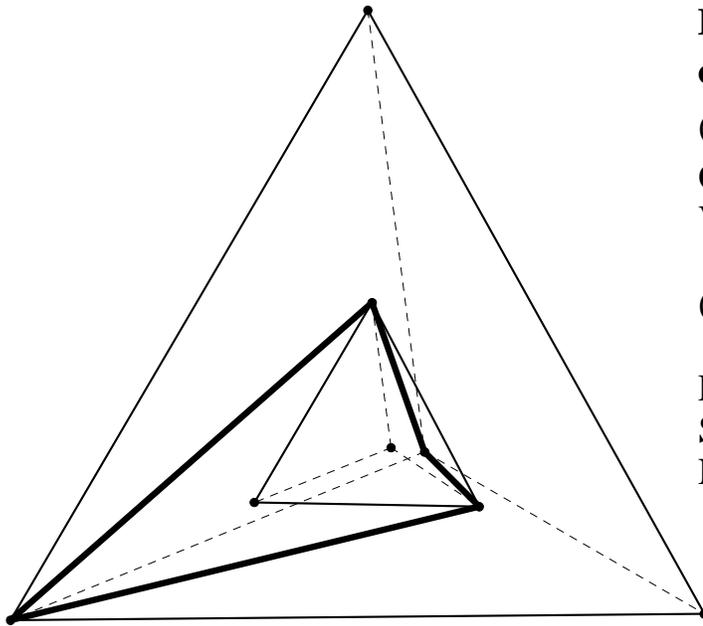
For each polyhedron, we give a geometric diagram of one face (each of the polyhedra with vertices on two orbits have just one face orbit under  $G^+(P)$ ), as well as a combinatorial map of the abstract polyhedron showing the automorphisms  $\rho_0, \rho_1, \rho_2, \rho_0\rho_1, \rho_0\rho_2, \rho_1\rho_2$ , and  $\rho_0\rho_1\rho_2$ .

The maps use the vertex labelling shown below. For polyhedra with vertices on two orbits, there are five possible combinations of  $S$  and  $S^*$ . When  $S$  is an icosahedron or dodecahedron, the vertex labelling for  $S^*$  corresponds to that of  $S$ , in the same way as it does when  $S$  is an octahedron, as shown below.

For polyhedra generated from index 1 polyhedra, the diagrams are the same regardless of whether  $S$  is the inner or outer vertex orbit. For the four 'sporadic' polyhedra not generated from an index 1 polyhedron, the diagrams assume that  $S$  is the inner vertex orbit, so that  $r > 1$ .







**Face of the polyhedron  
of type  $\{4,3\}_6$**

$(f_0, f_1, f_2) = (8, 12, 6)$

Orientable genus = 0

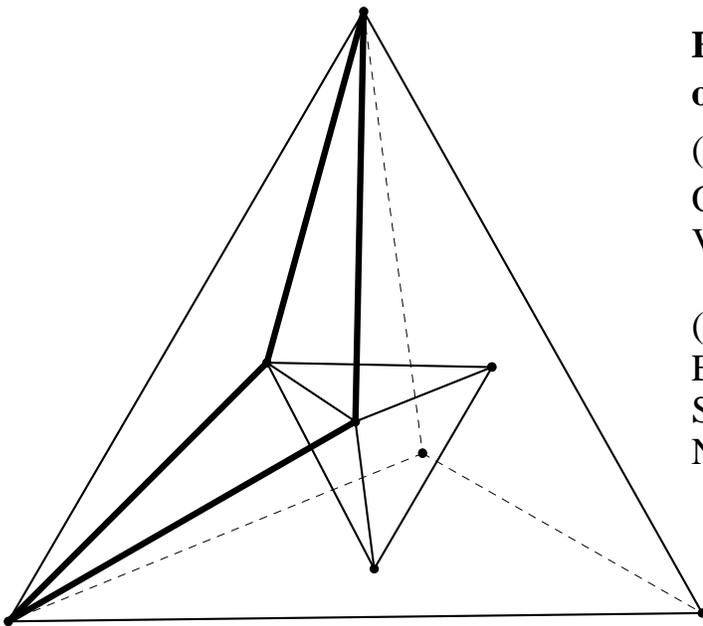
Vertices of two tetrahedra,  
aligned

(generated from Petrie  
of tetrahedron)

Edge length = 1

Shape of  $[r,l]$

Non-planar faces



**Face of the polyhedron  
of type  $\{4,3\}_6$**

$(f_0, f_1, f_2) = (8, 12, 6)$

Orientable genus = 0

Vertices of two tetrahedra,  
inverted

(generated from cube)

Edge length = 1

Shape of  $[r,r]$

Non-planar faces

**Map of the polyhedron of type  $\{4,3\}_6$**

$(f_0, f_1, f_2) = (8, 12, 6)$

Orientable genus = 0

Vertices of two tetrahedra, aligned  
(generated from Petrie of tetrahedron)

Edge length = 1

Shape of  $[r,l]$

Non-planar faces

When all vertices in orbit  $S^*$  are relabelled  
from  $n^*$  to  $(9-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{4,3\}_6$**

$(f_0, f_1, f_2) = (8, 12, 6)$

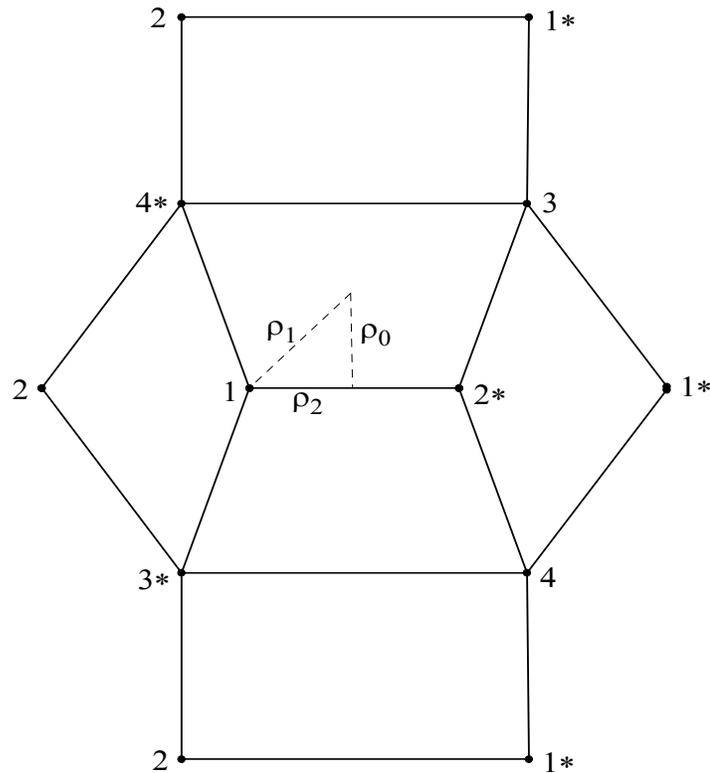
Orientable genus = 0

Vertices of two tetrahedra, inverted  
(generated from cube)

Edge length = 1

Shape of  $[r,r]$

Non-planar faces



$\rho_0: (1,2^*)(2,1^*)(3,4^*)(4,3^*)$

$\rho_1: (1)(2,4)(3)(1^*)(2^*,4^*)(3^*)$

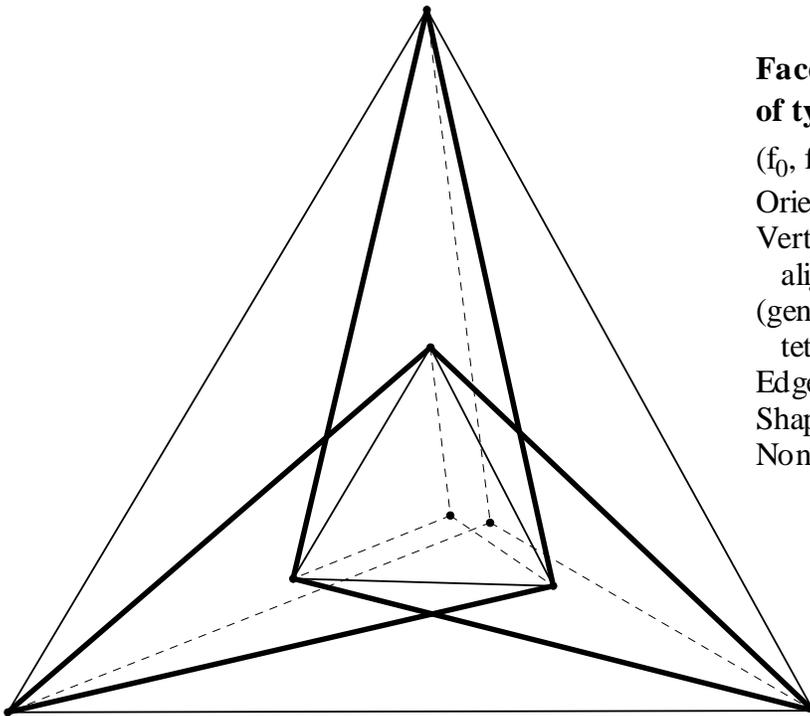
$\rho_2: (1)(2)(3,4)(1^*)(2^*)(3^*,4^*)$

$\rho_0\rho_1: (1,2^*,3,4^*)(2,3^*,4,1^*)$

$\rho_0\rho_2: (1,2^*)(2,1^*)(3,3^*)(4,4^*)$

$\rho_1\rho_2: (1)(2,4,3)(1^*)(2^*,4^*,3^*)$

$\rho_0\rho_1\rho_2: (1,2^*,3,1^*,2,3^*)(4,4^*)$



**Face of the polyhedron  
of type  $\{6,3\}_4$**

$(f_0, f_1, f_2) = (8, 12, 4)$

Orientable genus = 1

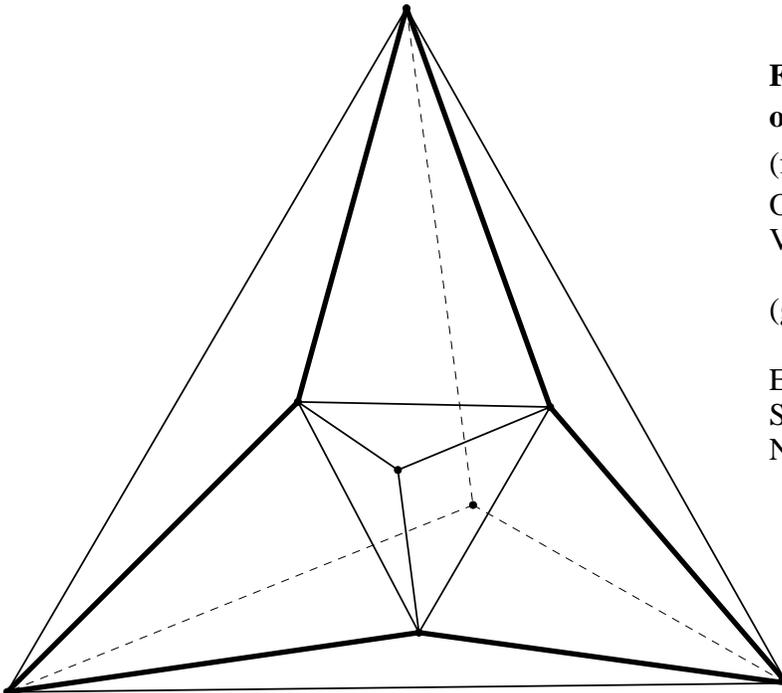
Vertices of two tetrahedra,  
aligned

(generated from  
tetrahedron)

Edge length = 1

Shape of  $[r,r]$

Non-planar faces



**Face of the polyhedron  
of type  $\{6,3\}_4$**

$(f_0, f_1, f_2) = (8, 12, 4)$

Orientable genus = 1

Vertices of two tetrahedra,  
inverted

(generated from Petrie  
of cube)

Edge length = 1

Shape of  $[r,l]$

Non-planar faces

**Map of the polyhedron of type  $\{6,3\}_4$**

$(f_0, f_1, f_2) = (8, 12, 4)$

Orientable genus = 1

Vertices of two tetrahedra, aligned  
(generated from tetrahedron)

Edge length = 1

Shape of  $[r,r]$

Non-planar faces

When all vertices in orbit  $S^*$  are relabelled  
from  $n^*$  to  $(9-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{6,3\}_4$**

$(f_0, f_1, f_2) = (8, 12, 4)$

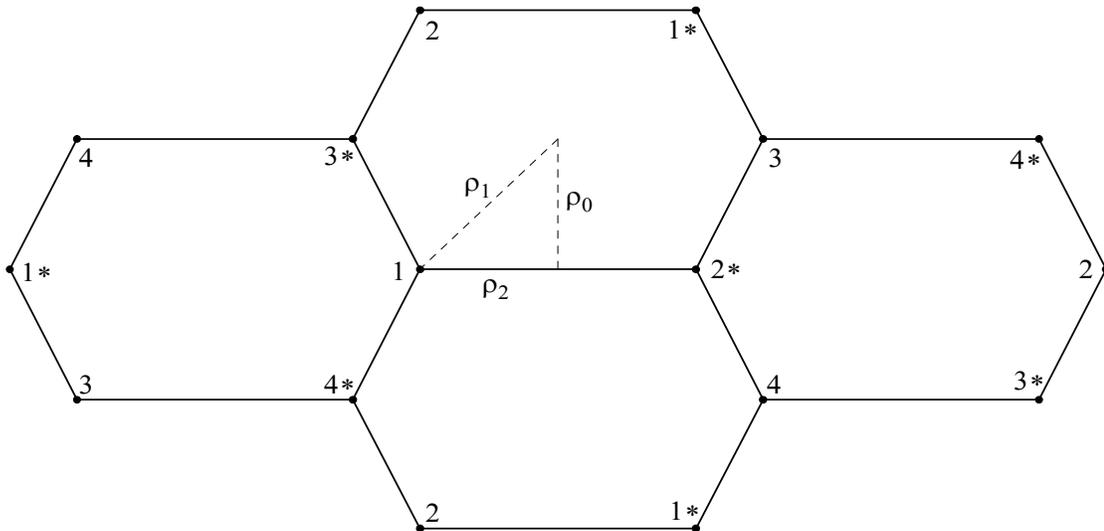
Orientable genus = 1

Vertices of two tetrahedra, inverted  
(generated from Petrie of cube)

Edge length = 1

Shape of  $[r,l]$

Non-planar faces



$\rho_0: (1,2^*)(2,1^*)(3,3^*)(4,4^*)$

$\rho_1: (1)(2,3)(4) (1^*)(2^*,3^*)(4^*)$

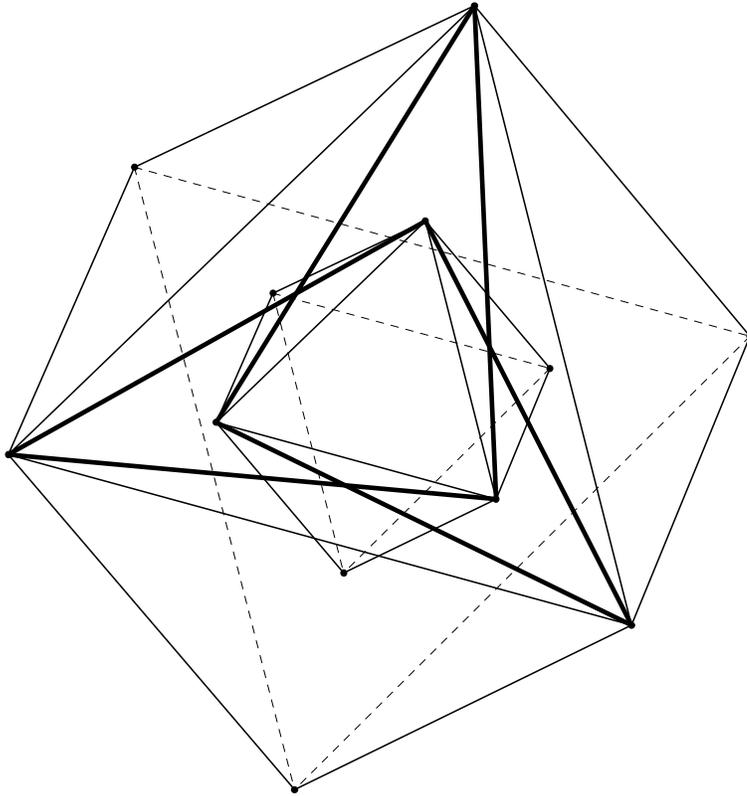
$\rho_2: (1)(2)(3,4) (1^*)(2^*)(3^*,4^*)$

$\rho_0\rho_1: (1,2^*,3,1^*,2,3^*)(4,4^*)$

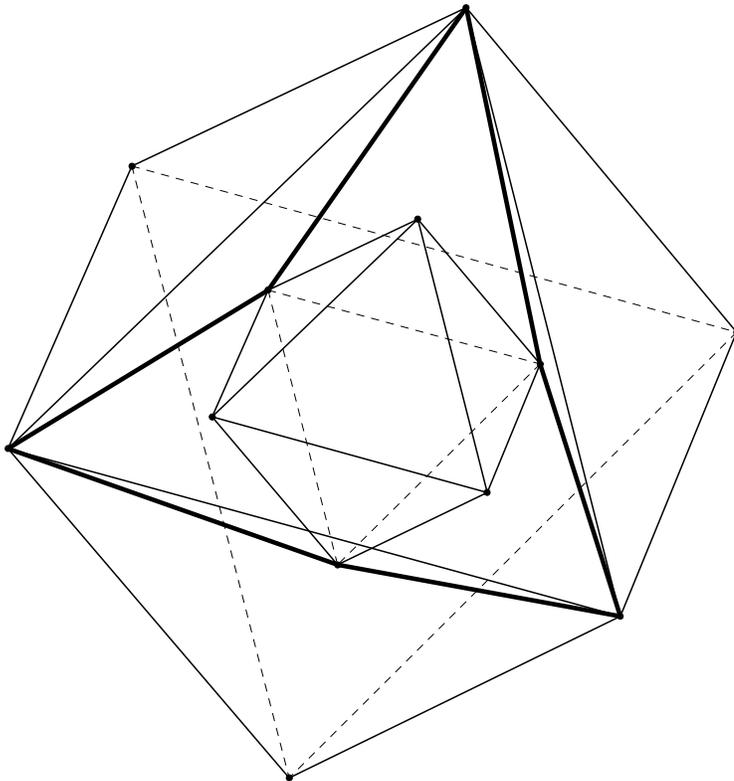
$\rho_0\rho_2: (1,2^*)(2,1^*)(3,4^*)(4,3^*)$

$\rho_1\rho_2: (1)(2,3,4) (1^*)(2^*,3^*,4^*)$

$\rho_0\rho_1\rho_2: (1,2^*,3,4^*)(2,3^*,4,1^*)$



**Face of the polyhedron**  
**of type  $\{6,4\}_6$**   
 $(f_0, f_1, f_2) = (12, 24, 8)$   
 Orientable genus = 3  
 Vertices of two octahedra  
 (generated from  
 octahedron)  
 Edge length = 1  
 Shape of  $[r,r]$   
 Non-planar faces



**Face of the polyhedron**  
**of type  $\{6,4\}_6$**   
 $(f_0, f_1, f_2) = (12, 24, 8)$   
 Orientable genus = 3  
 Vertices of two octahedra  
 (generated from Petrie  
 of octahedron)  
 Edge length = 1  
 Shape of  $[r,l]$   
 Non-planar faces

Map Classification Number: dual of R3.4

**Map of the polyhedron of type  $\{6,4\}_6$**

$(f_0, f_1, f_2) = (12, 24, 8)$

Orientable genus = 3

Vertices of two octahedra

(generated from octahedron)

Edge length = 1

Shape of  $[r,r]$

Non-planar faces

When all vertices in orbit  $S^*$  are relabelled from  $n^*$  to  $(7-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{6,4\}_6$**

$(f_0, f_1, f_2) = (12, 24, 8)$

Orientable genus = 3

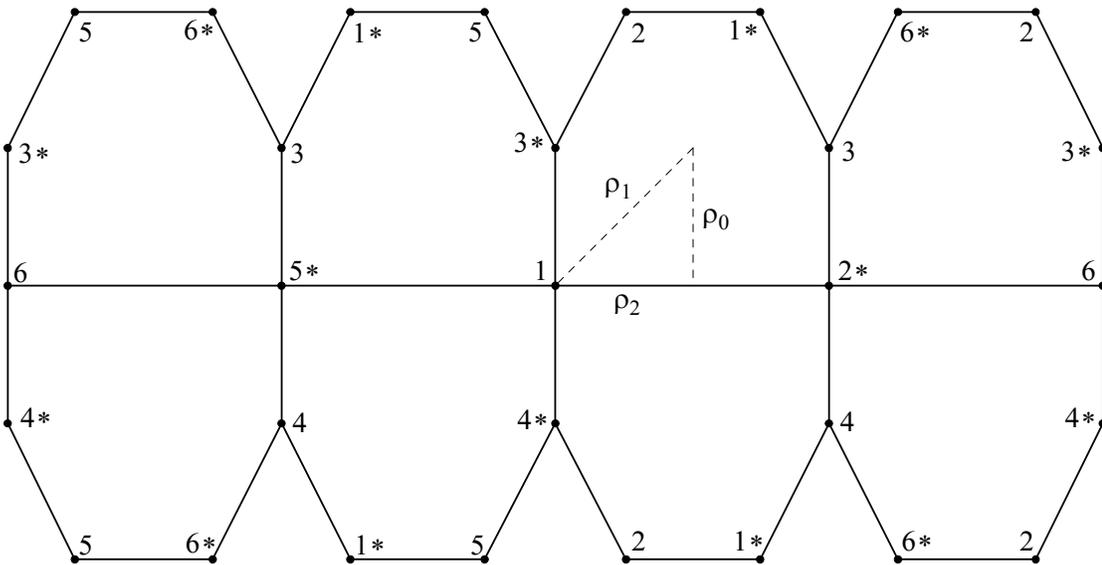
Vertices of two octahedra

(generated from Petrie of octahedron)

Edge length = 1

Shape of  $[r,l]$

Non-planar faces



$$\rho_0: (1,2^*)(2,1^*)(3,3^*)(4,4^*)(5,6^*)(6,5^*)$$

$$\rho_1: (1)(2,3)(4,5)(6) (1^*)(2^*,3^*)(4^*,5^*)(6^*)$$

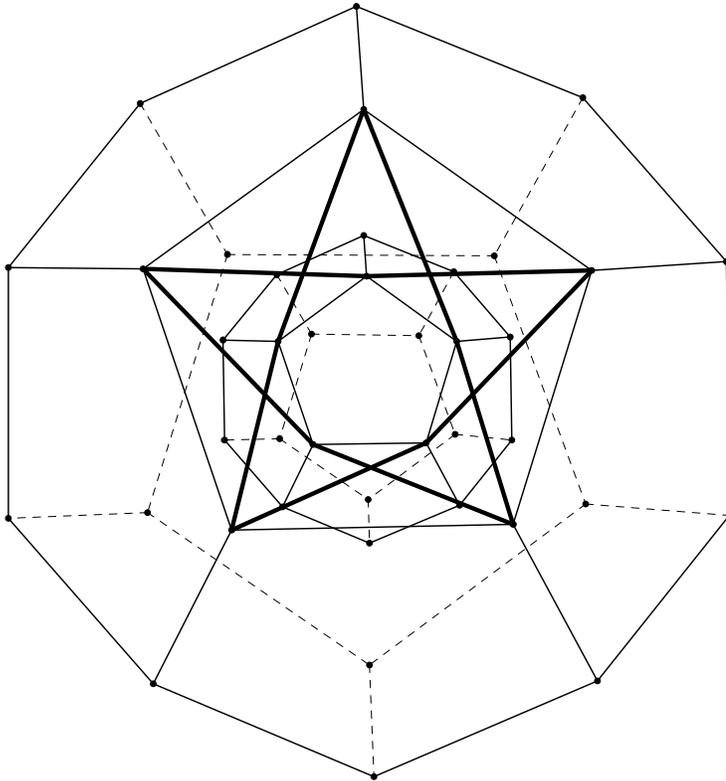
$$\rho_2: (1)(2)(3,4)(5)(6) (1^*)(2^*)(3^*,4^*)(5^*)(6^*)$$

$$\rho_0\rho_1: (1,2^*,3,1^*,2,3^*)(4,6^*,5,4^*,6,5^*)$$

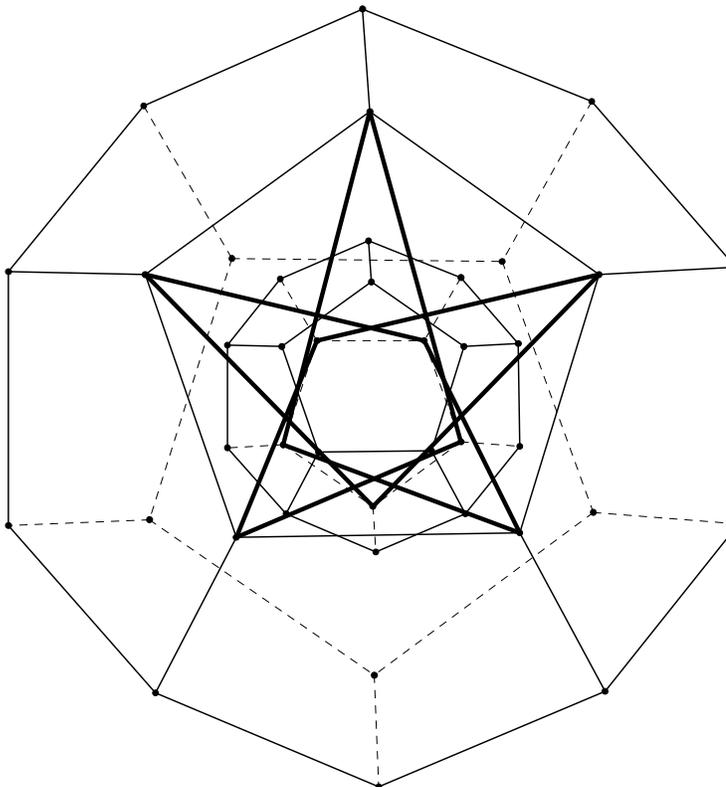
$$\rho_0\rho_2: (1,2^*)(2,1^*)(3,4^*)(4,3^*)(5,6^*)(6,5^*)$$

$$\rho_1\rho_2: (1)(2,3,5,4)(6) (1^*)(2^*,3^*,5^*,4^*)(6^*)$$

$$\rho_0\rho_1\rho_2: (1,2^*,3,6^*,5,4^*)(2,3^*,6,5^*,4,1^*)$$



**Face of the polyhedron**  
**of type  $\{10,3\}_{10}$**   
 $(f_0, f_1, f_2) = (40, 60, 12)$   
 Orientable genus = 5  
 Vertices of two dodecahedra  
 (generated from dodecahedron)  
 Edge length = 1  
 Shape of  $[r,r]$   
 Non-planar faces



**Face of the polyhedron**  
**of type  $\{10,3\}_{10}$**   
 $(f_0, f_1, f_2) = (40, 60, 12)$   
 Orientable genus = 5  
 Vertices of two dodecahedra  
 (generated from Petrie of great  
 stellated dodecahedron)  
 Edge length = 4  
 Shape of  $[r,l]$   
 Non-planar faces

Map Classification Number: dual of R5.2

**Map of the polyhedron of type  $\{10,3\}_{10}$**

$(f_0, f_1, f_2) = (40, 60, 12)$

Orientable genus = 5

Vertices of two dodecahedra  
(generated from dodecahedron)

Edge length = 1

Shape of  $[r,r]$

Non-planar faces

When all vertices in orbit  $S^*$  are relabelled  
from  $n^*$  to  $(21-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{10,3\}_{10}$**

$(f_0, f_1, f_2) = (40, 60, 12)$

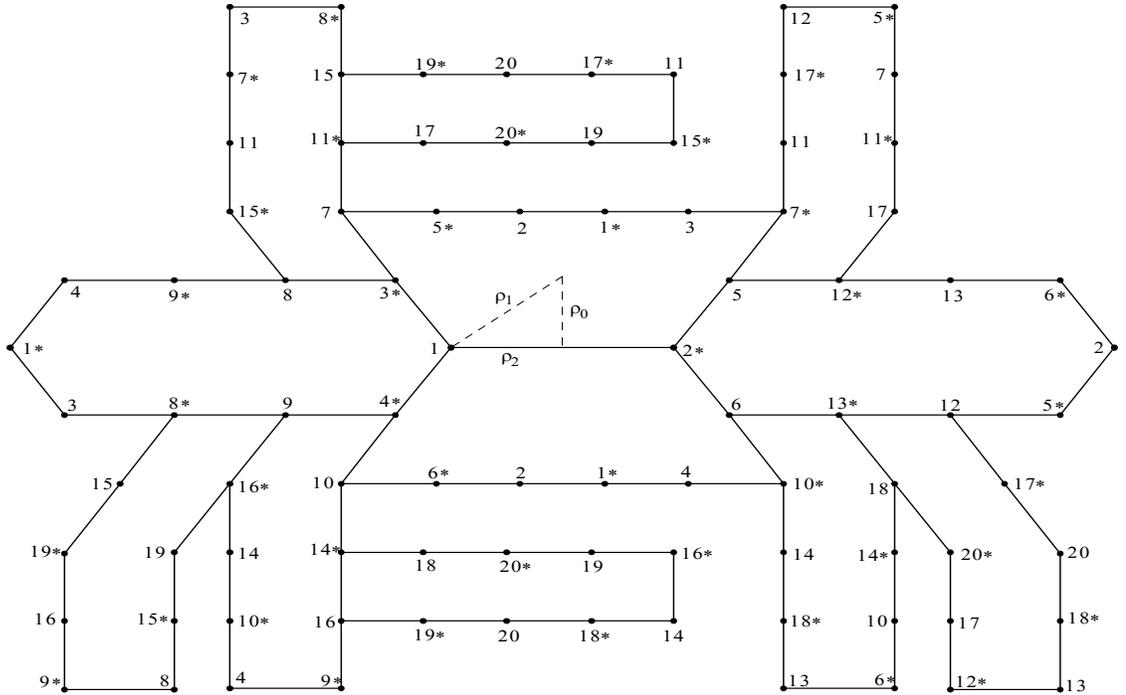
Orientable genus = 5

Vertices of two dodecahedra  
(generated from Petrie of great stellated dodecahedron)

Edge length = 4

Shape of  $[r,l]$

Non-planar faces



$\rho_0: (1,2^*)(2,1^*)(3,5^*)(4,6^*)(5,3^*)(6,4^*)(7,7^*)(8,12^*)(9,13^*)(10,10^*)$   
 $(11,11^*)(12,8^*)(13,9^*)(14,14^*)(15,17^*)(16,18^*)(17,15^*)(18,16^*)(19,20^*)(20,19^*)$

$\rho_1: (1)(2,3)(4)(5,7)(6,8)(9,10)(11,12)(13,15)(14,16)(17)(18,19)(20)$   
 $(1^*)(2^*,3^*)(4^*)(5^*,7^*)(6^*,8^*)(9^*,10^*)(11^*,12^*)(13^*,15^*)(14^*,16^*)(17^*)(18^*,19^*)(20^*)$

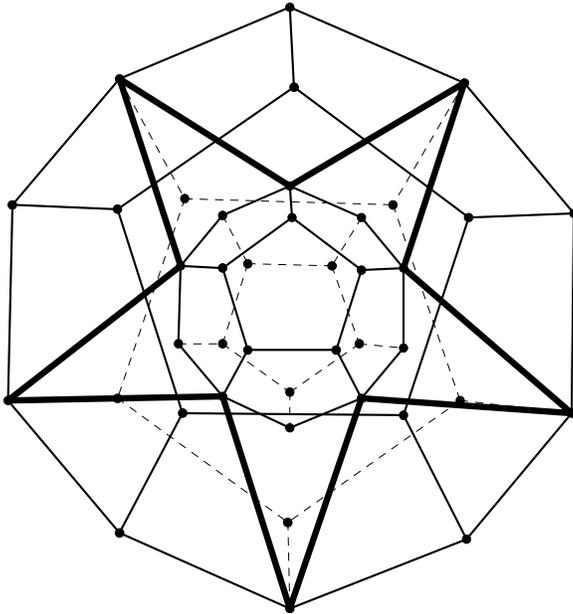
$\rho_2: (1)(2)(3,4)(5,6)(7,10)(8,9)(11,14)(12,13)(15,16)(17,18)(19)(20)$   
 $(1^*)(2^*)(3^*,4^*)(5^*,6^*)(7^*,10^*)(8^*,9^*)(11^*,14^*)(12^*,13^*)(15^*,16^*)(17^*,18^*)(19^*)(20^*)$

$\rho_0\rho_1: (1,2^*,5,7^*,3,1^*,2,5^*,7,3^*)(4,6^*,12,11^*,8,4^*,6,12^*,11,8^*)$   
 $(9,10^*,13,17^*,15,9^*,10,13^*,17,15^*)(14,18^*,20,19^*,16,14^*,18,20^*,19,16^*)$

$\rho_0\rho_2: (1,2^*)(2,1^*)(3,6^*)(4,5^*)(5,4^*)(6,3^*)(7,10^*)(8,13^*)(9,12^*)(10,7^*)$   
 $(11,14^*)(12,9^*)(13,8^*)(14,11^*)(15,18^*)(16,17^*)(17,16^*)(18,15^*)(19,20^*)(20,19^*)$

$\rho_1\rho_2: (1)(2,3,4)(5,8,10)(6,7,9)(11,16,13)(12,15,14)(17,19,18)(20)$   
 $(1^*)(2^*,3^*,4^*)(5^*,8^*,10^*)(6^*,7^*,9^*)(11^*,16^*,13^*)(12^*,15^*,14^*)(17^*,19^*,18^*)(20^*)$

$\rho_0\rho_1\rho_2: (1,2^*,5,12^*,17,20^*,19,16^*,9,4^*)(2,5^*,12,17^*,20,19^*,16,9^*,4,1^*)$   
 $(3,6^*,7,13^*,11,18^*,15,14^*,8,10^*)(6,7^*,13,11^*,18,15^*,14,8^*,10,3^*)$



**Face of the polyhedron  
of type  $\{10,3\}_{10}$**

$(f_0, f_1, f_2) = (40, 60, 12)$

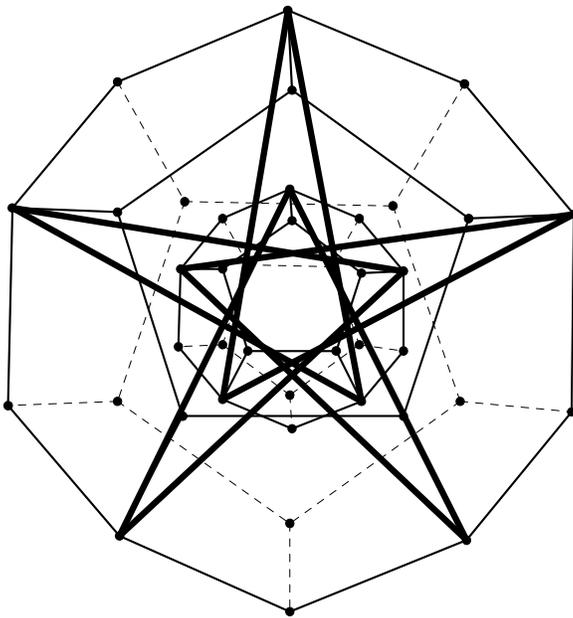
Orientable genus = 5

Vertices of two dodecahedra  
(generated from Petrie of  
dodecahedron)

Edge length = 1

Shape of  $[r,l]$

Non-planar faces



**Face of the polyhedron  
of type  $\{10,3\}_{10}$**

$(f_0, f_1, f_2) = (40, 60, 12)$

Orientable genus = 5

Vertices of two dodecahedra  
(generated from great  
stellated dodecahedron)

Edge length = 4

Shape of  $[r,r]$

Non-planar faces

Map Classification Number: dual of R5.2

When all vertices in orbit  $S^*$  are relabelled from  $n^*$  to  $(21-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{10,3\}_{10}$**

**Map of the polyhedron of type  $\{10,3\}_{10}$**

$(f_0, f_1, f_2) = (40, 60, 12)$

$(f_0, f_1, f_2) = (40, 60, 12)$

Orientable genus = 5

Orientable genus = 5

Vertices of two dodecahedra

Vertices of two dodecahedra

(generated from Petrie of dodecahedron)

(generated from great stellated dodecahedron)

Edge length = 1

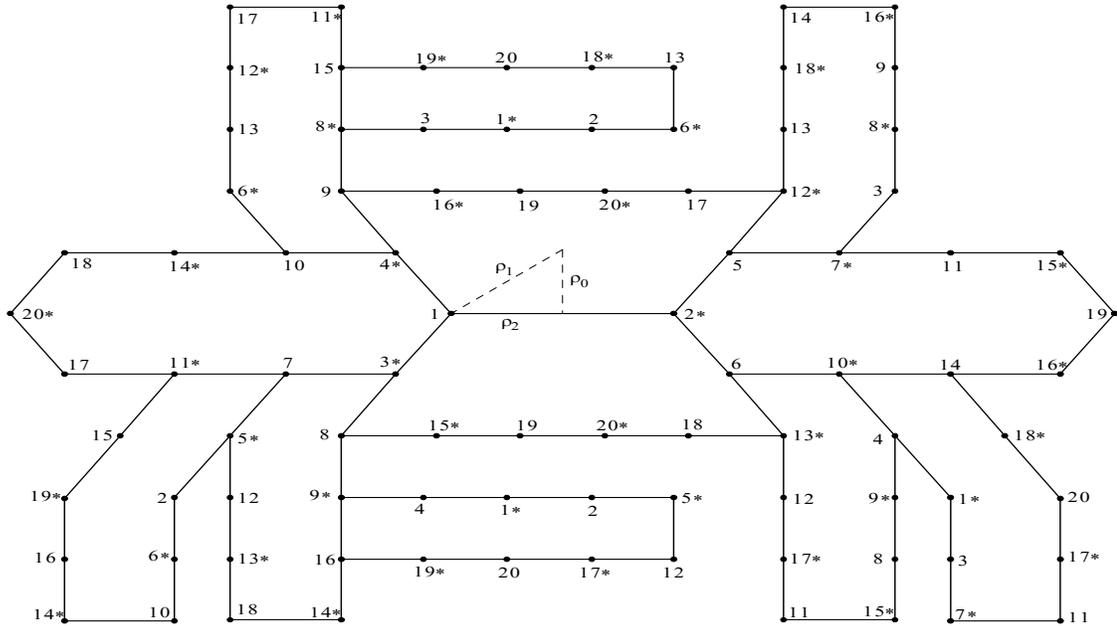
Edge length = 4

Shape of  $[r, l]$

Shape of  $[r, r]$

Non-planar faces

Non-planar faces



$\rho_0: (1, 2^*)(2, 1^*)(3, 6^*)(4, 5^*)(5, 4^*)(6, 3^*)(7, 10^*)(8, 13^*)(9, 12^*)(10, 7^*)$   
 $(11, 14^*)(12, 9^*)(13, 8^*)(14, 11^*)(15, 18^*)(16, 17^*)(17, 16^*)(18, 15^*)(19, 20^*)(20, 19^*)$

$\rho_1: (1)(2, 4)(3)(5, 9)(6, 10)(7, 8)(11, 15)(12, 16)(13, 14)(17, 19)(18)(20)$   
 $(1^*)(2^*, 4^*)(3^*)(5^*, 9^*)(6^*, 10^*)(7^*, 8^*)(11^*, 15^*)(12^*, 16^*)(13^*, 14^*)(17^*, 19^*)(18^*)(20^*)$

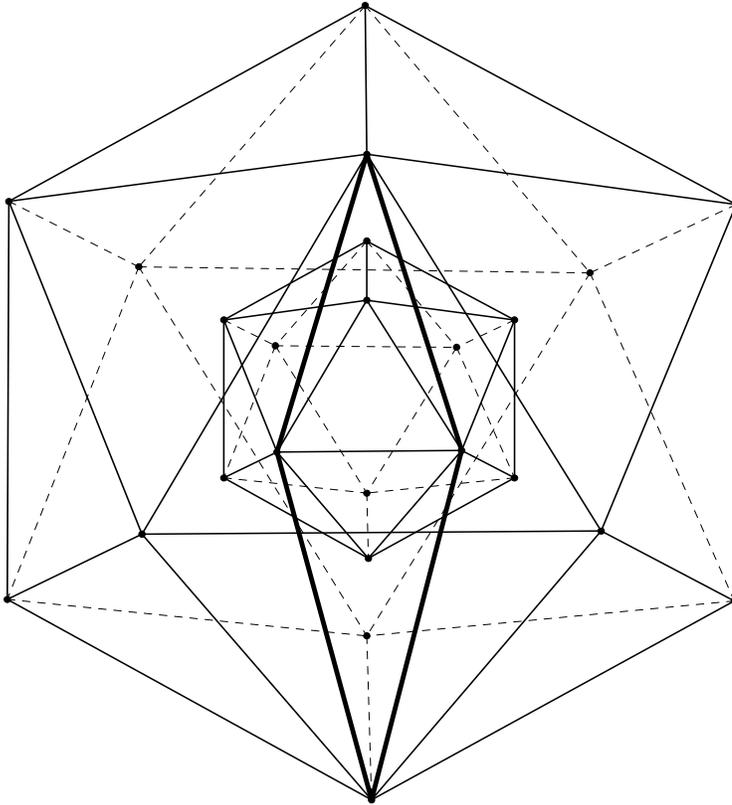
$\rho_2: (1)(2)(3, 4)(5, 6)(7, 10)(8, 9)(11, 14)(12, 13)(15, 16)(17, 18)(19)(20)$   
 $(1^*)(2^*)(3^*, 4^*)(5^*, 6^*)(7^*, 10^*)(8^*, 9^*)(11^*, 14^*)(12^*, 13^*)(15^*, 16^*)(17^*, 18^*)(19^*)(20^*)$

$\rho_0\rho_1: (1, 2^*, 5, 12^*, 17, 20^*, 19, 16^*, 9, 4^*)(2, 5^*, 12, 17^*, 20, 19^*, 16, 9^*, 4, 1^*)$   
 $(3, 6^*, 7, 13^*, 11, 18^*, 15, 14^*, 8, 10^*)(6, 7^*, 13, 11^*, 18, 15^*, 14, 8^*, 10, 3^*)$

$\rho_0\rho_2: (1, 2^*)(2, 1^*)(3, 5^*)(4, 6^*)(5, 3^*)(6, 4^*)(7, 7^*)(8, 12^*)(9, 13^*)(10, 10^*)$   
 $(11, 11^*)(12, 8^*)(13, 9^*)(14, 14^*)(15, 17^*)(16, 18^*)(17, 15^*)(18, 16^*)(19, 20^*)(20, 19^*)$

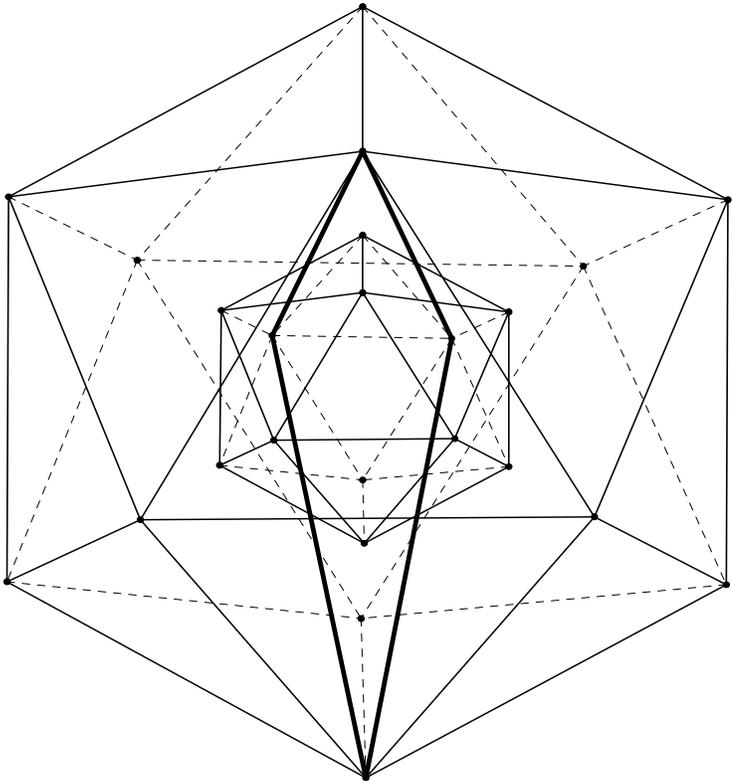
$\rho_1\rho_2: (1)(2, 4, 3)(5, 10, 8)(6, 9, 7)(11, 13, 16)(12, 14, 15)(17, 18, 19)(20)$   
 $(1^*)(2^*, 4^*, 3^*)(5^*, 10^*, 8^*)(6^*, 9^*, 7^*)(11^*, 13^*, 16^*)(12^*, 14^*, 15^*)(17^*, 18^*, 19^*)(20^*)$

$\rho_0\rho_1\rho_2: (1, 2^*, 5, 7^*, 3, 1^*, 2, 5^*, 7, 3^*)(4, 6^*, 12, 11^*, 8, 4^*, 6, 12^*, 11, 8^*)$   
 $(9, 10^*, 13, 17^*, 15, 9^*, 10, 13^*, 17, 15^*)(14, 18^*, 20, 19^*, 16, 14^*, 18, 20^*, 19, 16^*)$



**Face of the polyhedron  
of type  $\{4,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 30)$   
 Orientable genus = 4  
 Vertices of two icosahedra  
 (not generated from any  
 regular polyhedron)  
 Edge length = 1  
 Shape of  $[hr, sr]$   
 Planar faces iff  
 $r = (1 + \sqrt{5}) / 2$



**Face of the polyhedron  
of type  $\{4,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 30)$   
 Orientable genus = 4  
 Vertices of two icosahedra  
 (not generated from any  
 regular polyhedron)  
 Edge length = 2  
 Shape of  $[hr, sl]$   
 Non-planar faces

Map Classification Number: R4.2

**Map of the polyhedron of type  $\{4,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 30)$

Orientable genus = 4

Vertices of two icosahedra

(not generated from any regular polyhedron)

Edge length = 1

Shape of [hr,sr]

Planar faces iff  $r = (1+\sqrt{5})/2$

When all vertices in orbit  $S^*$  are relabelled from  $n^*$  to  $(13-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{4,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 30)$

Orientable genus = 4

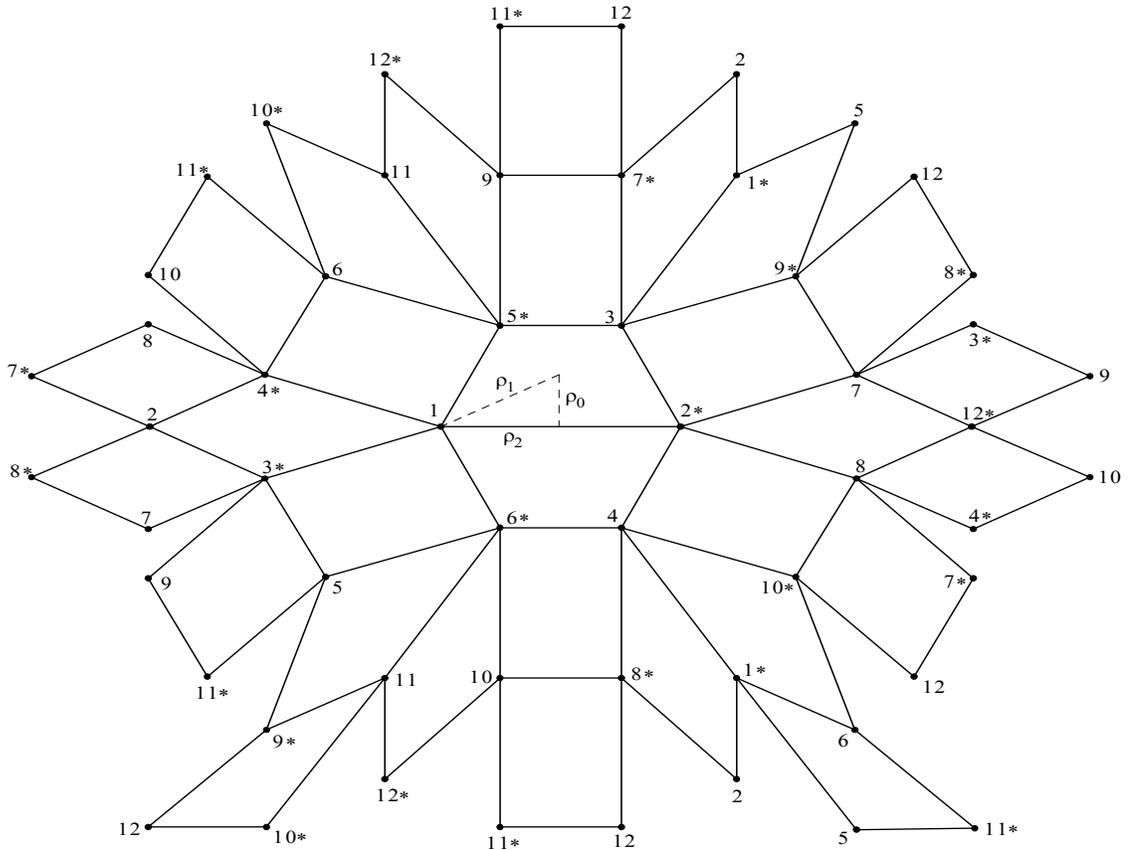
Vertices of two icosahedra

(not generated from any regular polyhedron)

Edge length = 2

Shape of [hr,s]

Non-planar faces



$\rho_0: (1,2^*)(2,12^*)(3,5^*)(4,6^*)(5,10^*)(6,9^*)(7,4^*)(8,3^*)(9,7^*)(10,8^*)(11,1^*)(12,11^*)$

$\rho_1: (1)(2,5)(3)(4,6)(7,9)(8,11)(10)(12) (1^*)(2^*,5^*)(3^*)(4^*,6^*)(7^*,9^*)(8^*,11^*)(10^*)(12^*)$

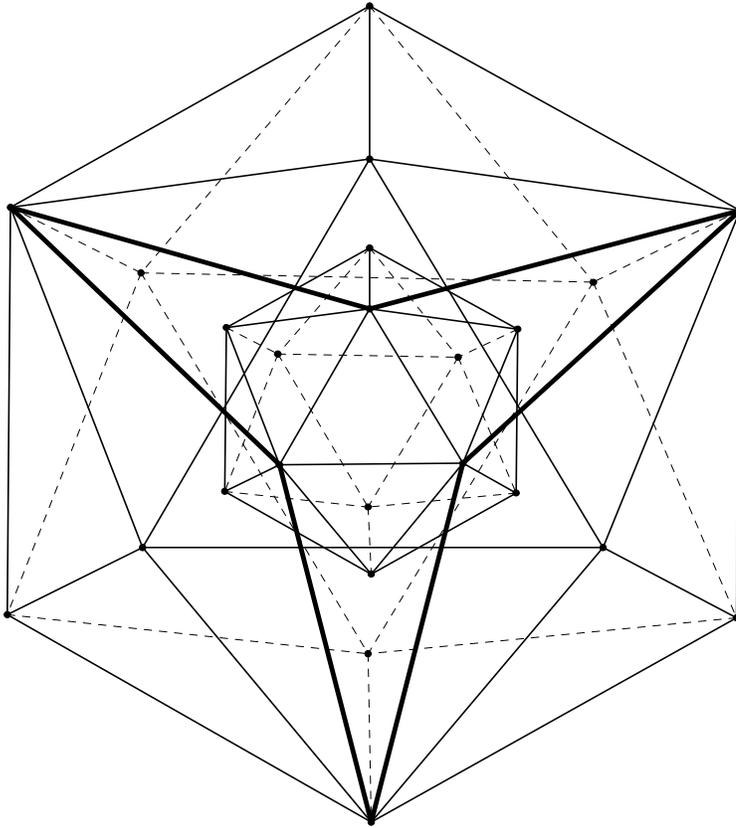
$\rho_2: (1)(2)(3,4)(5,6)(7,8)(9,10)(11)(12) (1^*)(2^*)(3^*,4^*)(5^*,6^*)(7^*,8^*)(9^*,10^*)(11^*)(12^*)$

$\rho_0\rho_1: (1,2^*,3,5^*)(2,10^*,5,12^*)(4,9^*,9,4^*)(6,6^*,7,7^*)(8,1^*,11,3^*)(10,8^*,12,11^*)$

$\rho_0\rho_2: (1,2^*)(2,12^*)(3,6^*)(4,5^*)(5,9^*)(6,10^*)(7,3^*)(8,4^*)(9,8^*)(10,7^*)(11,1^*)(12,11^*)$

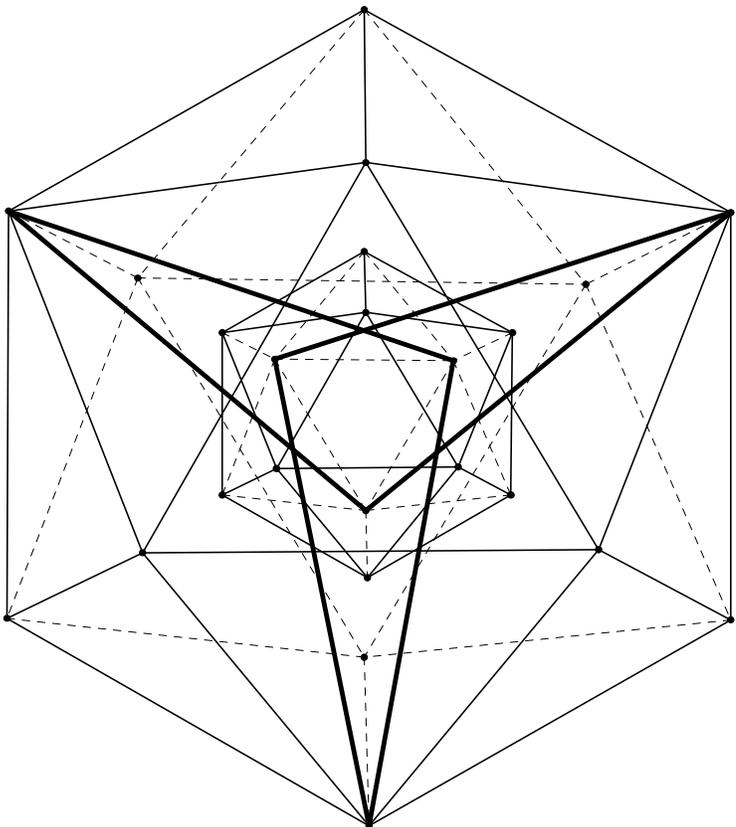
$\rho_1\rho_2: (1)(2,5,4,3,6)(7,11,8,9,10)(12)(1^*)(2^*,5^*,4^*,3^*,6^*)(7^*,11^*,8^*,9^*,10^*)(12^*)$

$\rho_0\rho_1\rho_2: (1,2^*,3,9^*,5,6^*)(2,10^*,9,8^*,6,12^*)(4,5^*,7,1^*,11,3^*)(8,7^*,12,11^*,10,4^*)$



**Face of the polyhedron  
of type  $\{6,5\}_4$**

$(f_0, f_1, f_2) = (24, 60, 20)$   
 Orientable genus = 9  
 Vertices of two icosahedra  
 (not generated from any  
 regular polyhedron)  
 Edge length = 1  
 Shape of  $[hr, s]$   
 Planar faces iff  $r = 2 + \sqrt{5}$



**Face of the polyhedron  
of type  $\{6,5\}_4$**

$(f_0, f_1, f_2) = (24, 60, 20)$   
 Orientable genus = 9  
 Vertices of two icosahedra  
 (not generated from any  
 regular polyhedron)  
 Edge length = 2  
 Shape of  $[hr, sr]$   
 Non-planar faces

Map Classification Number: dual of R9.16

**Map of the polyhedron of type  $\{6,5\}_4$**

$(f_0, f_1, f_2) = (24, 60, 20)$

Orientable genus = 9

Vertices of two icosahedra

(not generated from any regular polyhedron)

Edge length = 1

Shape of  $[hr,sl]$

Planar faces iff  $r = 2 + \sqrt{5}$

When all vertices in orbit  $S^*$  are relabelled from  $n^*$  to  $(13-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{6,5\}_4$**

$(f_0, f_1, f_2) = (24, 60, 20)$

Orientable genus = 9

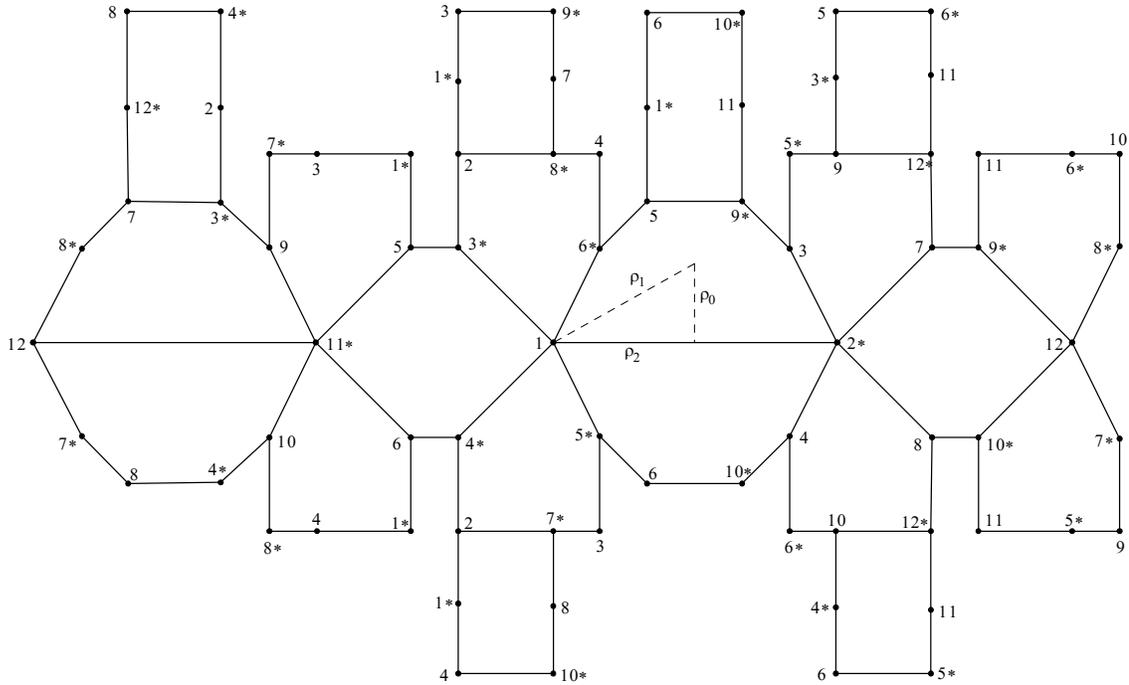
Vertices of two icosahedra

(not generated from any regular polyhedron)

Edge length = 2

Shape of  $[hr, sr]$

Non-planar faces



$\rho_0: (1,2^*)(2,12^*)(3,6^*)(4,5^*)(5,9^*)(6,10^*)(7,3^*)(8,4^*)(9,8^*)(10,7^*)(11,1^*)(12,11^*)$

$\rho_1: (1)(2,6)(3,5)(4)(7,11)(8,10)(9)(12)(1^*)(2^*,6^*)(3^*,5^*)(4^*)(7^*,11^*)(8^*,10^*)(9^*)(12^*)$

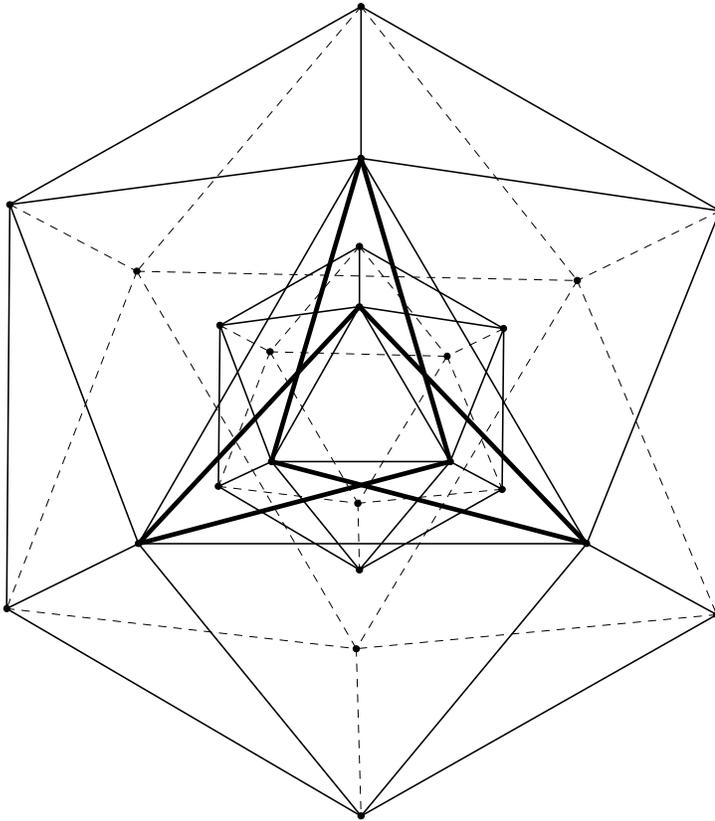
$\rho_2: (1)(2)(3,4)(5,6)(7,8)(9,10)(11)(12)(1^*)(2^*)(3^*,4^*)(5^*,6^*)(7^*,8^*)(9^*,10^*)(11^*)(12^*)$

$\rho_0\rho_1: (1,2^*,3,9^*,5,6^*)(2,10^*,9,8^*,6,12^*)(4,5^*,7,1^*,11,3^*)(8,7^*,12,11^*,10,4^*)$

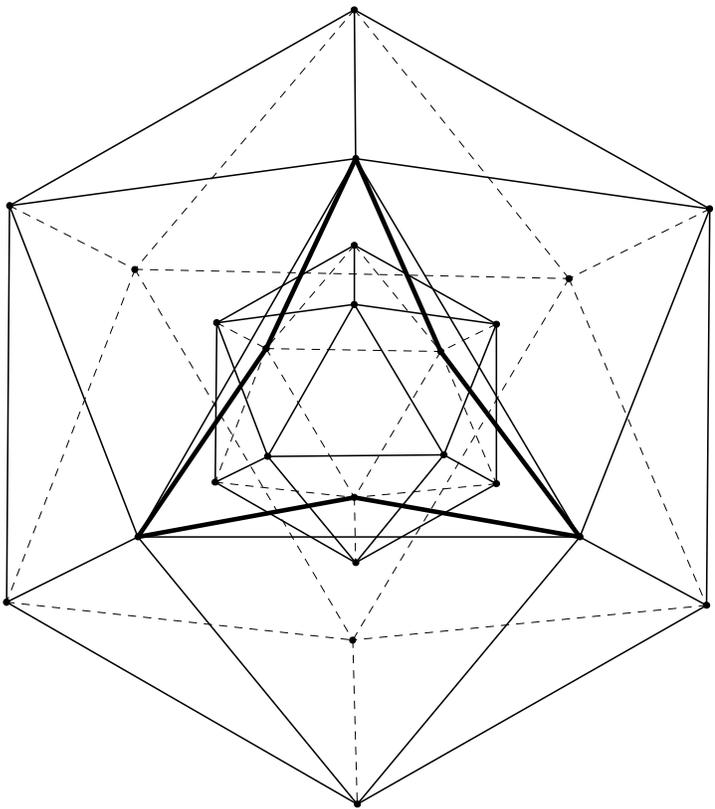
$\rho_0\rho_2: (1,2^*)(2,12^*)(3,5^*)(4,6^*)(5,10^*)(6,9^*)(7,4^*)(8,3^*)(9,7^*)(10,8^*)(11,1^*)(12,11^*)$

$\rho_1\rho_2: (1)(2,6,3,4,5)(7,10,9,8,11)(12)(1^*)(2^*,6^*,3^*,4^*,5^*)(7^*,10^*,9^*,8^*,11^*)(12^*)$

$\rho_0\rho_1\rho_2: (1,2^*,3,5^*)(2,10^*,5,12^*)(4,9^*,9,4^*)(6,6^*,7,7^*)(8,1^*,11,3^*)(10,8^*,12,11^*)$



**Face of the polyhedron  
of type  $\{6,5\}_{10}$**   
 $(f_0, f_1, f_2) = (24, 60, 20)$   
Orientable genus = 9  
Vertices of two icosahedra  
(generated from icosahedron)  
Edge length = 1  
Shape of  $[hr, hr]$   
Non-planar faces



**Face of the polyhedron  
of type  $\{6,5\}_{10}$**   
 $(f_0, f_1, f_2) = (24, 60, 20)$   
Orientable genus = 9  
Vertices of two icosahedra  
(generated from Petrie of small  
stellated dodecahedron)  
Edge length = 2  
Shape of  $[hr, h]$   
Non-planar faces

Map Classification Number: dual of R9.15

**Map of the polyhedron of type  $\{6,5\}_{10}$**

$(f_0, f_1, f_2) = (24, 60, 20)$

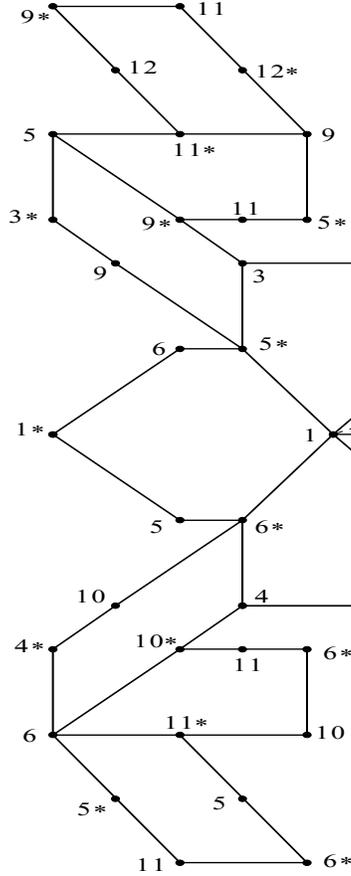
Orientable genus = 9

Vertices of two icosahedra  
(generated from icosahedron)

Edge length = 1

Shape of  $[hr, hr]$

Non-planar faces



When all vertices in orbit  $S^*$  are relabelled from  $n^*$  to  $(13-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{6,5\}_{10}$**

$(f_0, f_1, f_2) = (24, 60, 20)$

Orientable genus = 9

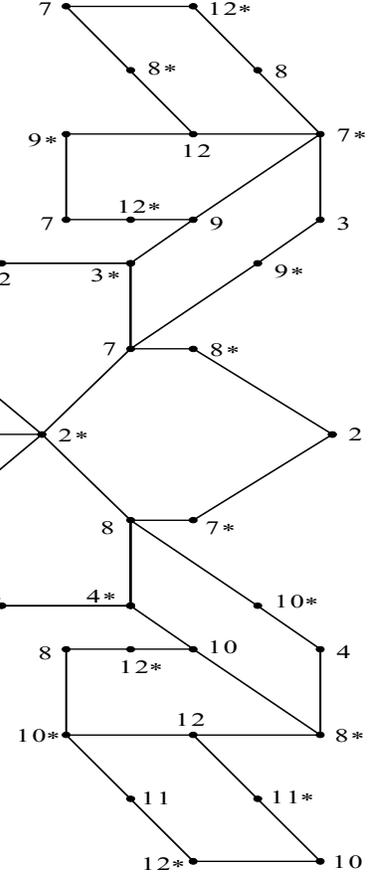
Vertices of two icosahedra

(generated from Petrie of small stellated dodecahedron)

Edge length = 2

Shape of  $[hr, hl]$

Non-planar faces



$\rho_0: (1, 2^*)(2, 1^*)(3, 3^*)(4, 4^*)(5, 7^*)(6, 8^*)(7, 5^*)(8, 6^*)(9, 9^*)(10, 10^*)(11, 12^*)(12, 11^*)$

$\rho_1: (1)(2, 3)(4, 5)(6)(7)(8, 9)(10, 11)(12) (1^*)(2^*, 3^*)(4^*, 5^*)(6^*)(7^*)(8^*, 9^*)(10^*, 11^*)(12^*)$

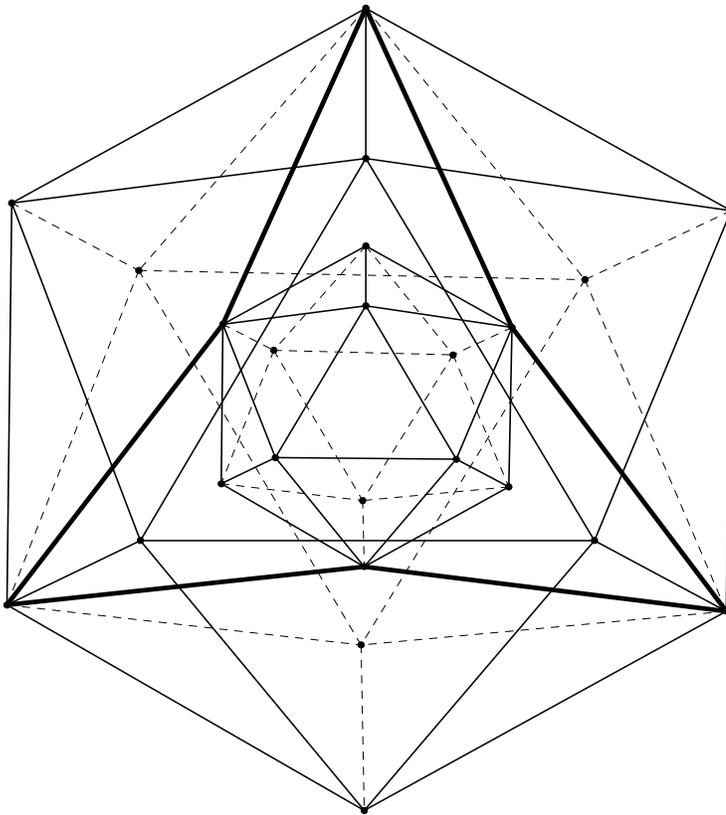
$\rho_2: (1)(2)(3, 4)(5, 6)(7, 8)(9, 10)(11)(12) (1^*)(2^*)(3^*, 4^*)(5^*, 6^*)(7^*, 8^*)(9^*, 10^*)(11^*)(12^*)$

$\rho_0\rho_1: (1, 2^*, 3, 1^*, 2, 3^*)(4, 7^*, 5, 4^*, 7, 5^*)(6, 8^*, 9, 6^*, 8, 9^*)(10, 12^*, 11, 10^*, 12, 11^*)$

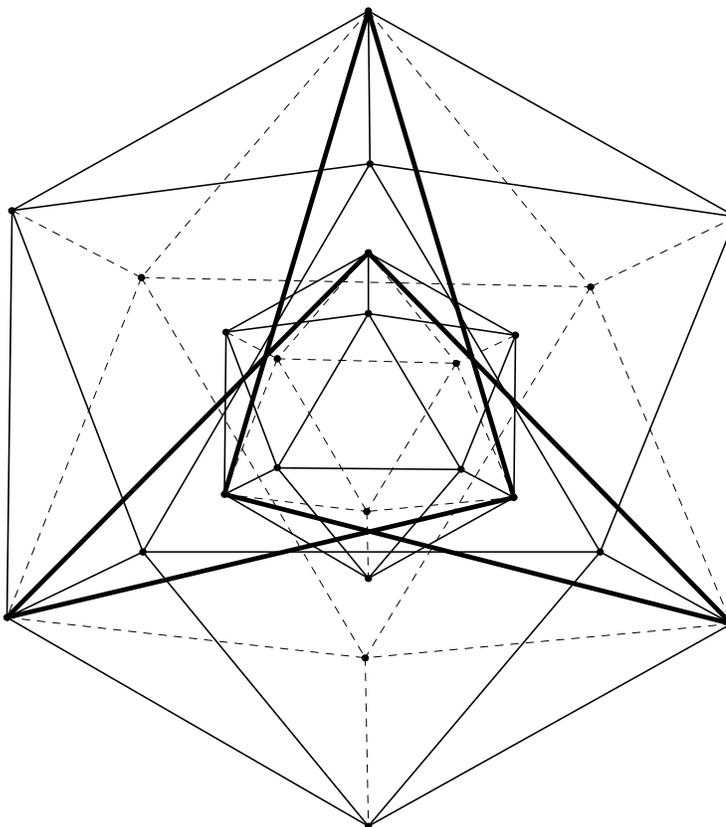
$\rho_0\rho_2: (1, 2^*)(2, 1^*)(3, 4^*)(4, 3^*)(5, 8^*)(6, 7^*)(7, 6^*)(8, 5^*)(9, 10^*)(10, 9^*)(11, 12^*)(12, 11^*)$

$\rho_1\rho_2: (1)(2, 3, 5, 6, 4)(7, 9, 11, 10, 8)(12)(1^*)(2^*, 3^*, 5^*, 6^*, 4^*)(7^*, 9^*, 11^*, 10^*, 8^*)(12^*)$

$\rho_0\rho_1\rho_2: (1, 2^*, 3, 7^*, 9, 12^*, 11, 10^*, 6, 4^*)(2, 3^*, 7, 9^*, 12, 11^*, 10, 6^*, 4, 1^*)(5, 8^*)(8, 5^*)$



**Face of the polyhedron  
of type  $\{6,5\}_{10}$**   
 $(f_0, f_1, f_2) = (24, 60, 20)$   
Orientable genus = 9  
Vertices of two icosahedra  
(generated from Petrie of  
great dodecahedron)  
Edge length = 1  
Shape of  $[sr, s]$   
Non-planar faces



**Face of the polyhedron  
of type  $\{6,5\}_{10}$**   
 $(f_0, f_1, f_2) = (24, 60, 20)$   
Orientable genus = 9  
Vertices of two icosahedra  
(generated from great  
icosahedron)  
Edge length = 2  
Shape of  $[sr, sr]$   
Non-planar faces

Map Classification Number: dual of R9.15

**Map of the polyhedron of type  $\{6,5\}_{10}$**

$(f_0, f_1, f_2) = (24, 60, 20)$

Orientable genus = 9

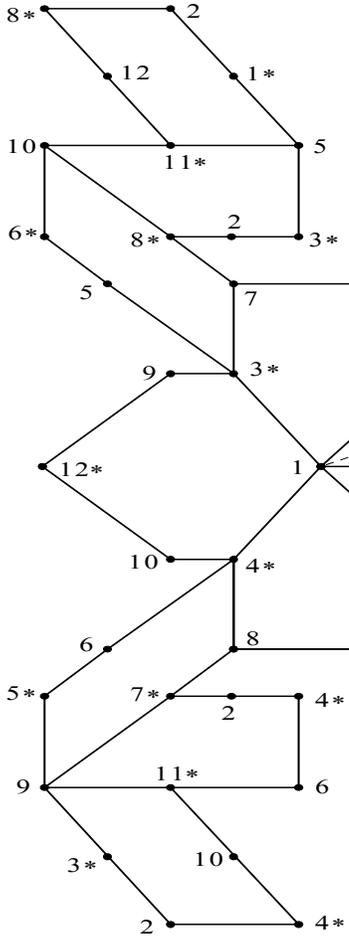
Vertices of two icosahedra

(generated from Petrie of great dodecahedron)

Edge length = 1

Shape of  $[sr,sl]$

Non-planar faces



When all vertices in orbit  $S^*$  are relabelled from  $n^*$  to  $(13-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{6,5\}_{10}$**

$(f_0, f_1, f_2) = (24, 60, 20)$

Orientable genus = 9

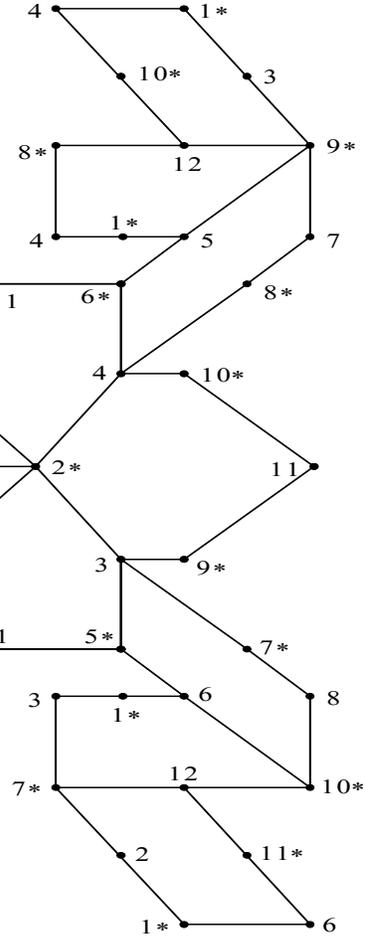
Vertices of two icosahedra

(generated from great icosahedron)

Edge length = 2

Shape of  $[sr,sl]$

Non-planar faces



$\rho_0: (1,2^*)(2,1^*)(3,4^*)(4,3^*)(5,8^*)(6,7^*)(7,6^*)(8,5^*)(9,10^*)(10,9^*)(11,12^*)(12,11^*)$

$\rho_1: (1)(2,6)(3,5)(4)(7,11)(8,10)(9)(12) (1^*)(2^*,6^*)(3^*,5^*)(4^*)(7^*,11^*)(8^*,10^*)(9^*)(12^*)$

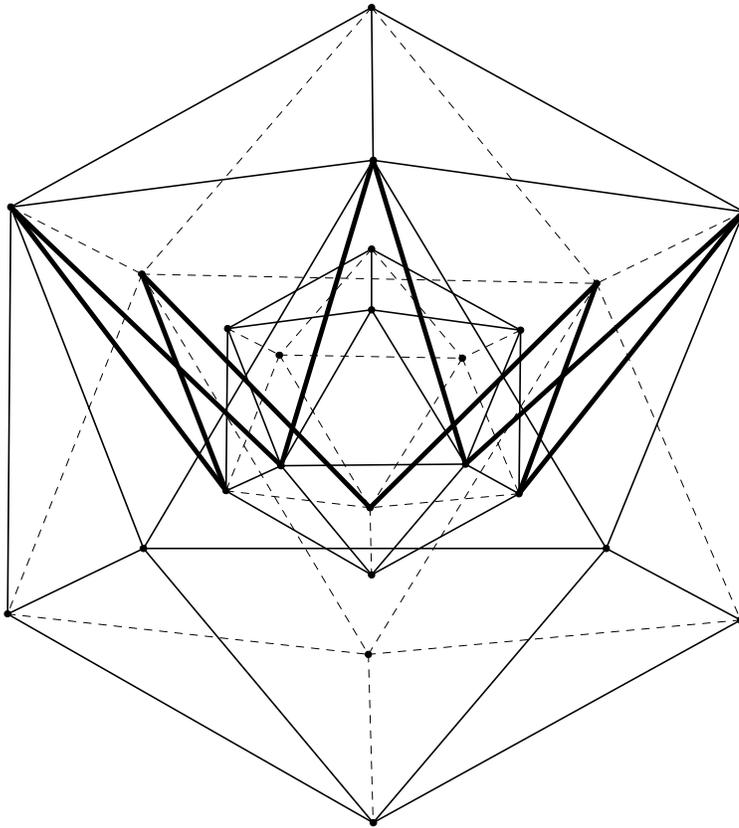
$\rho_2: (1)(2)(3,4)(5,6)(7,8)(9,10)(11)(12) (1^*)(2^*)(3^*,4^*)(5^*,6^*)(7^*,8^*)(9^*,10^*)(11^*)(12^*)$

$\rho_0\rho_1: (1,2^*,7,12^*,11,6^*)(2,7^*,12,11^*,6,1^*)(3,8^*,9,10^*,5,4^*)(4,3^*,8,9^*,10,5^*)$

$\rho_0\rho_2: (1,2^*)(2,1^*)(3,3^*)(4,4^*)(5,7^*)(6,8^*)(7,5^*)(8,6^*)(9,9^*)(10,10^*)(11,12^*)(12,11^*)$

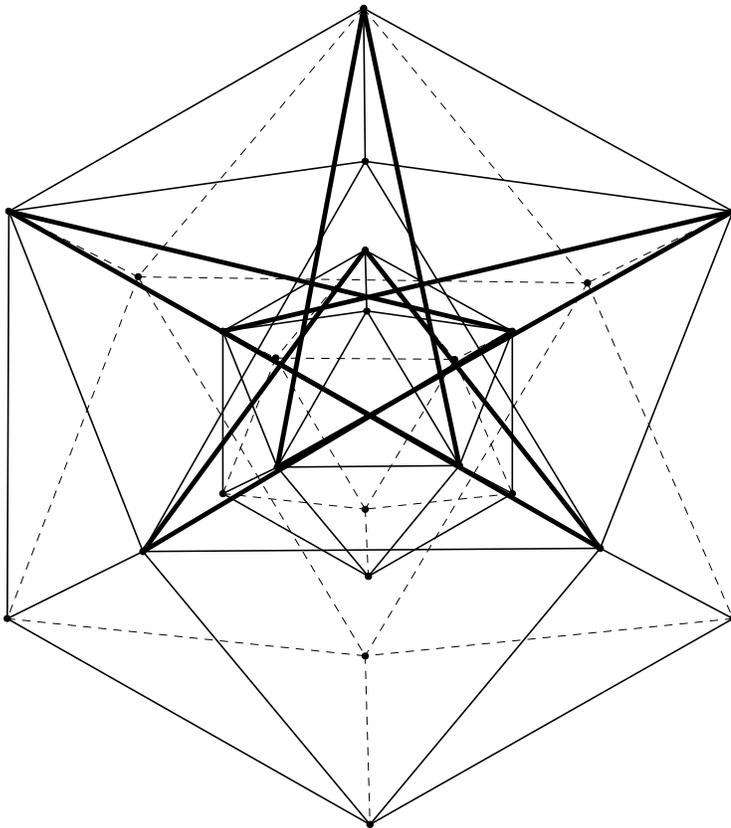
$\rho_1\rho_2: (1)(2,6,3,4,5)(7,10,9,8,11)(12)(1^*)(2^*,6^*,3^*,4^*,5^*)(7^*,10^*,9^*,8^*,11^*)(12^*)$

$\rho_0\rho_1\rho_2: (1,2^*,7,9^*,5,1^*,2,7^*,9,5^*)(3,3^*)(4,8^*,12,11^*,6,4^*,8,12^*,11,6^*)(10,10^*)$



**Face of the polyhedron  
of type  $\{10,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 12)$   
 Orientable genus = 13  
 Vertices of two icosahedra  
 (generated from Petrie of  
 icosahedron)  
 Edge length = 1  
 Shape of  $[hr,hl]$   
 Non-planar faces



**Face of the polyhedron  
of type  $\{10,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 12)$   
 Orientable genus = 13  
 Vertices of two icosahedra  
 (generated from small  
 stellated dodecahedron)  
 Edge length = 2  
 Shape of  $[hr,hr]$   
 Non-planar faces

Map Classification Number: dual of R13.8

**Map of the polyhedron of type  $\{10,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 12)$

Orientable genus = 13

Vertices of two icosahedra

(generated from Petrie of icosahedron)

Edge length = 1

Shape of [hr,hl]

Non-planar faces

When all vertices in orbit  $S^*$  are relabelled from  $n^*$  to  $(13-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{10,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 12)$

Orientable genus = 13

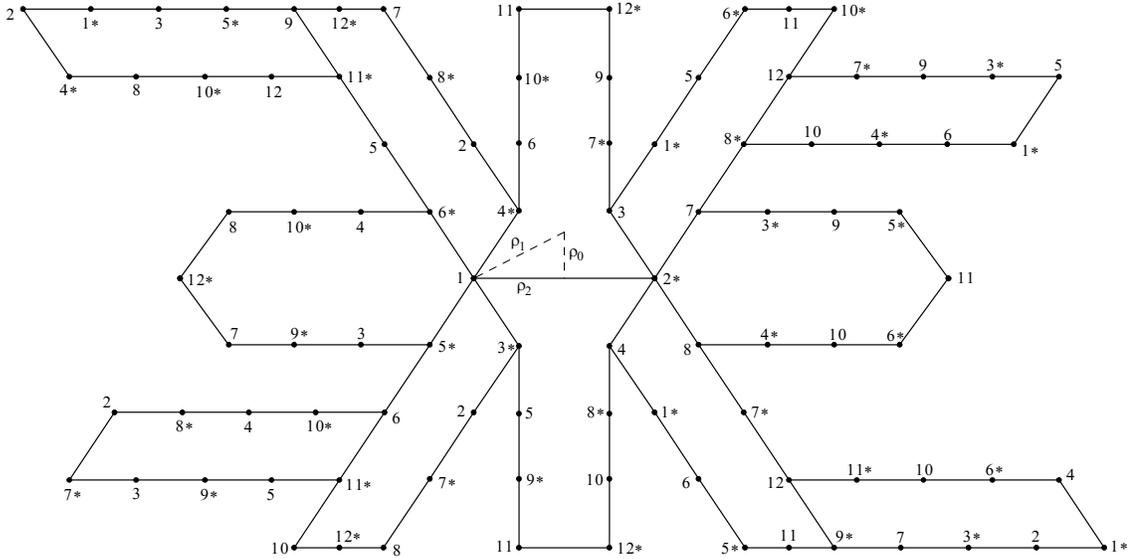
Vertices of two icosahedra

(generated from small stellated dodecahedron)

Edge length = 2

Shape of [hr,hr]

Non-planar faces



$$\rho_0: (1,2^*)(2,1^*)(3,4^*)(4,3^*)(5,8^*)(6,7^*)(7,6^*)(8,5^*)(9,10^*)(10,9^*)(11,12^*)(12,11^*)$$

$$\rho_1: (1)(2,4)(3,6)(5)(7,10)(8)(9,11)(12) (1^*)(2^*,4^*)(3^*,6^*)(5^*)(7^*,10^*)(8^*)(9^*,11^*)(12^*)$$

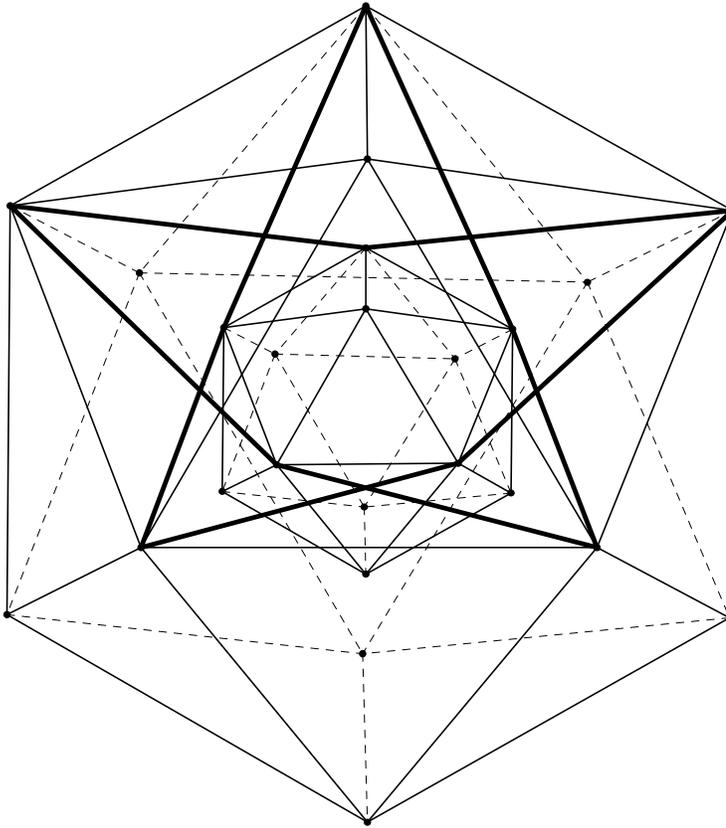
$$\rho_2: (1)(2)(3,4)(5,6)(7,8)(9,10)(11)(12) (1^*)(2^*)(3^*,4^*)(5^*,6^*)(7^*,8^*)(9^*,10^*)(11^*)(12^*)$$

$$\rho_0\rho_1: (1,2^*,3,7^*,9,12^*,11,10^*,6,4^*)(2,3^*,7,9^*,12,11^*,10,6^*,4,1^*)(5,8^*)(8,5^*)$$

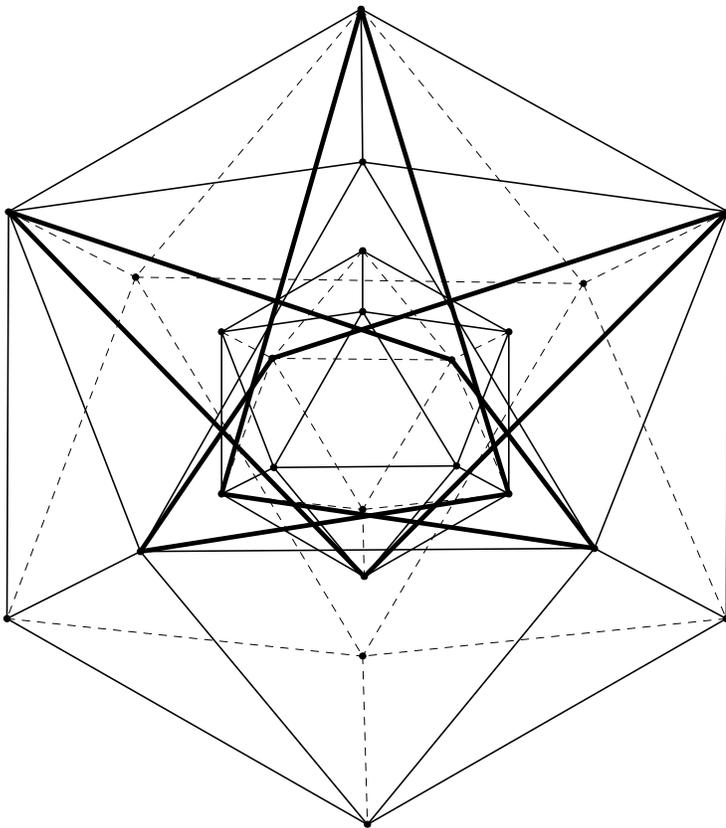
$$\rho_0\rho_2: (1,2^*)(2,1^*)(3,3^*)(4,4^*)(5,7^*)(6,8^*)(7,5^*)(8,6^*)(9,9^*)(10,10^*)(11,12^*)(12,11^*)$$

$$\rho_1\rho_2: (1)(2,4,6,5,3)(7,8,10,11,9)(12)(1^*)(2^*,4^*,6^*,5^*,3^*)(7^*,8^*,10^*,11^*,9^*)(12^*)$$

$$\rho_0\rho_1\rho_2: (1,2^*,3,1^*,2,3^*)(4,7^*,5,4^*,7,5^*)(6,8^*,9,6^*,8,9^*)(10,12^*,11,10^*,12,11^*)$$



**Face of the polyhedron  
of type  $\{10,5\}_6$**   
 $(f_0, f_1, f_2) = (24, 60, 12)$   
Orientable genus = 13  
Vertices of two icosahedra  
(generated from great  
dodecahedron)  
Edge length = 1  
Shape of  $[sr, sr]$   
Non-planar faces



**Face of the polyhedron  
of type  $\{10,5\}_6$**   
 $(f_0, f_1, f_2) = (24, 60, 12)$   
Orientable genus = 13  
Vertices of two icosahedra  
(generated from Petrie of  
great icosahedron)  
Edge length = 2  
Shape of  $[sr, sl]$   
Non-planar faces

Map Classification Number: dual of R13.8

**Map of the polyhedron of type  $\{10,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 12)$

Orientable genus = 13

Vertices of two icosahedra

(generated from great dodecahedron)

Edge length = 1

Shape of  $[sr, sr]$

Non-planar faces

When all vertices in orbit  $S^*$  are relabelled from  $n^*$  to  $(13-n)^*$ , this becomes ...

**Map of the polyhedron of type  $\{10,5\}_6$**

$(f_0, f_1, f_2) = (24, 60, 12)$

Orientable genus = 13

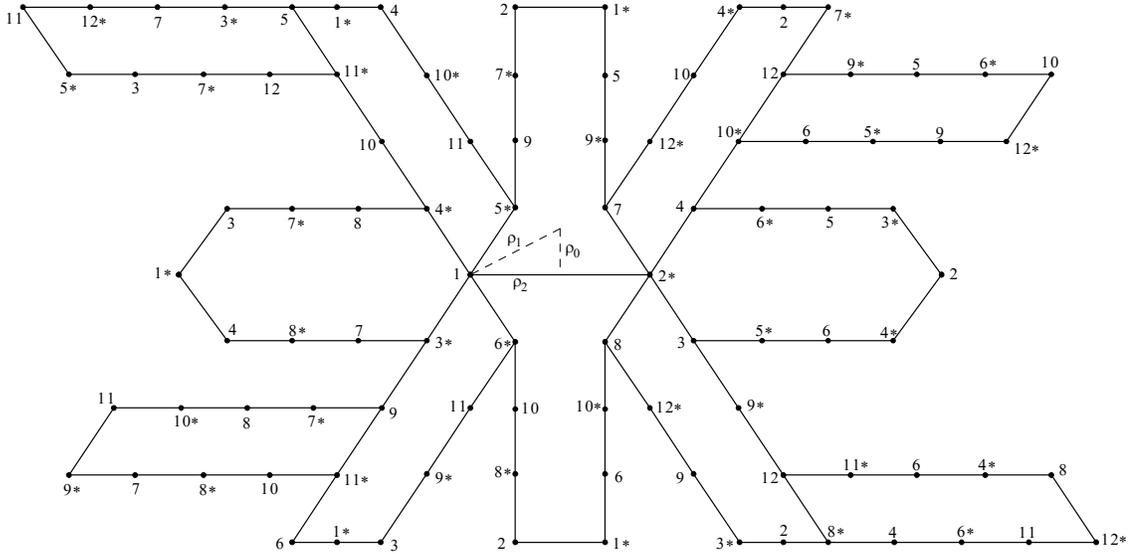
Vertices of two icosahedra

(generated from Petrie of great icosahedron)

Edge length = 2

Shape of  $[sr, s]$

Non-planar faces



$$\rho_0: (1,2^*)(2,1^*)(3,3^*)(4,4^*)(5,7^*)(6,8^*)(7,5^*)(8,6^*)(9,9^*)(10,10^*)(11,12^*)(12,11^*)$$

$$\rho_1: (1)(2,5)(3)(4,6)(7,9)(8,11)(10)(12) (1^*)(2^*,5^*)(3^*)(4^*,6^*)(7^*,9^*)(8^*,11^*)(10^*)(12^*)$$

$$\rho_2: (1)(2)(3,4)(5,6)(7,8)(9,10)(11)(12) (1^*)(2^*)(3^*,4^*)(5^*,6^*)(7^*,8^*)(9^*,10^*)(11^*)(12^*)$$

$$\rho_0\rho_1: (1,2^*,7,9^*,5,1^*,2,7^*,9,5^*)(3,3^*)(4,8^*,12,11^*,6,4^*,8,12^*,11,6^*)(10,10^*)$$

$$\rho_0\rho_2: (1,2^*)(2,1^*)(3,4^*)(4,3^*)(5,8^*)(6,7^*)(7,6^*)(8,5^*)(9,10^*)(10,9^*)(11,12^*)(12,11^*)$$

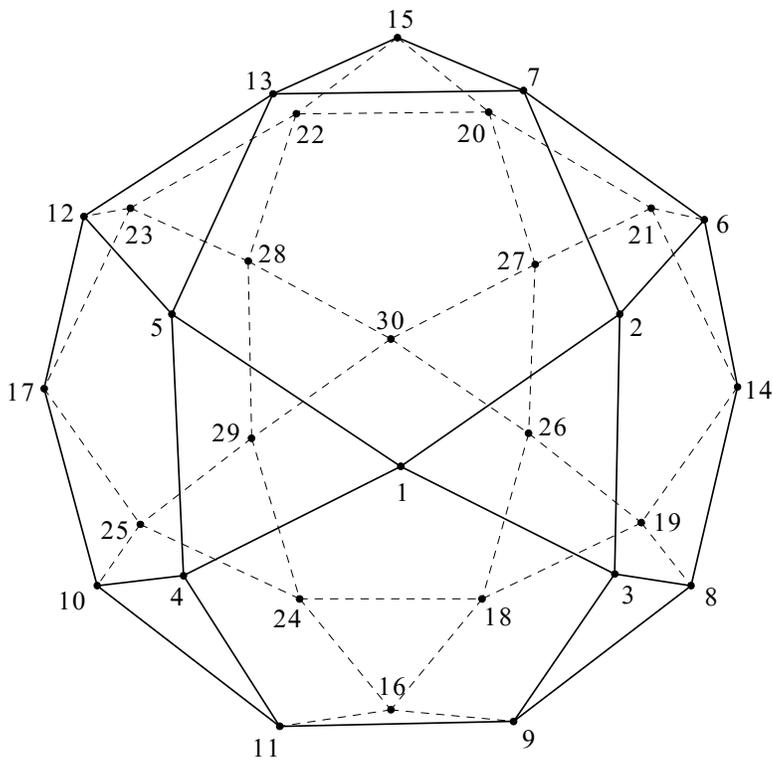
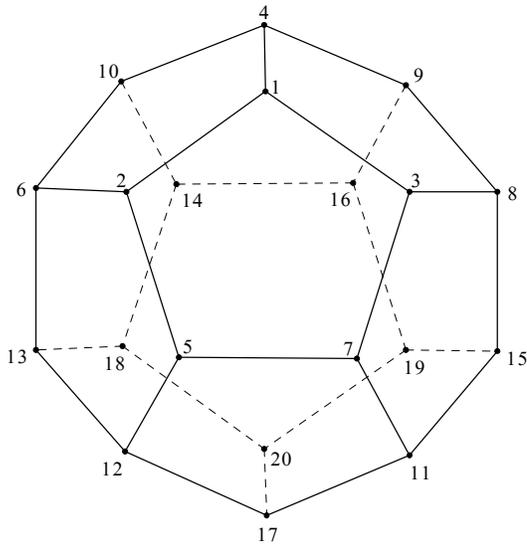
$$\rho_1\rho_2: (1)(2,5,4,3,6)(7,11,8,9,10)(12)(1^*)(2^*,5^*,4^*,3^*,6^*)(7^*,11^*,8^*,9^*,10^*)(12^*)$$

$$\rho_0\rho_1\rho_2: (1,2^*,7,12^*,11,6^*)(2,7^*,12,11^*,6,1^*)(3,8^*,9,10^*,5,4^*)(4,3^*,8,9^*,10,5^*)$$

## Appendix 2: Diagrams and maps for Polyhedra with vertices on one orbit

For each polyhedron, there is a geometric diagram of one or two faces, depending on the number of face orbits under  $G(P)$  (not  $G^+(P)$ ), as well as a combinatorial map of the abstract polyhedron showing the automorphisms  $\rho_0, \rho_1, \rho_2, \rho_0\rho_1, \rho_0\rho_2, \rho_1\rho_2$ , and  $\rho_0\rho_1\rho_2$ .

For polyhedra with vertices on one orbit,  $S$  is either a dodecahedron or an icosidodecahedron. The vertex labelling for these that have been used in the maps is shown below.



**Face of the polyhedron of type  $\{6,6\}_6$**

$(f_0, f_1, f_2) = (20, 60, 20)$

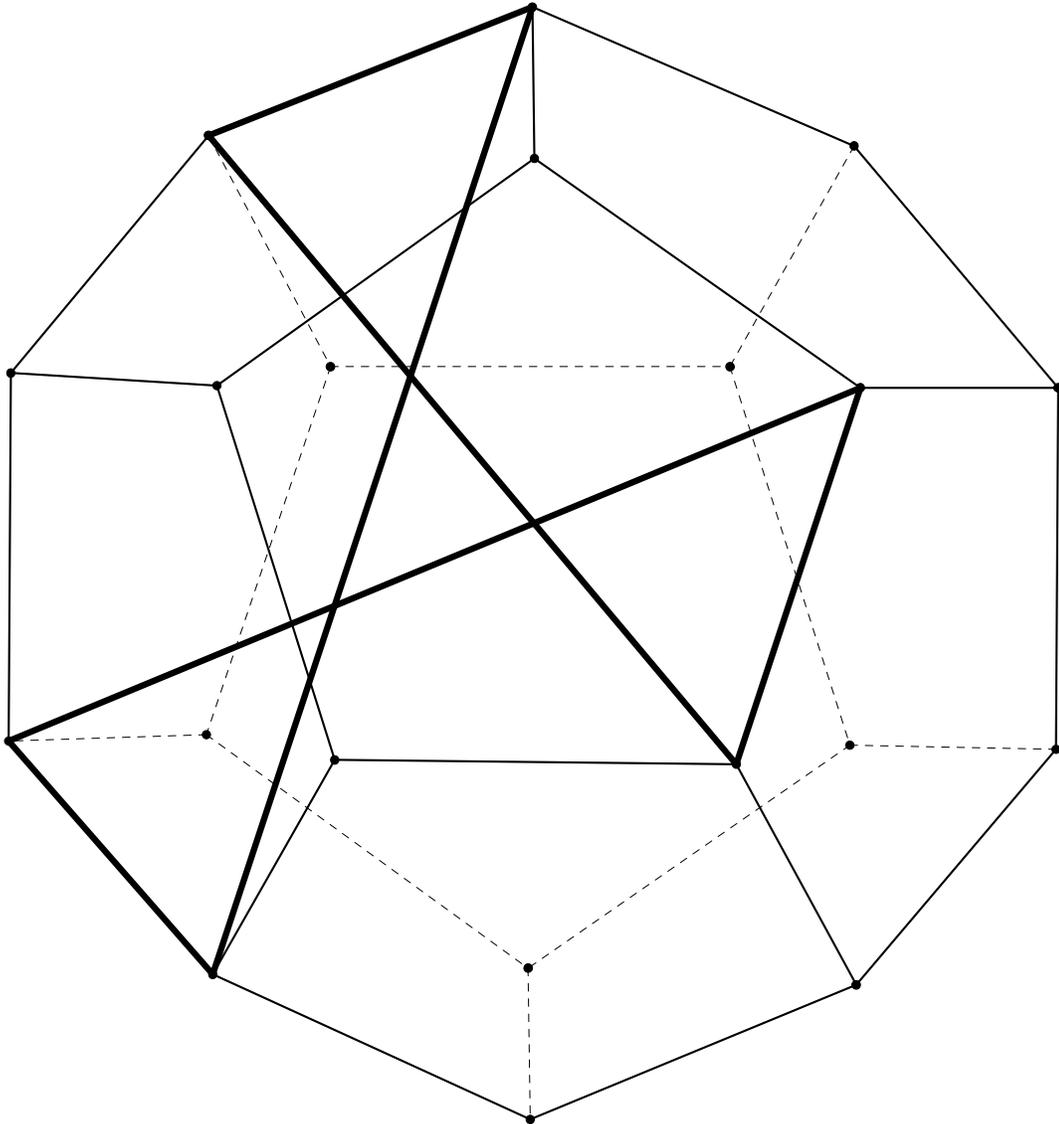
Orientable genus = 11

Vertices of one dodecahedron

Edge lengths = 1 & 4

Shape of  $[r,r,r,r]$

Planar faces



Map Classification Number: R11.5

**Map of the polyhedron of type  $\{6,6\}_6$**

$(f_0, f_1, f_2) = (20, 60, 20)$

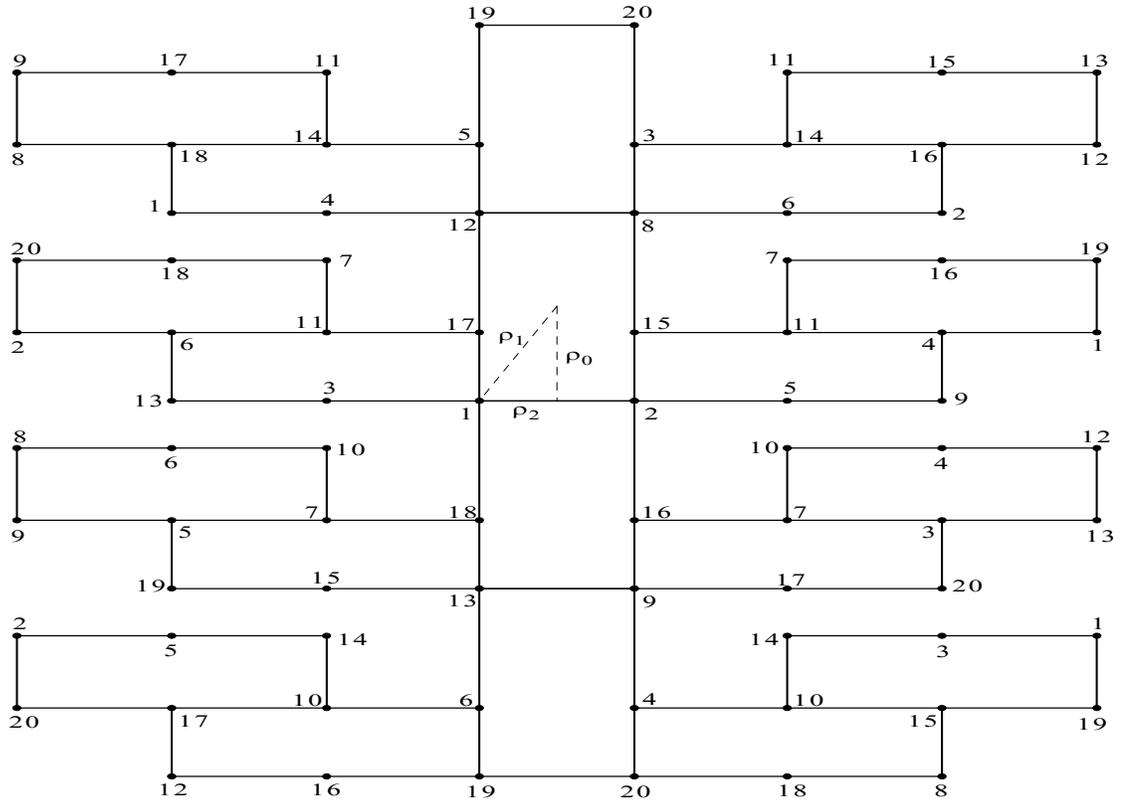
Orientable genus = 11

Vertices of one dodecahedron

Edge lengths = 1 & 4

Shape of  $[r,r,r,r]$

Planar faces



$\rho_0$ : (1,2)(3,5)(4,6)(7)(8,12)(9,13)(10)(11)(14)(15,17)(16,18)(19,20)

$\rho_1$ : (1)(2,17)(3,18)(4,19)(5,10)(6,9)(7)(8)(11,16)(12,15)(13)(14)(20)

$\rho_2$ : (1)(2)(3,4)(5,6)(7,10)(8,9)(11,14)(12,13)(15,16)(17,18)(19)(20)

$\rho_0\rho_1$ : (1,2,15,8,12,17)(3,16,11,18,5,10)(4,20,19,6,13,9)(7)(14)

$\rho_0\rho_2$ : (1,2)(3,6)(4,5)(7,10)(8,13)(9,12)(11,14)(15,18)(16,17)(19,20)

$\rho_1\rho_2$ : (1)(2,17,3,19,4,18)(5,9,8,6,10,7)(11,14,16,12,13,15)(20)

$\rho_0\rho_1\rho_2$ : (1,2,15,11,14,18)(3,20,19,6,10,7)(4,16,8)(5,13,17)(9,12)

**Face of the polyhedron of type  $\{6,6\}_6$**

$(f_0, f_1, f_2) = (20, 60, 20)$

Non-orientable genus = 22

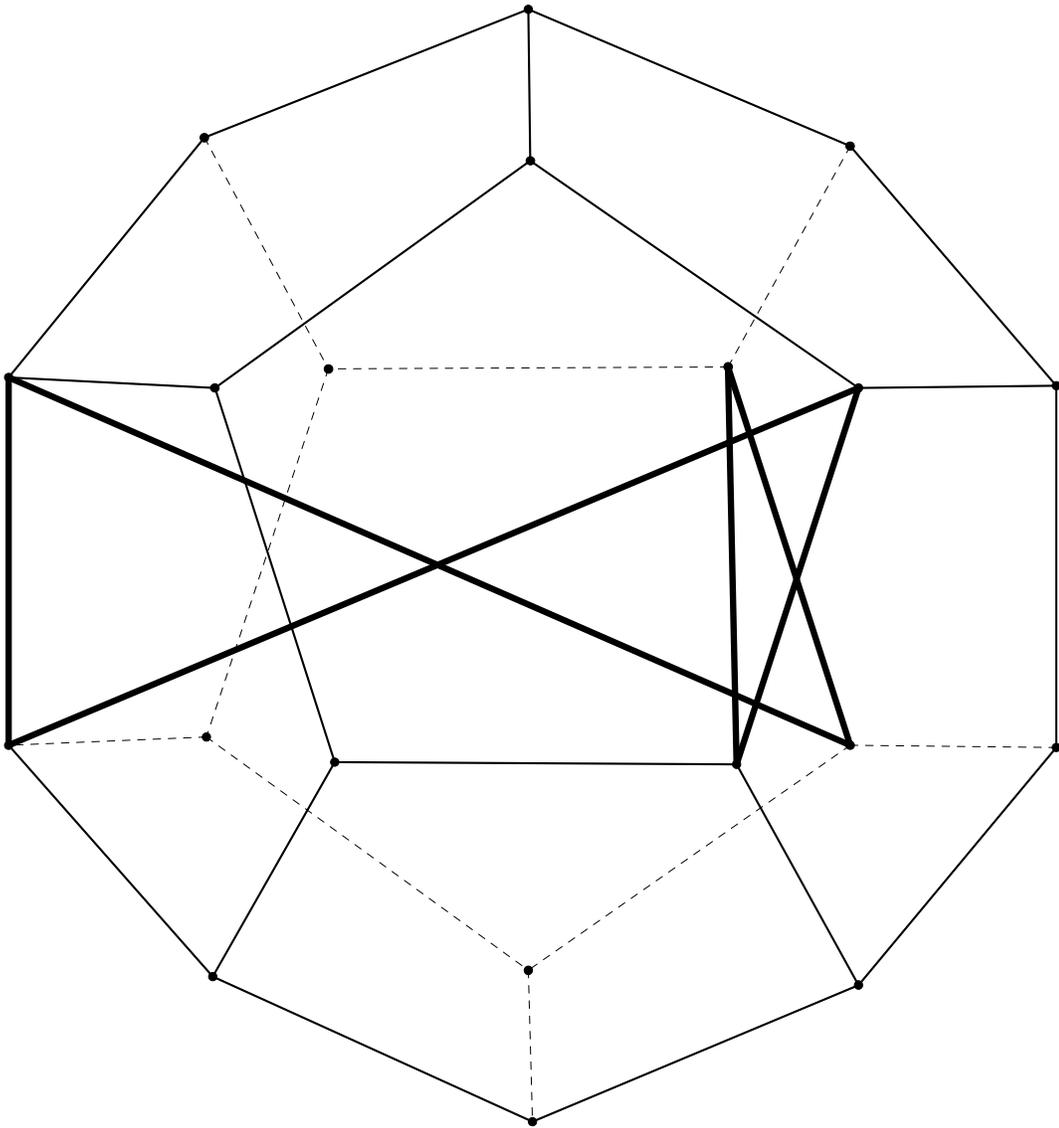
Vertices of one dodecahedron

Edge lengths = 1 & 4

Shape of  $[r,l,r,l]$  &  $[l,r,l,r]$

One face orbit under  $G(P)$

Non-planar faces



Map Classification Number: N22.3

**Map of the polyhedron of type  $\{6,6\}_6$**

$(f_0, f_1, f_2) = (20, 60, 20)$

Non-orientable genus = 22

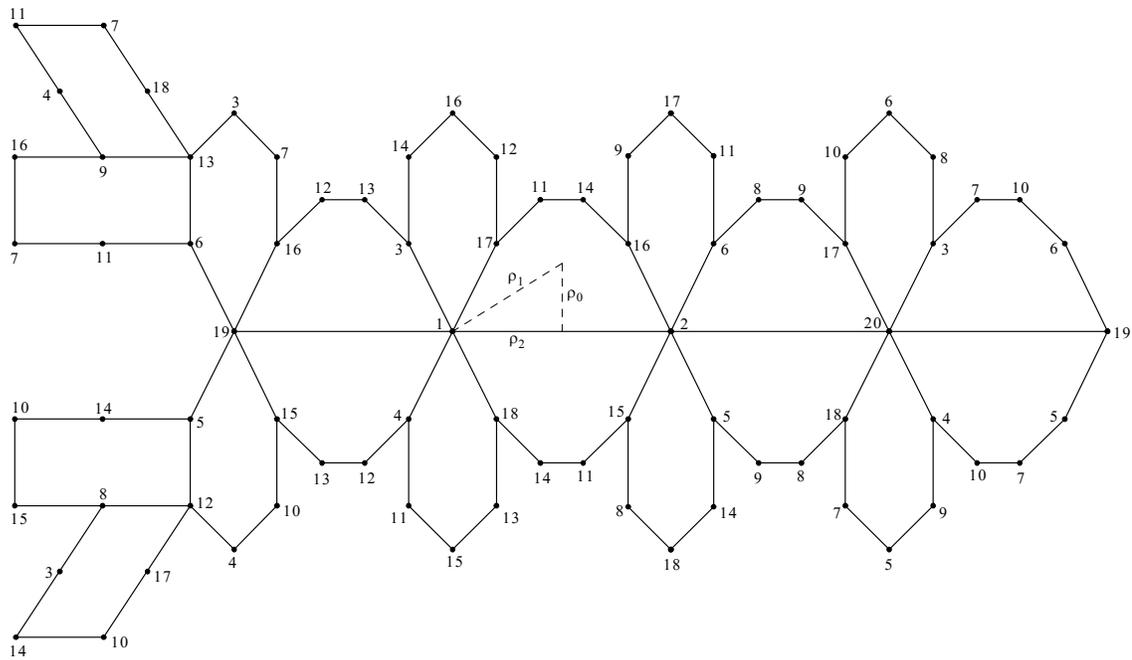
Vertices of one dodecahedron

Edge lengths = 1 & 4

Shape of  $[r,l,r,l]$  &  $[l,r,l,r]$

One face orbit under  $G(P)$

Non-planar faces



$\rho_0: (1,2)(3,6)(4,5)(7,10)(8,13)(9,12)(11,14)(15,18)(16,17)(19,20)$

$\rho_1: (1)(2,17)(3,18)(4,19)(5,10)(6,9)(7)(8)(11,16)(12,15)(13)(14)(20)$

$\rho_2: (1)(2)(3,4)(5,6)(7,10)(8,9)(11,14)(12,13)(15,16)(17,18)(19)(20)$

$\rho_0\rho_1: (1,2,16,14,11,17)(3,15,9)(4,20,19,5,7,10)(6,12,18)(8,13)$

$\rho_0\rho_2: (1,2)(3,5)(4,6)(7)(8,12)(9,13)(10)(11)(14)(15,17)(16,18)(19,20)$

$\rho_1\rho_2: (1)(2,17,3,19,4,18)(5,9,8,6,10,7)(11,14,16,12,13,15)(20)$

$\rho_0\rho_1\rho_2: (1,2,16,9,13,18)(3,20,19,5,12,8)(4,15,14,17,6,7)(10)(11)$

**Face of the polyhedron of type  $\{6,4\}_5$**

$(f_0, f_1, f_2) = (30, 60, 20)$

Non-orientable genus = 12

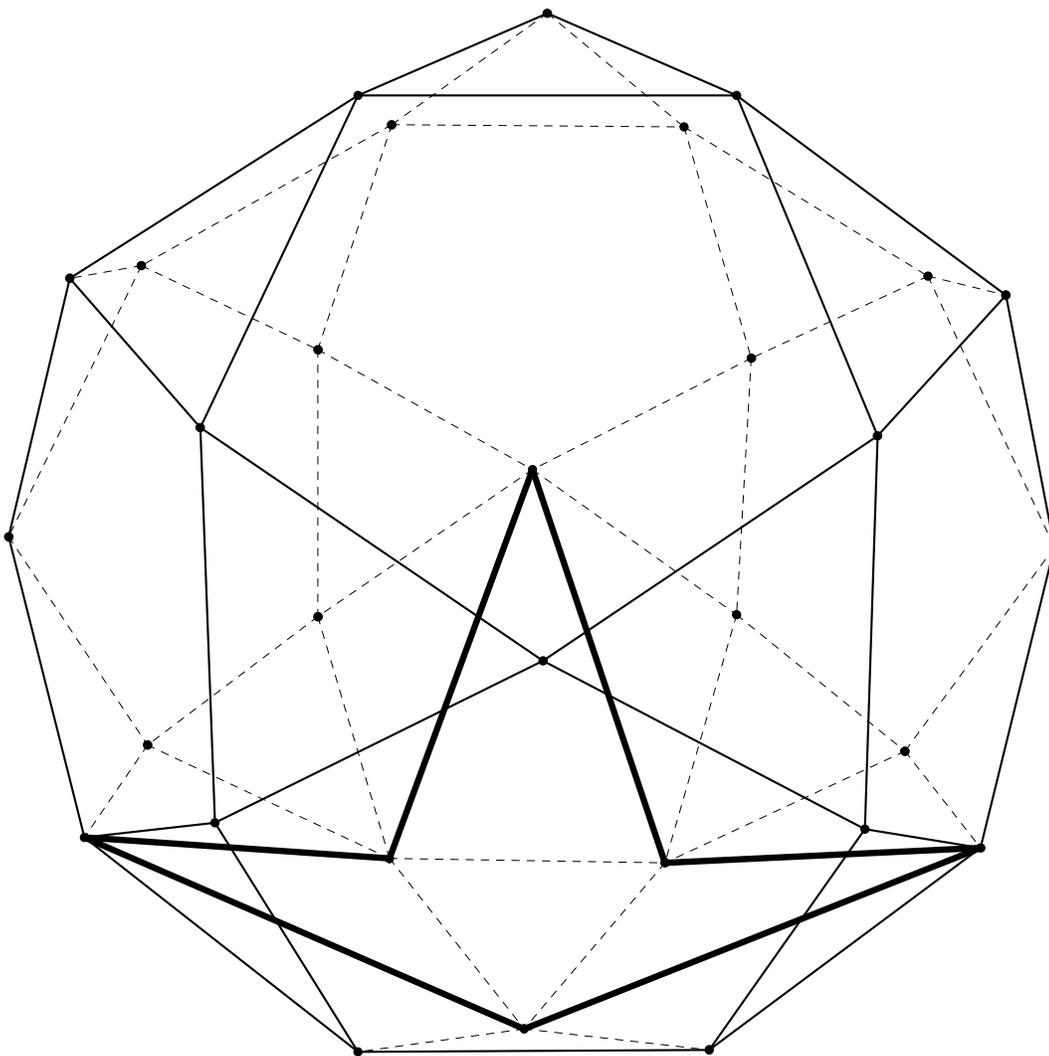
Vertices of one icosidodecahedron

Edge length = d

Directed type

Shape of  $[r,l,r,l]$

Non-planar faces



Map Classification Number: dual of N12.1

**Map of the polyhedron of type  $\{6,4\}_5$**

$(f_0, f_1, f_2) = (30, 60, 20)$

Non-orientable genus = 12

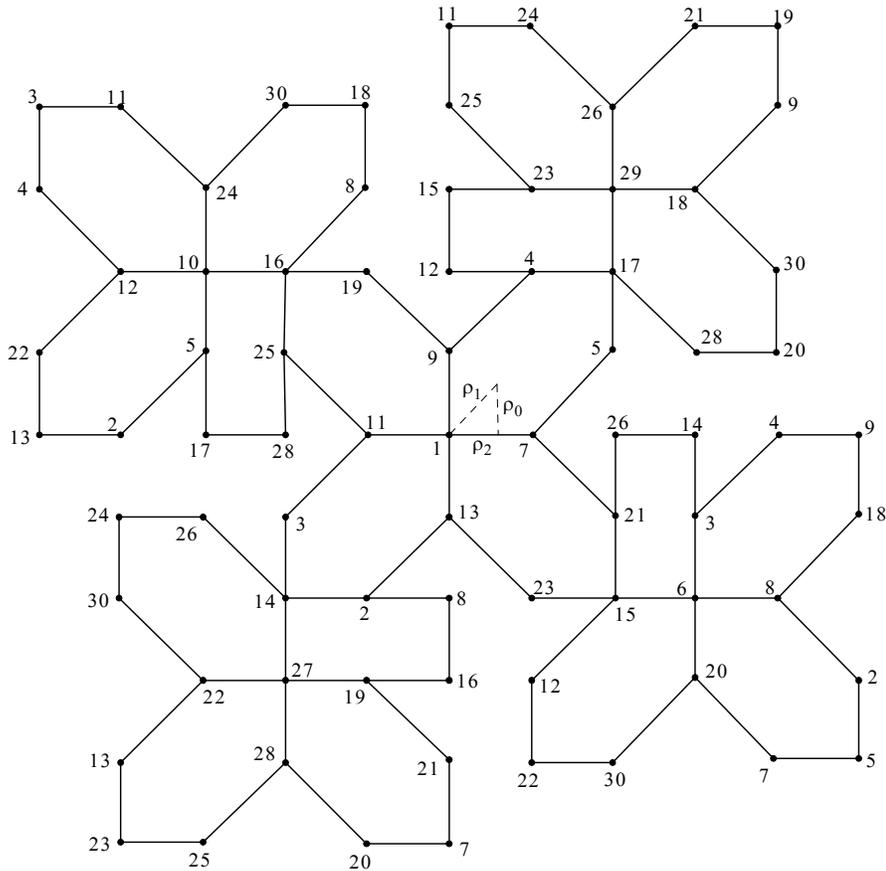
Vertices of one icosidodecahedron

Edge length = d

Directed type

Shape of  $[r,l,r,l]$

Non-planar faces



$\rho_0$ : (1,7)(2,19)(3,28)(4,17)(5,9)(6,25)(8,16)(10,18)(11,20)(12,29)(13,21)(14,27)(15,23)(22,26)(24,30)

$\rho_1$ : (1)(2,3)(4,5)(6,8)(7,9)(10,12)(11,13)(14)(15,16)(17)(18,20)(19,21)(22,24)(23,25)(26,27)(28,29)(30)

$\rho_2$ : (1)(2,19)(3,25)(4,23)(5,21)(6,28)(7)(8,27)(9,13)(10,26)(11)(12,29)(14,16)(15,17)(18,22)(20)(24)(30)

$\rho_0\rho_1$ : (1,7,5,17,4,9)(2,28,12,18,11,21)(3,19,13,20,10,29)(6,16,23)(8,25,15)(14,27,22,30,24,26)

$\rho_0\rho_2$ : (1,7)(2)(3,6)(4,15)(5,13)(8,14)(9,21)(10,22)(11,20)(12)(16,27)(17,23)(18,26)(19)(24,30)(25,28)(29)

$\rho_1\rho_2$ : (1)(2,21,4,25)(3,23,5,19)(6,29,10,27)(7,9,11,13)(8,26,12,28)(14,15,17,16)(18,24,22,20)(30)

$\rho_0\rho_1\rho_2$ : (1,7,5,2,13)(3,15,4,6,12)(8,22,11,21,17)(9,20,10,14,23)(16,27,25,19,28)(18,30,24,26,29)

**Faces of the polyhedron of type  $\{5,4\}_6$**

$(f_0, f_1, f_2) = (30, 60, 24)$

Orientable genus = 4

Vertices of one icosidodecahedron

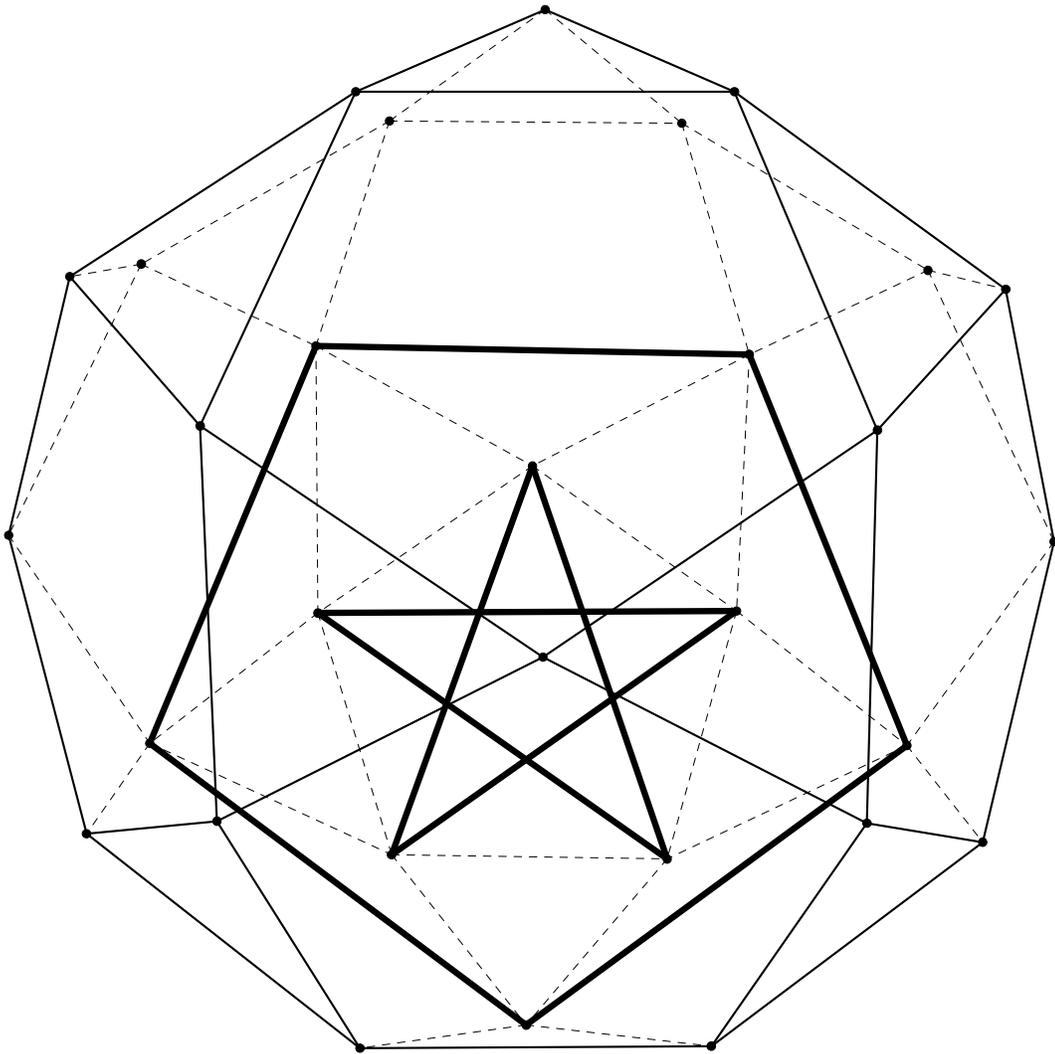
Edge length = d

Directed type

Shape of  $[r,r,r,r]$  &  $[l,l,l,l]$

Two face orbits under  $G(P)$

Planar faces



Map Classification Number: dual of R4.2

**Map of the polyhedron of type  $\{5,4\}_6$**

$(f_0, f_1, f_2) = (30, 60, 24)$

Orientable genus = 4

Vertices of one icosidodecahedron

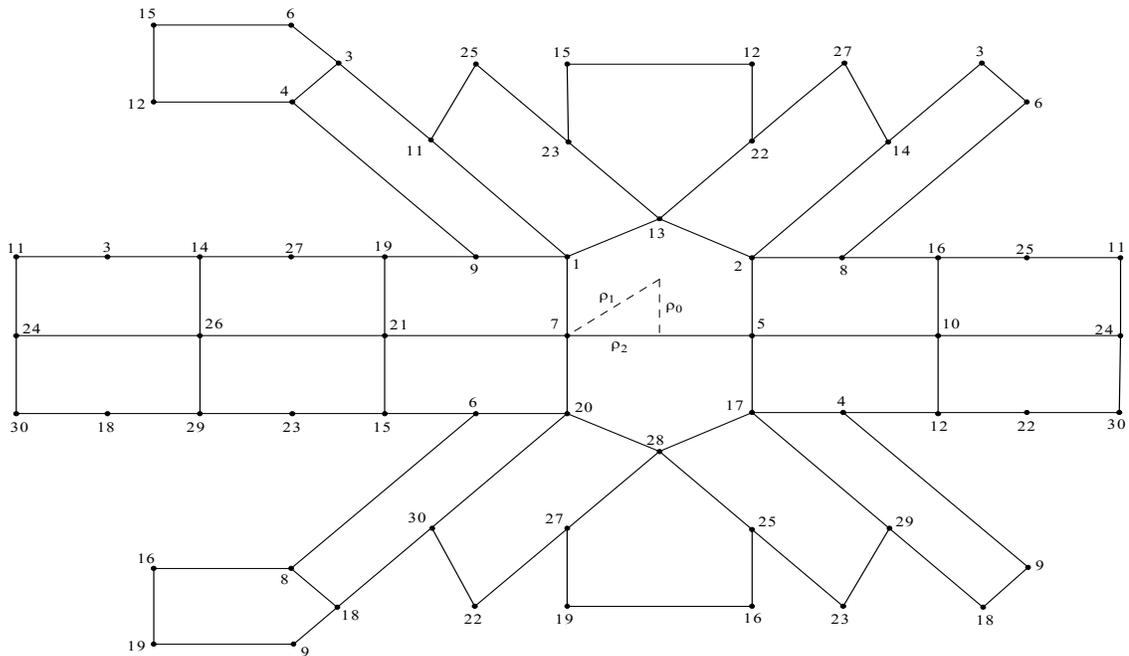
Edge length = d

Directed type

Shape of  $[r,r,r,r]$  &  $[l,l,l,l]$

Two face orbits under  $G(P)$

Planar faces



$\rho_0$ : (1,2)(3,4)(5,7)(8,9)(10,21)(11,14)(12,15)(13)(16,19)(17,20)(18)(22,23)(24,26)(25,27)(28)(29,30)

$\rho_1$ : (1,5)(2,13)(3,12)(4)(6,15)(7)(8,23)(9,17)(10,11)(14,22)(16,25)(18,29)(19,28)(20,21)(24)(26,30)(27)

$\rho_2$ : (1,20)(2,17)(3,18)(4,8)(5)(6,9)(7)(10)(11,30)(12,16)(13,28)(14,29)(15,19)(21)(22,25)(23,27)(24)(26)

$\rho_0\rho_1$ : (1,7,5,2,13)(3,15,4,6,12)(8,22,11,21,17)(9,20,10,14,23)(16,27,25,19,28)(18,30,24,26,29)

$\rho_0\rho_2$ : (1,17)(2,20)(3,18)(4,9)(5,7)(6,8)(10,21)(11,29)(12,19)(13,28)(14,30)(15,16)(22,27)(23,25)(24,26)

$\rho_1\rho_2$ : (1,21,20,5)(2,9,15,28)(3,29,22,16)(4,23,27,8)(6,17,13,19)(7)(10,11,26,30)(12,25,14,18)(24)

$\rho_0\rho_1\rho_2$ : (1,10,14,18,15,28)(2,8,6,20,7,5)(3,30,21,17,13,16)(4,22,19)(9,12,27)(11,24,26,29,23,25)

**Face of the polyhedron of type  $\{4,6\}_5$**

$(f_0, f_1, f_2) = (20, 60, 30)$

Non-orientable genus = 12

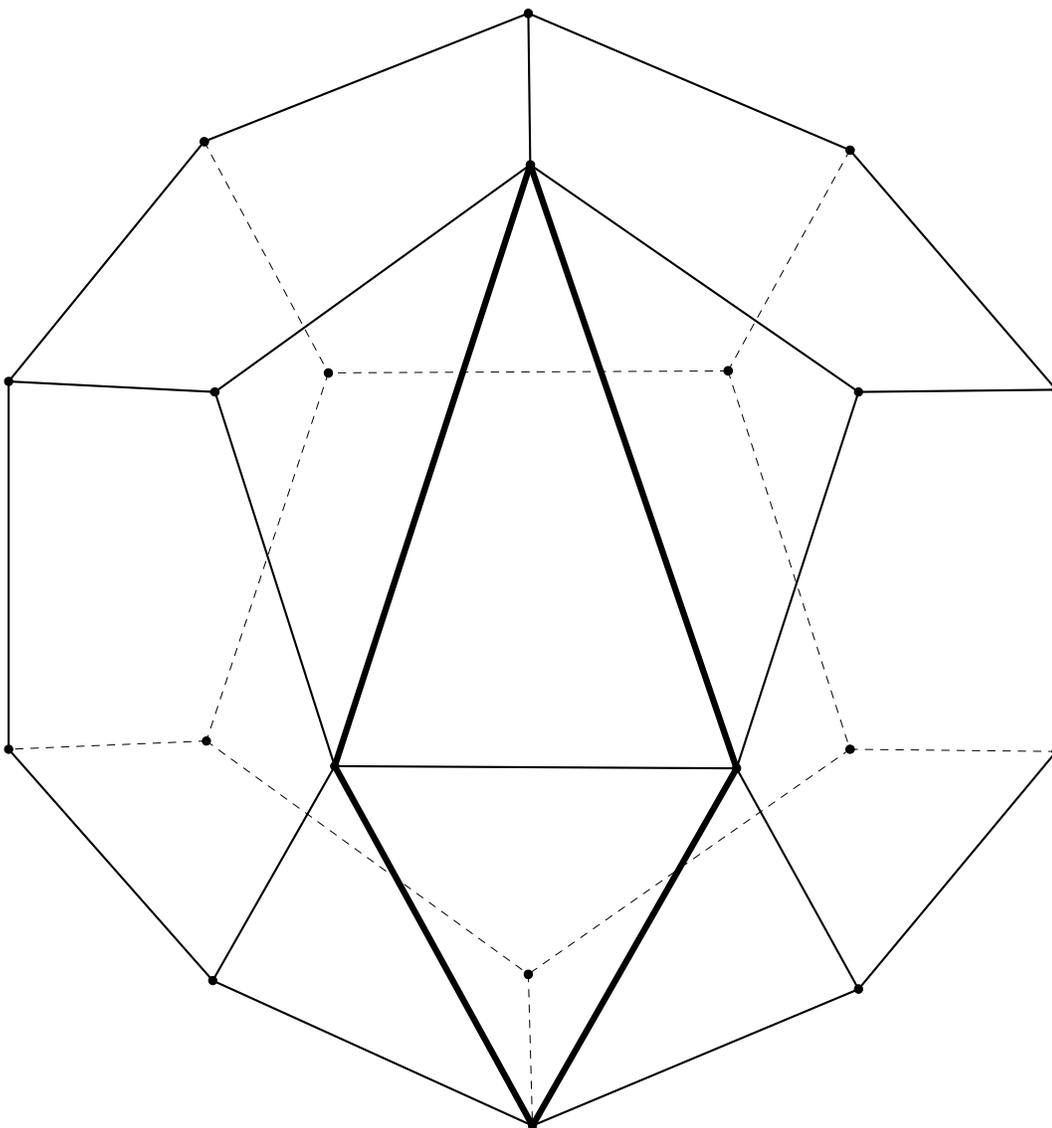
Vertices of one dodecahedron

Edge length = 2

Directed type

Shape of  $[hl, f, hl, f]$

Non-planar faces



Map Classification Number: N12.1

**Map of the polyhedron of type  $\{4,6\}_5$**

$(f_0, f_1, f_2) = (20, 60, 30)$

Non-orientable genus = 12

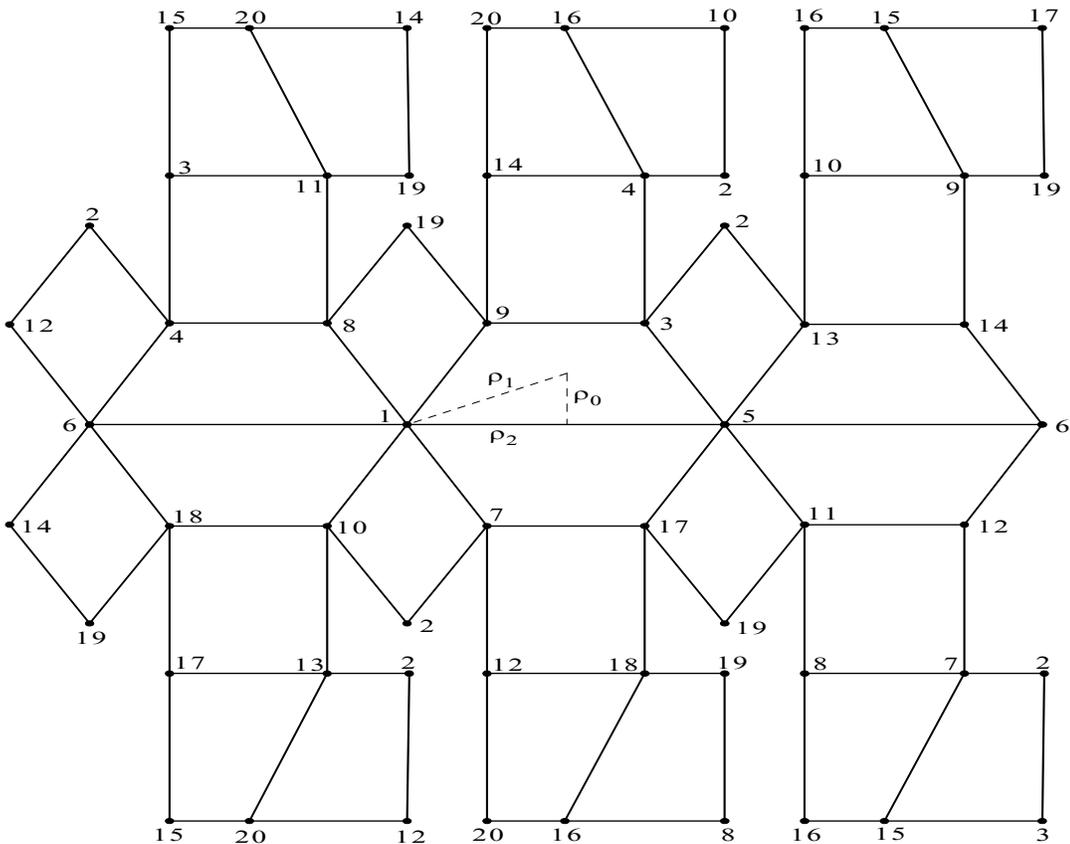
Vertices of one dodecahedron

Edge length = 2

Directed type

Shape of  $[hl, f, hl, f]$

Non-planar faces



$\rho_0$ : (1,5)(2,19)(3,9)(4,14)(6)(7,17)(8,13)(10,11)(12,18)(15)(16,20)

$\rho_1$ : (1)(2,4)(3)(5,9)(6,10)(7,8)(11,15)(12,16)(13,14)(17,19)(18)(20)

$\rho_2$ : (1)(2,19)(3,17)(4,18)(5)(6)(7,9)(8,10)(11,13)(12,14)(15)(16)(20)

$\rho_0\rho_1$ : (1,5,3,9)(2,14,8,17)(4,19,7,13)(6,11,15,10)(12,20,16,18)

$\rho_0\rho_2$ : (1,5)(2)(3,7)(4,12)(6)(8,11)(9,17)(10,13)(14,18)(15)(16,20)(19)

$\rho_1\rho_2$ : (1)(2,17,3,19,4,18)(5,9,8,6,10,7)(11,14,16,12,13,15)(20)

$\rho_0\rho_1\rho_2$ : (1,5,3,2,7)(4,12,8,6,11)(9,13,15,10,17)(14,20,16,18,19)

**Faces of the polyhedron of type  $\{5,6\}_4$**

$(f_0, f_1, f_2) = (20, 60, 24)$

Orientable genus = 9

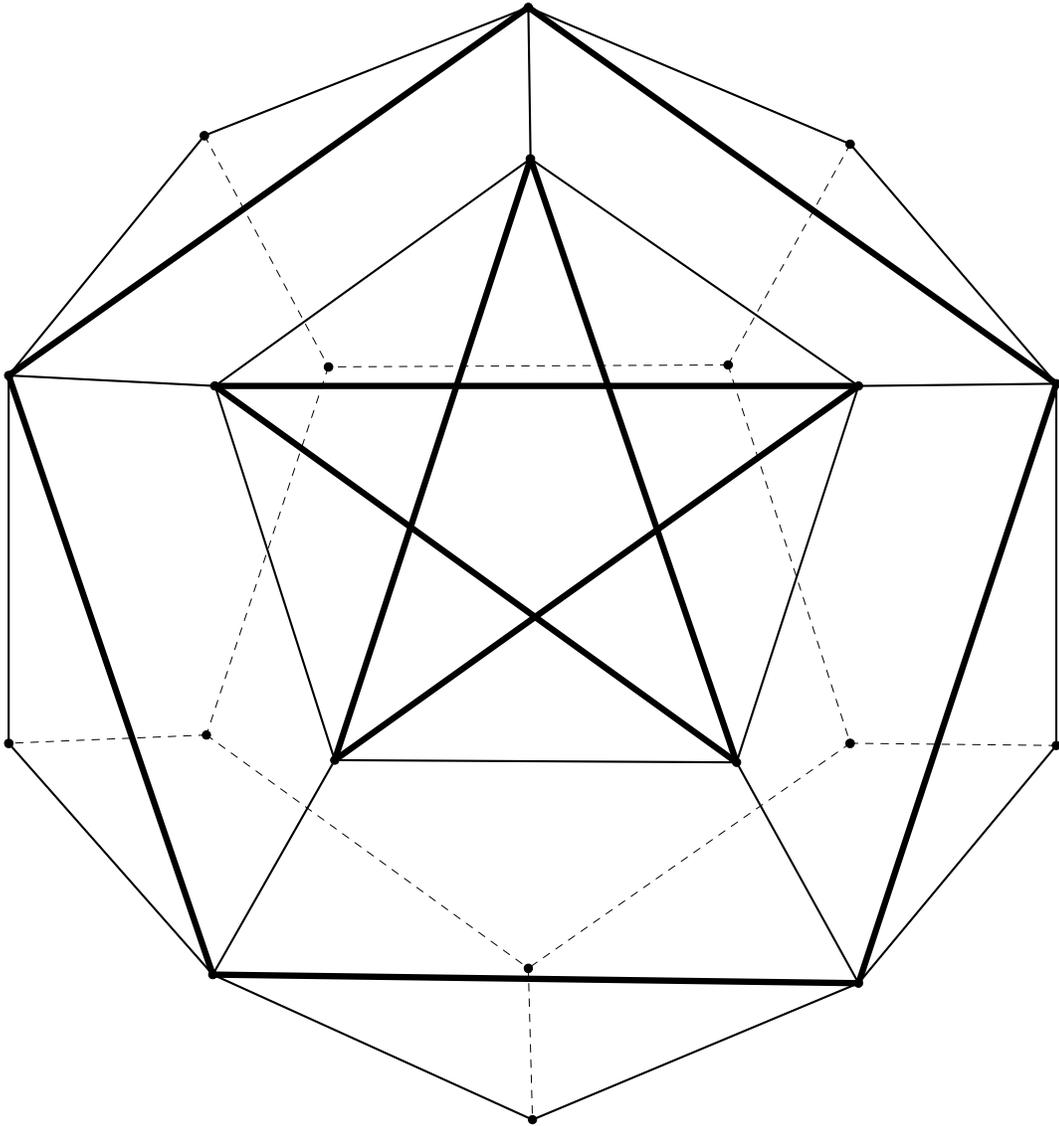
Vertices of one dodecahedron

Edge length = 2

Directed type

Shape of  $[h, h, h, h]$  &  $[f, f, f, f]$

Planar faces



Map Classification Number: R9.16

**Map of the polyhedron of type  $\{5,6\}_4$**

$(f_0, f_1, f_2) = (20, 60, 24)$

Orientable genus = 9

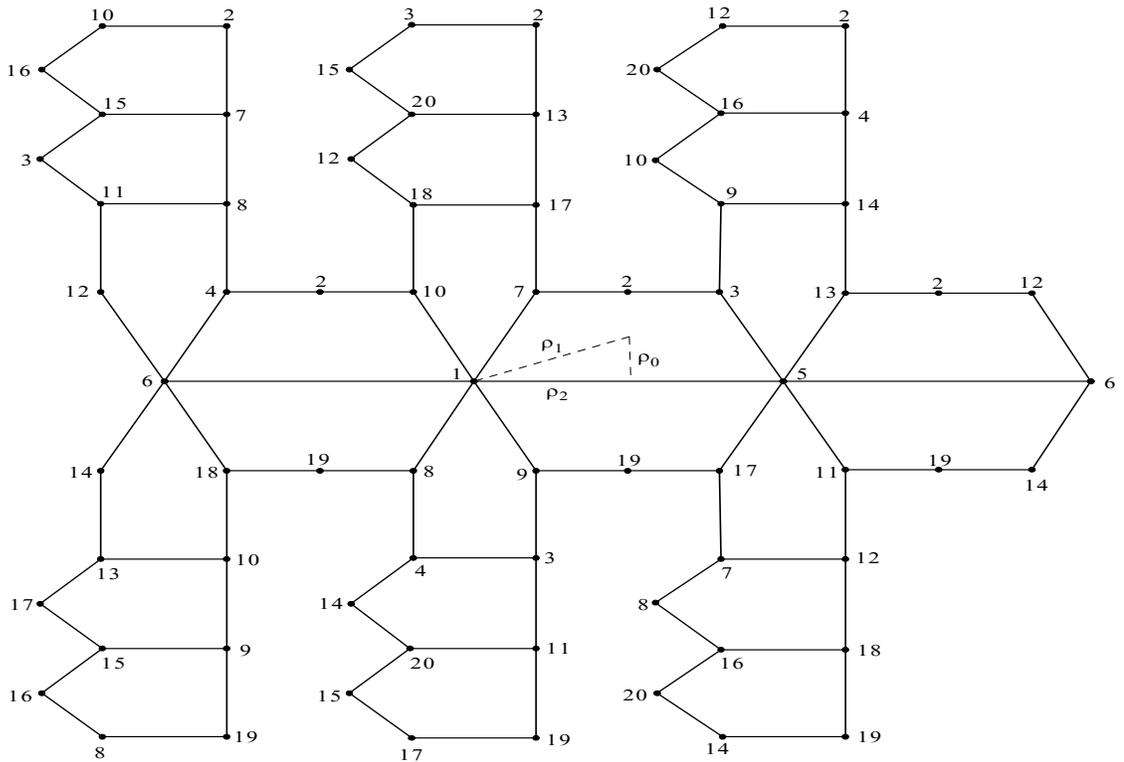
Vertices of one dodecahedron

Edge length = 2

Directed type

Shape of  $[hl,hl,hl,hl]$  &  $[f,f,f,f]$

Planar faces



$\rho_0$ : (1,5)(2)(3,7)(4,12)(6)(8,11)(9,17)(10,13)(14,18)(15)(16,20)(19)

$\rho_1$ : (1)(2,3)(4)(5,7)(6,8)(9,10)(11,12)(13,15)(14,16)(17)(18,19)(20)

$\rho_2$ : (1)(2,19)(3,17)(4,18)(5)(6)(7,9)(8,10)(11,13)(12,14)(15)(16)(20)

$\rho_0\rho_1$ : (1,5,3,2,7)(4,12,8,6,11)(9,13,15,10,17)(14,20,16,18,19)

$\rho_0\rho_2$ : (1,5)(2,19)(3,9)(4,14)(6)(7,17)(8,13)(10,11)(12,18)(15)(16,20)

$\rho_1\rho_2$ : (1)(2,18,4,19,3,17)(5,7,10,6,8,9)(11,15,13,12,16,14)(20)

$\rho_0\rho_1\rho_2$ : (1,5,3,9)(2,14,8,17)(4,19,7,13)(6,11,15,10)(12,20,16,18)

**Face of the polyhedron of type  $\{6,4\}_{10}$**

$(f_0, f_1, f_2) = (30, 60, 20)$

Orientable genus = 6

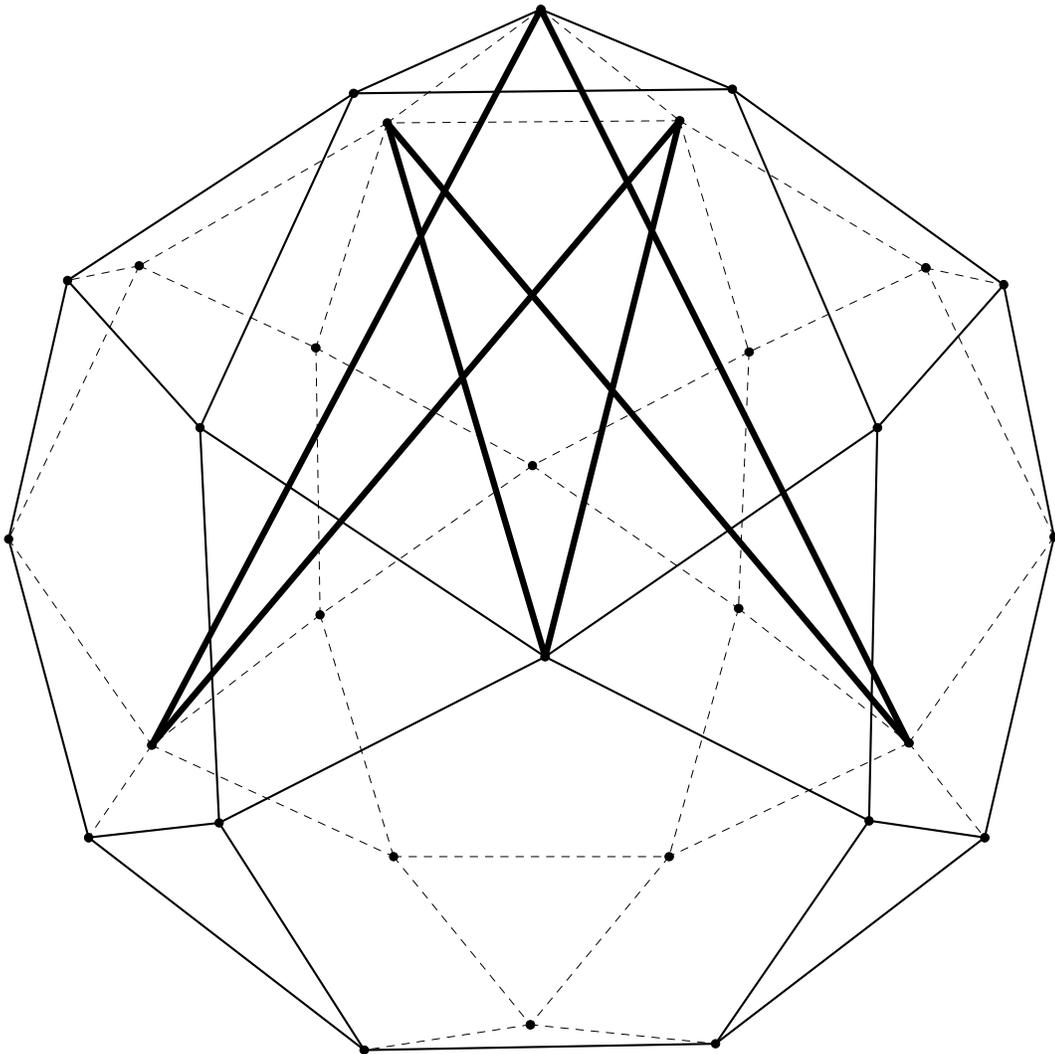
Vertices of one icosidodecahedron

Edge length =  $2d$

Bicolor type

Shape of  $[r,r,r,r]$

Non-planar faces



Map Classification Number: dual of R6.2

**Map of the polyhedron of type  $\{6,4\}_{10}$**

$(f_0, f_1, f_2) = (30, 60, 20)$

Orientable genus = 6

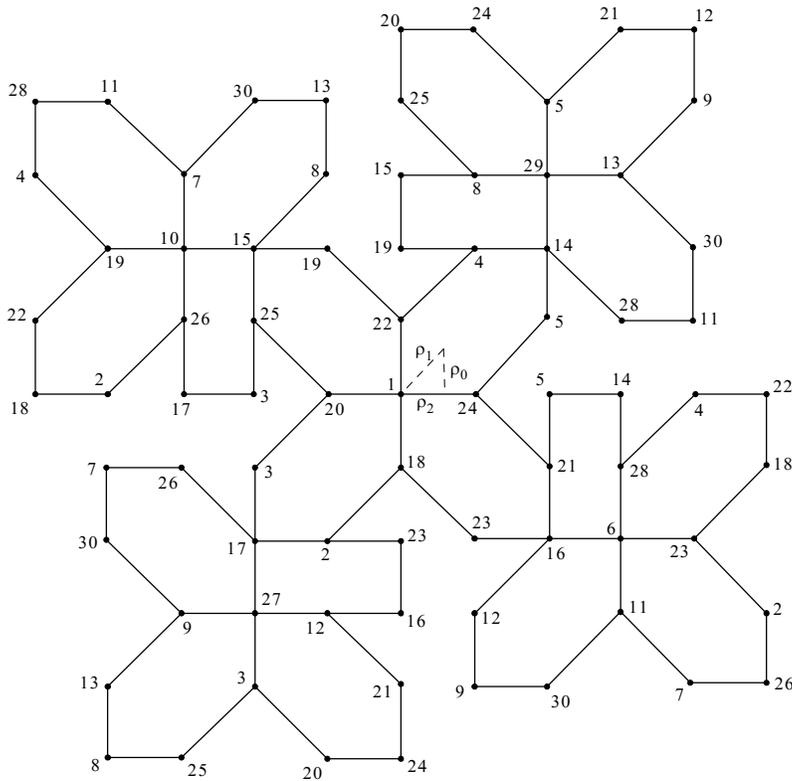
Vertices of one icosidodecahedron

Edge length =  $2d$

Bicolor type

Shape of  $[r,r,r,r]$

Non-planar faces



$p_0$ : (1,24)(2,12)(3)(4,14)(5,22)(6)(7,30)(8,15)(9,26)(10,13)(11)(16,23)(17,27)(18,21)(19,29)(20)(25)(28)

$p_1$ : (1)(2,3)(4,5)(6,8)(7,9)(10,12)(11,13)(14)(15,16)(17)(18,20)(19,21)(22,24)(23,25)(26,27)(28,29)(30)

$p_2$ : (1)(2,19)(3,25)(4,23)(5,21)(6,28)(7)(8,27)(9,13)(10,26)(11)(12,29)(14,16)(15,17)(18,22)(20)(24)(30)

$p_0p_1$ : (1,24,5,14,4,22)(2,3,12,13,11,10)(6,15,23,25,16,8)(7,26,17,27,9,30)(18,20,21,29,28,19)

$p_0p_2$ : (1,24)(2,29)(3,25)(4,16)(5,18)(6,28)(7,30)(8,17)(9,10)(11)(12,19)(13,26)(14,23)(15,27)(20)(21,22)

$p_1p_2$ : (1)(2,21,4,25)(3,23,5,19)(6,29,10,27)(7,9,11,13)(8,26,12,28)(14,15,17,16)(18,24,22,20)(30)

$p_0p_1p_2$ : (1,24,5,29,13,30,7,26,2,18)(3,16,4,25,12,28,15,27,6,19)(8,9,11,10,17,23,22,20,21,14)

**Faces of the polyhedron of type  $\{10,4\}_6$**

$(f_0, f_1, f_2) = (30, 60, 12)$

Non-orientable genus = 20

Vertices of one icosidodecahedron

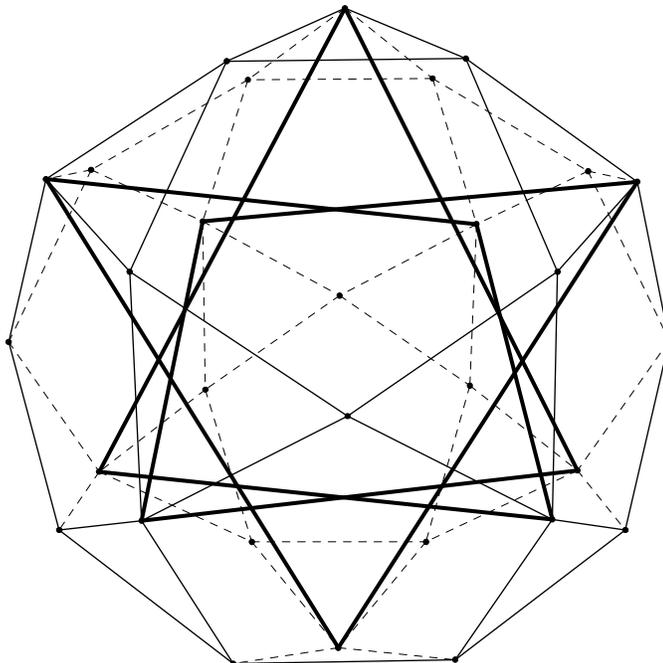
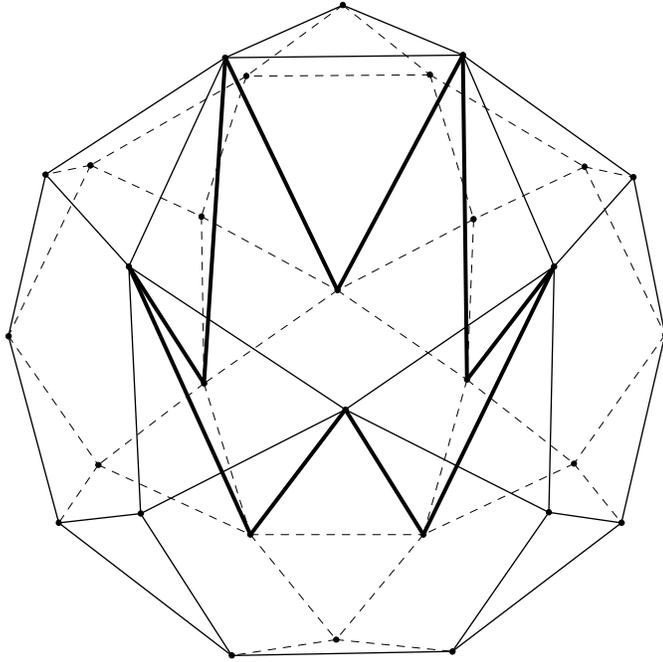
Edge length =  $2d$

Bicolor type

Shape of  $[r,1,r,1]$  &  $[1,r,1,r]$

Two face orbits under  $G(P)$

Non-planar faces



Map Classification Number: dual of N20.1

**Map of the polyhedron of type  $\{10,4\}_6$**

$(f_0, f_1, f_2) = (30, 60, 12)$

Non-orientable genus = 20

Vertices of one icosidodecahedron

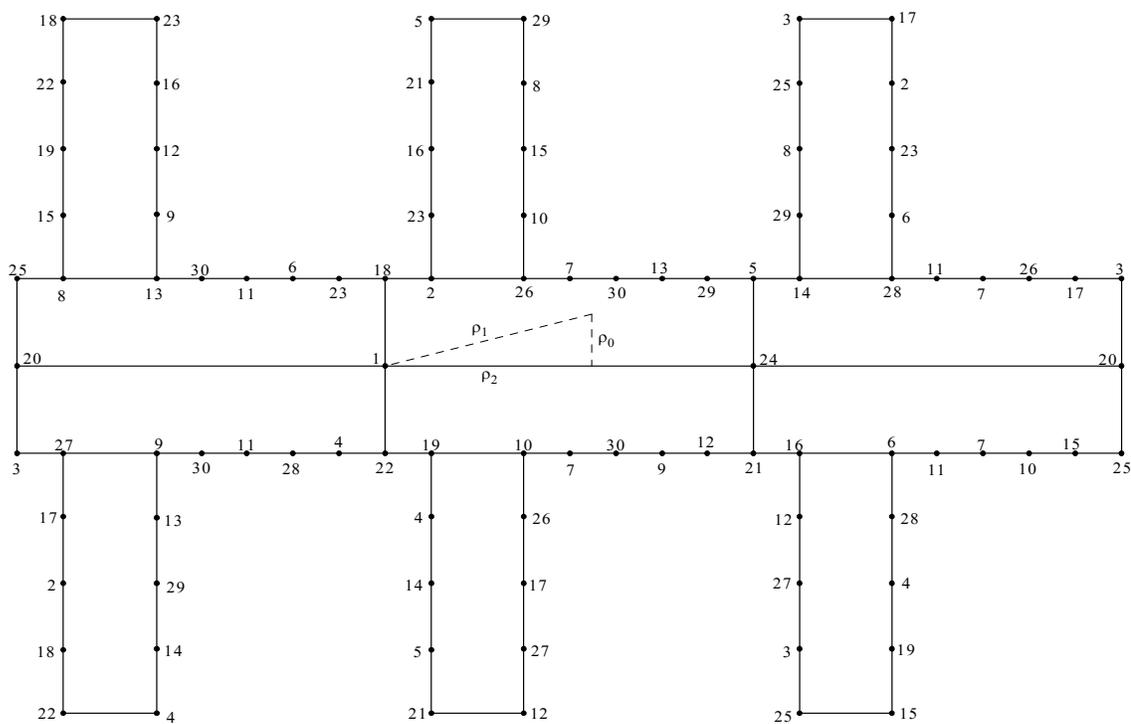
Edge length =  $2d$

Bicolor type

Shape of  $[r,l,r,l]$  &  $[l,r,l,r]$

Two face orbits under  $G(P)$

Non-planar faces



$\rho_0$ : (1,24)(2,29)(3,25)(4,16)(5,18)(6,28)(7,30)(8,17)(9,10)(11)(12,19)(13,26)(14,23)(15,27)(20)(21,22)

$\rho_1$ : (1)(2,5)(3,4)(6,12)(7,13)(8,10)(9,11)(14,17)(15)(16)(18,24)(19,25)(20,22)(21,23)(26,29)(27,28)(30)

$\rho_2$ : (1)(2,19)(3,25)(4,23)(5,21)(6,28)(7)(8,27)(9,13)(10,26)(11)(12,29)(14,16)(15,17)(18,22)(20)(24)(30)

$\rho_0\rho_1$ : (1,24,5,29,13,30,7,26,2,18)(3,16,4,25,12,28,15,27,6,19)(8,9,11,10,17,23,22,20,21,14)

$\rho_0\rho_2$ : (1,24)(2,12)(3)(4,14)(5,22)(6)(7,30)(8,15)(9,26)(10,13)(11)(16,23)(17,27)(18,21)(19,29)(20)(25)(28)

$\rho_1\rho_2$ : (1)(2,25,4,21)(3,19,5,23)(6,27,10,29)(7,13,11,9)(8,28,12,26)(14,16,17,15)(18,20,22,24)(30)

$\rho_0\rho_1\rho_2$ : (1,24,5,14,4,22)(2,3,12,13,11,10)(6,15,23,25,16,8)(7,26,17,27,9,30)(18,20,21,29,28,19)

**Face of the polyhedron of type  $\{4,6\}_{10}$**

$(f_0, f_1, f_2) = (20, 60, 30)$

Orientable genus = 6

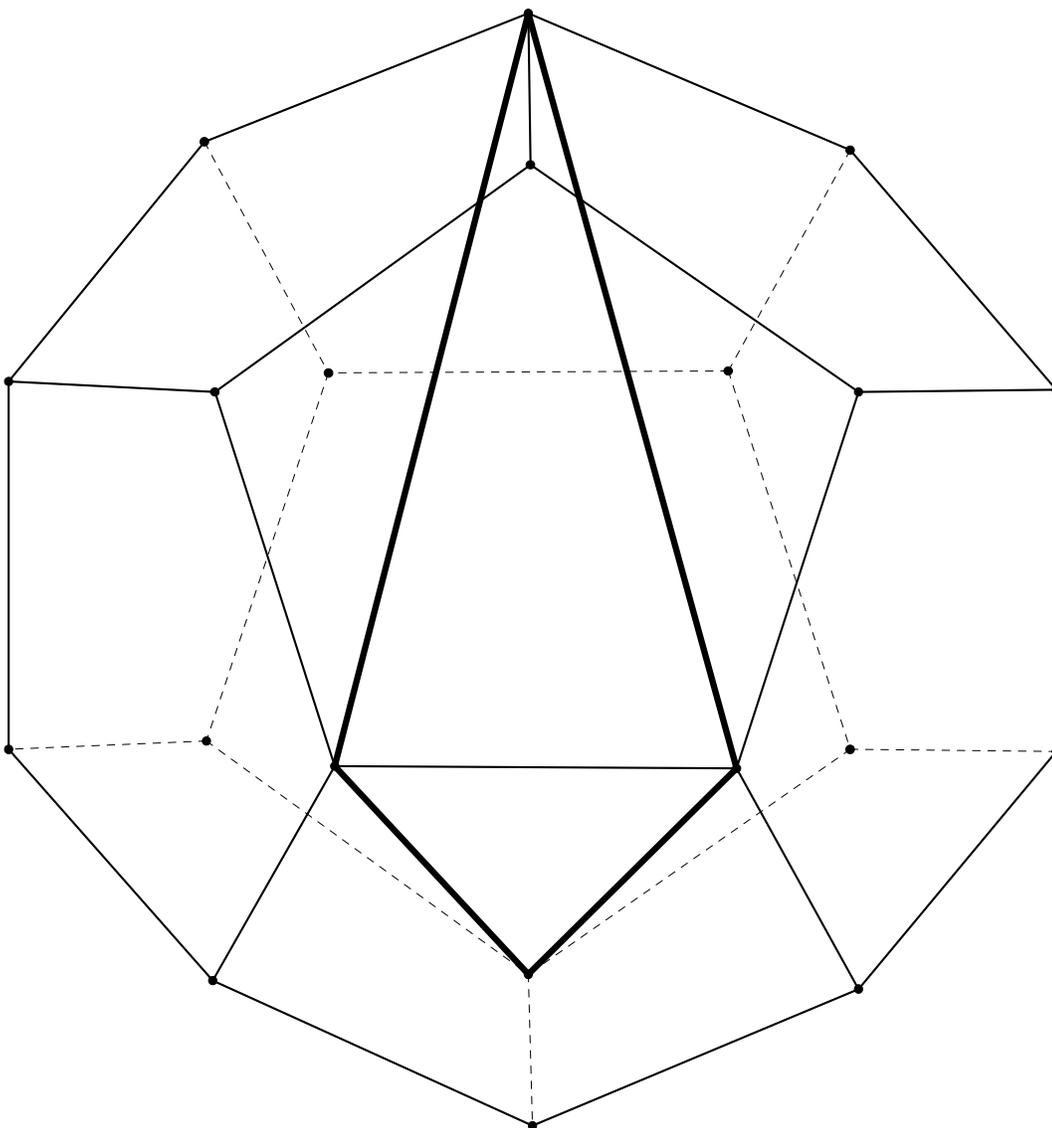
Vertices of one dodecahedron

Edge length = 3

Bicolor type

Shape of  $[h_l, f, h_l, f]$

Non-planar faces



Map Classification Number: R6.2

**Map of the polyhedron of type  $\{4,6\}_{10}$**

$(f_0, f_1, f_2) = (20, 60, 30)$

Orientable genus = 6

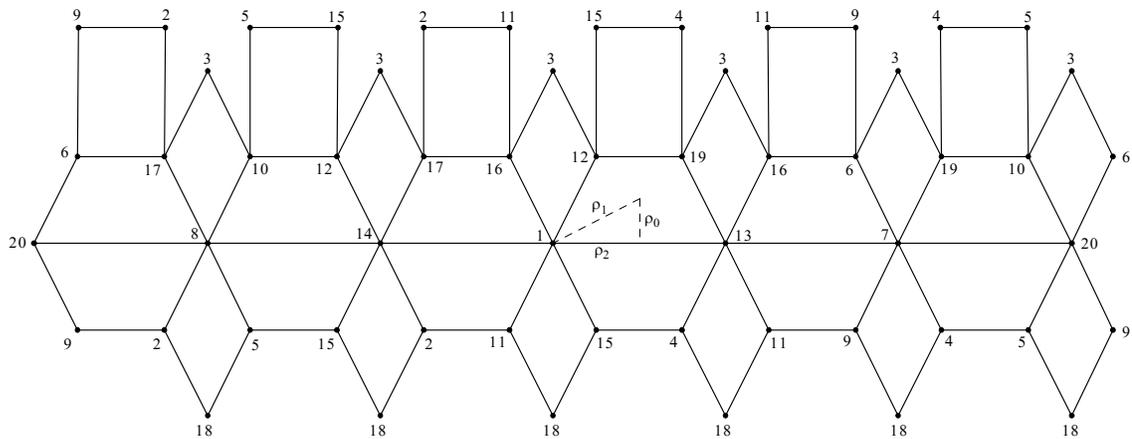
Vertices of one dodecahedron

Edge length = 3

Bicolor type

Shape of  $[hl, f, hl, f]$

Non-planar faces



$\rho_0: (1,13)(2,9)(3)(4,15)(5)(6,17)(7,14)(8,20)(10)(11)(12,19)(16)(18)$

$\rho_1: (1)(2)(3,4)(5,6)(7,10)(8,9)(11,14)(12,13)(15,16)(17,18)(19)(20)$

$\rho_2: (1)(2,17)(3,18)(4,19)(5,10)(6,9)(7)(8)(11,16)(12,15)(13)(14)(20)$

$\rho_0\rho_1: (1,13,19,12)(2,9,20,8)(3,15,16,4)(5,17,18,6)(7,10,14,11)$

$\rho_0\rho_2: (1,13)(2,6)(3,18)(4,12)(5,10)(7,14)(8,20)(9,17)(11,16)(15,19)$

$\rho_1\rho_2: (1)(2,18,4,19,3,17)(5,7,10,6,8,9)(11,15,13,12,16,14)(20)$

$\rho_0\rho_1\rho_2: (1,13,19,3,6,20,8,2,18,15)(4,12,16,7,10,17,9,5,14,11)$

**Faces of the polyhedron of type  $\{10,6\}_4$**

$(f_0, f_1, f_2) = (20, 60, 12)$

Non-orientable genus = 30

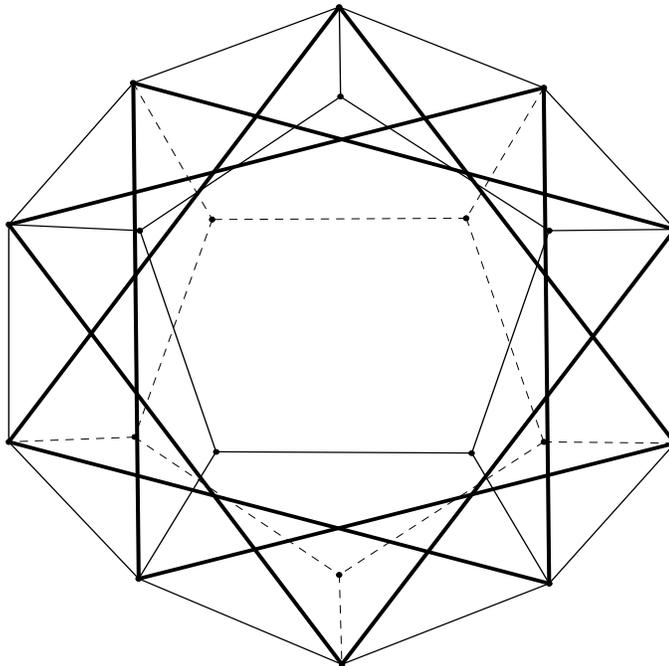
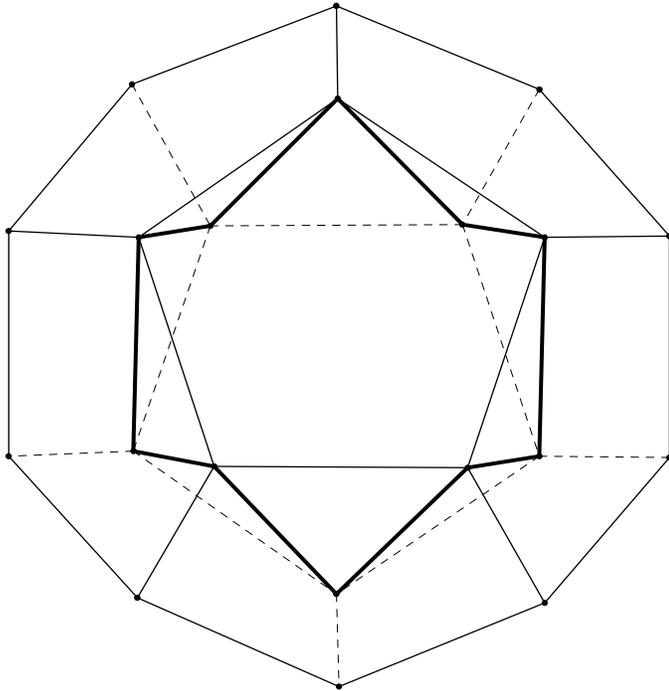
Vertices of one dodecahedron

Edge length = 3

Bicolor type

Shape of  $[h_1, h_r, h_1, h_r]$  &  $[f, f, f, f]$

Non-planar faces





# Appendix 3

This Appendix contains diagrams showing that each of the seven structures that are Petrie dual to the seven structures shown in Figure 5.1 either is not vertex faithful or else has an underlying map that is not combinatorially regular. Thus none of the seven structures Petrie dual to those in Figure 5.1 is a regular polyhedron.

The edges in Figs. 1-3 have been shrunk slightly to show the underlying structure more clearly.

None of these three figures represents the entire face, as they each have an odd number of edges, whereas any face with edges of different lengths must have an even number of edges. Thus each of the three faces that are partially shown self-intersects at a vertex, and so does not have the non-repetitive property of faces. Therefore these three structures are not vertex faithful.

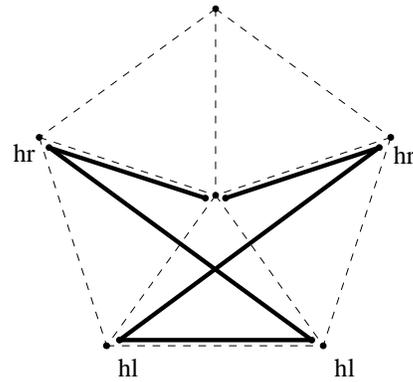


Fig. 1 Portion of face [hr,hl,hl,hr] on icosahedron.

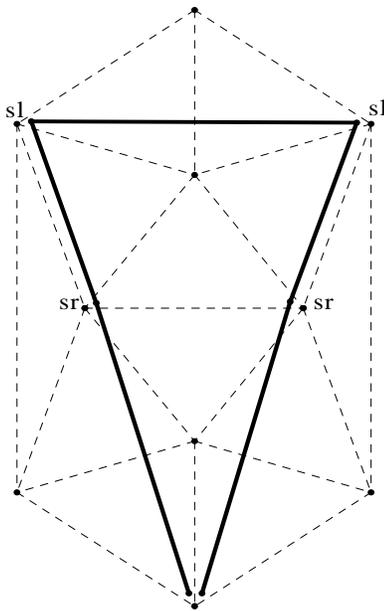


Fig. 2 Portion of face [sr,sr,sl,sl] on icosahedron

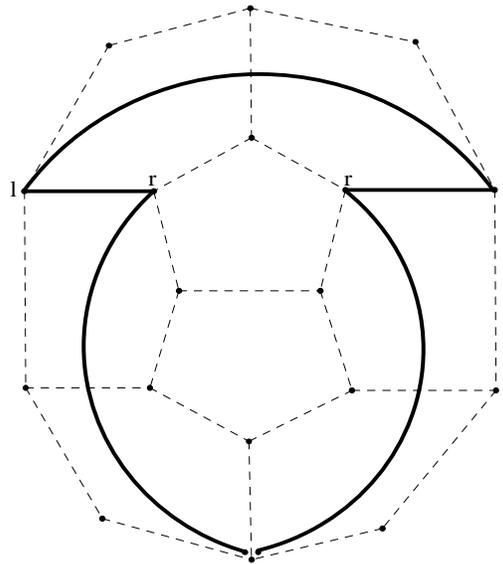


Fig. 3 Portion of face [r,r,l,l] on dodecahedron

For each of Figs. 4 - 7, we define the involution  $\rho_1$  on the flag  $(A, \{A,B\}, F_1)$ , as shown.  $F_2$  and  $F_3$  are neighboring faces of  $F_1$ , having edges  $\{A,B\}$  and  $\{A,C\}$ , respectively, in common with  $F_1$ . Thus  $\rho_1$  maps the face boundary of  $F_2$  onto the face boundary of  $F_3$ , so that  $\rho_1(A) = A$ ,  $\rho_1(B) = C$ , and  $\rho_1(D) = B$ . Being an involution, this also gives  $\rho_1(B) = D$ , so that  $\rho_1$  maps  $B$  to both  $C$  and  $D$ . Thus  $\rho_1$  is not well-defined, so that the underlying map is not combinatorially regular, and the structure can not be a regular polyhedron.

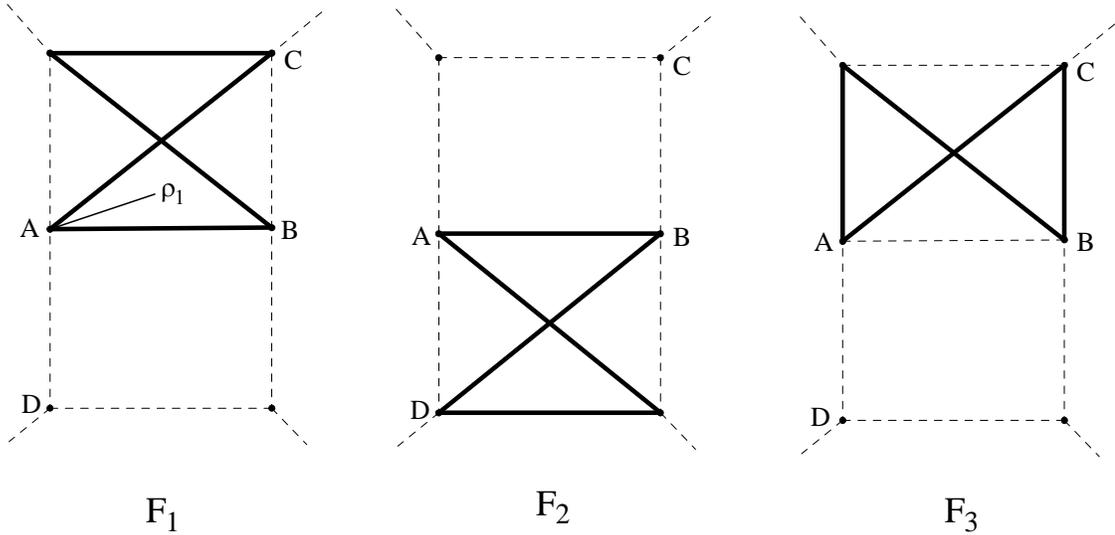


Fig. 4 Complete face boundaries of three faces of structure  $[r,l,l,r]$  on the cube.

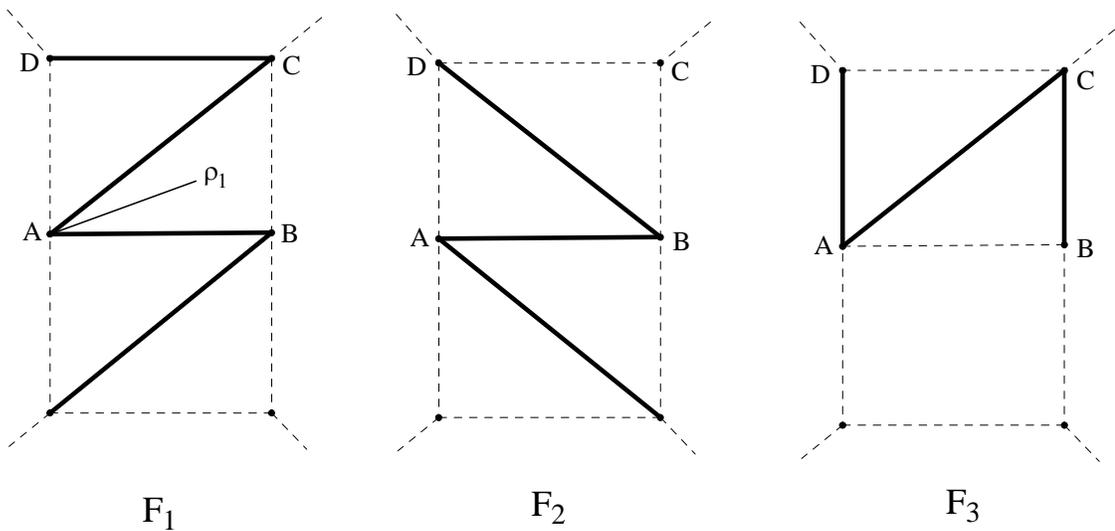


Fig. 5 Portions of three faces of structure  $[r,l,r,l]$  &  $[l,r,l,r]$  on the cube.

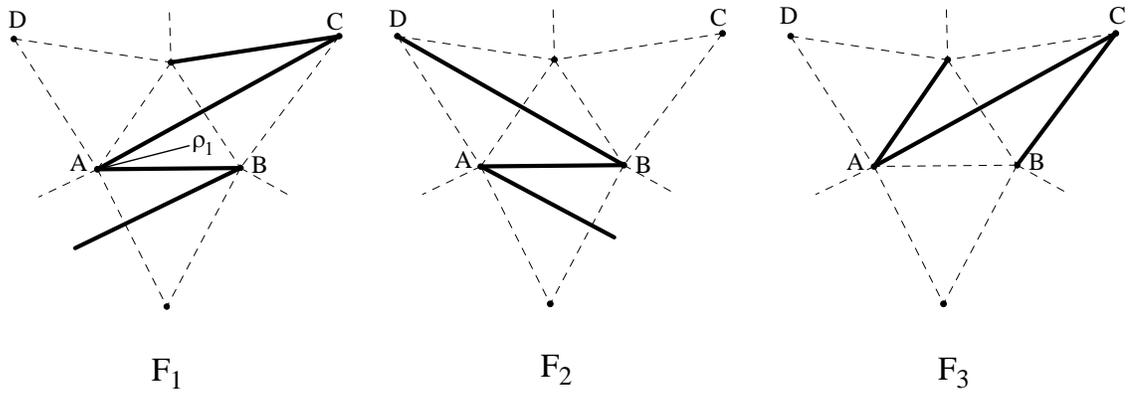


Fig. 6 Portions of three faces of structure  $[hr,h,hr,h]$  &  $[hl,hr,h,hr]$  on the icosahedron.

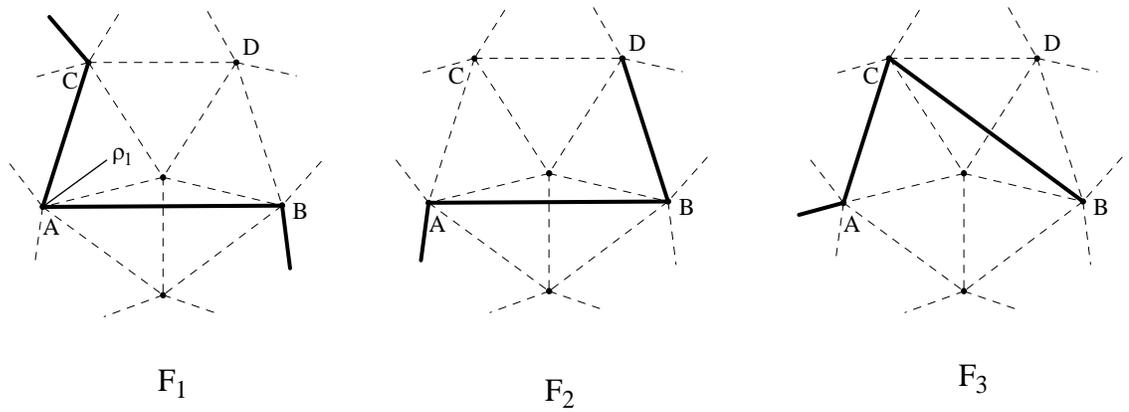


Fig. 7 Portions of three faces on structure  $[sr,s,sr,s]$  &  $[sl,sr,s,sl]$  on the icosahedron.

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