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An alternative characterization of robust stability and stability radius for linear time delay systems

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An Alternative Characterization of Robust Stability and Stability Radius for Linear Time Delay Systems

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Stability of Linear Time-Delay Systems

- Linear time delayed system:

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{x}(t-\tau),\ \tau \geq 0 \qquad (1)$$

is stable if the zeros of

$$det(sI - A - Be^{-s\tau}) = 0$$

are in $\mathscr{C}_g$, (open left half $s-$plane, good region).

Lemma 1: The system (1) is stable if there exists $R = R'$ and $0 < P_1 = P_1',\ P_2,\ P_3$, that satisfy the following LMI:

$$\begin{pmatrix} Y_{11} & Y_{12} & \tau P_2'B \\ Y_{12}' & Y_{22} & \tau_1 P_3'B \\ \tau B'P_2 & \tau B'P_3 & -\tau R \end{pmatrix} < 0,$$

where $Y_{11} = (A+B)P_2 + P_2'(A+B)$,
$Y_{12} = P_1 - P_2' + (A+B)P_3$, and $Y_{22} = -P_3 - P_3' + \tau R$.

Robust Stability

- Structured perturbation:

$$\dot{\mathbf{x}}(t) = (A + E_A\Delta_A F_A)\,\mathbf{x}(t) + (B + E_B\Delta_B F_B)\,\mathbf{x}(t-\tau) \quad (2)$$

where $\Delta_A \in \mathscr{R}^{m\times p_A},\ \Delta_B \in \mathscr{R}^{m\times p_B},\ E_A,\ E_B \in \mathscr{R}^{n\times m}$, $F_A \in \mathscr{R}^{p_A\times n}$, and $F_B \in \mathscr{R}^{p_B\times n}$.

Theorem 1: If there exist $P > 0,\ Q > 0,\ X,\ Y$ and Z and positive scalars e_1 and e_2 such that

$$\begin{pmatrix} Y_{11} & -Y + PB & \bar{\tau}A'Z & PE & PE \\ -Y' + B'P & -Q_{22} & \bar{\tau}B'Z & 0 & 0 \\ \bar{\tau}ZA & \bar{\tau}ZB & -\bar{\tau}Z & \bar{\tau}ZE & \bar{\tau}ZE \\ E'P & 0 & \bar{\tau}E'Z & -e_1 I & 0 \\ E'P & 0 & \bar{\tau}E'Z & 0 & -e_2 I \end{pmatrix} < 0,$$

$$\begin{pmatrix} X & Y \\ Y' & Z \end{pmatrix} \geq 0 \qquad (3)$$

where $Y_{11} = A'P + PA + \bar{\tau}X + Y + Y' + Q + e_1F_0'F_0$, and $Q_{22} = Q - e_2F_1'F_1$. The system (2) is asymptotically stable for any time delay τ satisfying $0 \leq \tau \leq \bar{\tau}$ and all admissible uncertainties.

Stability Radius Preliminaries

- For the regular dynamical system with perturbation

$$\dot{\mathbf{x}}(t) = (A + E\Delta F)\mathbf{x}(\mathbf{t}) \qquad (4)$$

The (structured) real stability radius is defined as

$$r_R(A,E,F) = \inf_{s\in C_0} \inf_{\Delta}\{\bar{\sigma}(\Delta) : \Delta \in \mathscr{R}^{m\times p}\ |I - \Delta F(sI-A)^{-1}E| = 0\}.$$

Since $\mu_{\mathscr{R}}(M) = (\inf_\Delta\{\bar{\sigma}(\Delta) : \Delta \in \mathscr{R}^{m\times p}\ |I - \Delta M| = 0\})^{-1}$
then, $r_R(A,E,F) = (\sup_{s\in\mathscr{C}_0} \mu_{\mathscr{R}}[F(sI-A)^{-1}E])^{-1}$

- Qui, L., Davison, E.J., Young, P.M., Doyle, J.C., and Bernhardson, B., (1995):

$$\mu_{\mathscr{R}}(M) := \inf_{\gamma\in(0,1]} \sigma_2\left[\begin{pmatrix} Re(M) & -\gamma\, Im(M) \\ \gamma^{-1}\, Im(M) & Re(M) \end{pmatrix}\right]$$

$\sigma_2(M)$ is the second largest singular values of $M \in \mathscr{C}^{p\times m}$.

Stability Radius of Single-Delay Systems

- The real stability radius of the system (2) is given by

$$r_R(A,B,E,F_A,F_B) = \inf_{s\in\mathscr{C}_0} \inf_{\Delta}\{\bar{\sigma}(\Delta) : \Delta \in \mathscr{R}^{m\times(p_A+p_B)},\ |I - \Delta M(s)| = 0\}$$

where $\Delta = [\Delta_A \quad \Delta_B]$,

$$M(s) = \begin{pmatrix} M_1(s) \\ M_2(s) \end{pmatrix} = \begin{pmatrix} F_A \\ F_B e^{-\tau s} \end{pmatrix} (sI - A - Be^{-\tau s})^{-1}E. \quad (5)$$

Using Rekasius Transformation:
$e^{-s\tau_i} = \frac{1-sT_i}{1+sT_i},\ \ \tau_i \in \mathscr{R}^+,\ T_i \in \mathscr{R},\ \ i = 1, 2.$
Exact for: $s = jw \Rightarrow \tau_i = \frac{2}{w}[\tan^{-1}(wT_i) + l\pi]$,
$l = 0, 1, 2, \ldots$. (5) becomes $M(w,\tau) = \begin{pmatrix} M_1(w,\tau) \\ M_2(w,\tau) \end{pmatrix}$, with

$$M_1(w,\tau) = [1 + j\tan(\tfrac{\tau w}{2} - l\pi)]F_A \times \Big[[jwI - (A+B)] + j\tan(\tfrac{\tau w}{2} - l\pi)[jwI - (A-B)]\Big]^{-1}E,$$

$$M_2(w,\tau) = [1 - \tan(\tfrac{\tau w}{2} - l\pi)]F_B \times \Big[[jwI - (A+B)] + j\tan(\tfrac{\tau w}{2} - l\pi)[jwI - (A-B)]\Big]^{-1}E, \quad 0 \leq \tau \leq \bar{\tau}.$$

Stability Radius of Metzlerian Systems

- Shafai,B. and Chen, J., (1993):
For a special class of nonnegative and metzlerian system

$$\dot{\mathbf{x}}(t) = (A + E\Delta F)\mathbf{x}(t) \qquad (6)$$
$$E \geq 0,\ F \geq 0, \text{ and } A \geq 0,\ A \text{ stable}, \Delta \in \mathscr{R}^{m\times p}$$

Lemma 2: The real and complex stability radii coincide

$$r_{\mathscr{R}}(A,E,F) = r_{\mathscr{C}}(A,E,F) = \frac{1}{\|\ FA^{-1}E\ \|_2}.$$

Metzlerian Delay System

- Consider the class of positive delay dynamic systems

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{x}(t-\tau) \qquad (7)$$

where A is metzlerian, B is nonnegative or metzlerian such that $A + B$ remains Metzlerian.
Theorem 2: The real and complex stability radii for the following uncertain Metzlerian delay system coincide

$$\dot{\mathbf{x}}(t) = (A + E\Delta F)\mathbf{x}(t) + B\mathbf{x}(t-\tau)$$

$$r_{\mathscr{C}}(A,B,E,F) = r_{\mathscr{R}}(A,B,E,F) = \frac{1}{\|\ F(A+B)^{-1}E\ \|}.$$

Example 1:
$\dot{\mathbf{x}}(t) = (A + E\Delta F)\,\mathbf{x}(t) + B\,\mathbf{x}(t-\tau)$,

$$A = \begin{pmatrix} -22.3578 & 1.6015 & 0.0138 & 2.3068 \\ 1.0000 & -8.0777 & 1.7000 & 4.0013 \\ 2.4217 & 3.6563 & -9.9694 & 2.0000 \\ 13.2160 & 3.5504 & 0.0519 & -11.6325 \end{pmatrix},$$

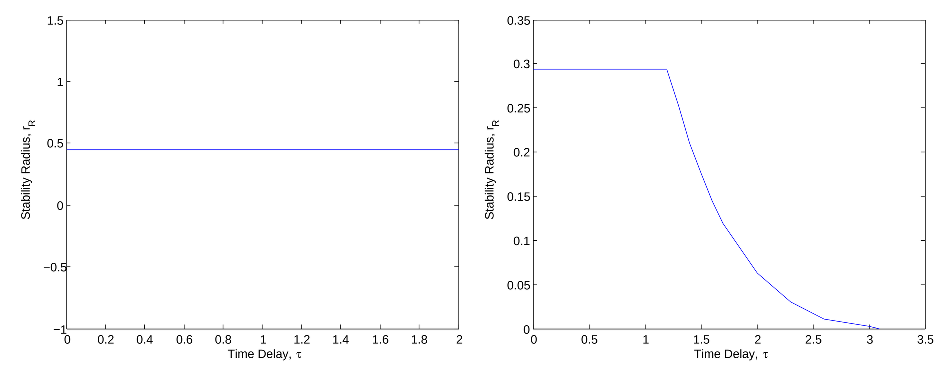
$$B = \begin{pmatrix} -1.0000 & 1.0000 & 0.5000 & 2.0000 \\ 0.0000 & -1.0000 & 1.0323 & 1.7200 \\ 2.4000 & 2.2000 & -1.0000 & 0.5648 \\ 1.2000 & 1.0000 & 3.0000 & -1.0000 \end{pmatrix}, \quad E = F = I.$$

$$r_R(A,E,F) = \frac{1}{\|\ F(A+B)^{-1}E\ \|_2} = 0.4488.$$

Example 2:
$\dot{\mathbf{x}}(t) = (A + E_A\Delta_A F_A)\,\mathbf{x}(t) + (B + E_B\Delta_B F_B)\,\mathbf{x}(t-\tau)$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\ E's = F's = I_2.$$

- Applying the LMI, the maximum time delay, $\bar{\tau} = \pi$.
- The real stability radius with respect to τ, $0 \leq \tau \leq \pi$.



Example 1 Example 2

Stability of Linear Two-Delay Systems

- Consider the two-delay retarded type linear system

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + \sum_{i=1}^{2} A_i\mathbf{x}(t-\tau_i),\ 0 \leq \tau_i \leq \bar{\tau}_i \qquad (8)$$

The system (8) is said to be stable if the zeros of

$$det(sI - A_0 - \sum_{i=1}^{2} A_i e^{-s\tau_i}) = 0 \text{ are in } \mathscr{C}_g.$$

- Structure perturbation:

$$\dot{\mathbf{x}}(t) = (A_0 + E_0\Delta_0F_0)\mathbf{x}(t) + \sum_{i=1}^{2}(A_i + E_i\Delta_iF_i)\mathbf{x}(t-\tau_i) \quad (9)$$

$\Delta_i \in \mathscr{R}^{m\times p_i},\ E_i \in \mathscr{R}^{n\times m}$, and $F_i \in \mathscr{R}^{p_i\times n},\ i = 1, 2$

The charactrestic equation: WLOG $E_i = E$

$$\det\left(I - \sum_{k=0}^{2}\Delta_iF_i(sI - \sum_{i=1}^{2}A_ie^{s\tau_i})^{-1}Ee^{s\tau_k}\right) = 0, \quad \tau_0 = 0$$

Stability Radius of Two-Delay Systems

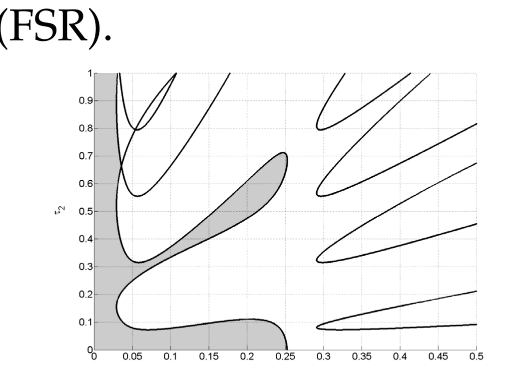
- The real stability radius of the system (9)

$$r_R(A_0,A_1,A_2,E,F_0,F_1,F_2) = \inf_{s\in\mathscr{C}_0}\inf_{\Delta}\{\bar{\sigma}(\Delta) : \Delta \in \mathscr{R}^{m\times(\sum_{k=0}^{2}p_k)},\ |I - \Delta M(s)| = 0\},$$

$\Delta = [\Delta_0\ \Delta_1\ \Delta_2]$,

$$M(s) = \begin{pmatrix} F_0 \\ F_1e^{-\tau_1 s} \\ F_2e^{-\tau_2 s} \end{pmatrix}(sI - A_0 - \sum_{i=1}^{2}A_ie^{-\tau_i s})^{-1}E$$

Definition 1: Fundamental Stability Region
Let the two time-delay system be zero delay stable. Then a subset of kernel and offspring curves such that any point on these curves can be connected to the origin by a continuous path in the stable region is called fundamental kernel and offspring curves. Furthermore, the region of stability bounded by τ_1, τ_2 axes and the fundamental kernel and offspring curves is called the fundamental stability region (FSR).



Fundamental Stability Region

Theorem 3: Let the two time delay system be zero delay robustly stable and let the box
$B = \{(\tau_1,\tau_2) : 0 \leq \tau_1 \leq \bar{\tau}_1,\ 0 \leq \tau_2 \leq \bar{\tau}_2\}$ be a subset of the FSR defined for (8). Then (9) is robustly stable if and only if the following two LMI's corresponding to two separate single delay systems are satisfied:
System 1 defined when $\tau_2 = 0$:

$$\dot{\mathbf{x}}(t) = (A_{02} + E\Delta_{02}F_{02})\mathbf{x}(t) + (A_1 + E\Delta_1F_1)\mathbf{x}(t-\tau_1),$$

where $A_{02} = A_0 + A_2,\ \Delta_{02} = [\Delta_0\ \Delta_2]$, and $F_{02} = \begin{pmatrix} F_0 \\ F_2 \end{pmatrix}$,
i.e., if there exist $P > 0,\ Q > 0,\ X,\ Y$ and Z and positive scalars e_1 and e_2 such that

$$\begin{pmatrix} Y_{11} & -Y + PA_1 & \bar{\tau}_1A_{02}'Z & PE & PE \\ -Y' + A_1'P & -Q_{22} & \bar{\tau}_1A_1'Z & 0 & 0 \\ \bar{\tau}_1ZA_{02} & \bar{\tau}_1ZA_1 & -\bar{\tau}_1Z & \bar{\tau}_1ZE & \bar{\tau}_1ZE \\ E'P & 0 & \bar{\tau}_1E'Z & -e_1I & 0 \\ E'P & 0 & \bar{\tau}_1E'Z & 0 & -e_2I \end{pmatrix} < 0,$$

$$\begin{pmatrix} X & Y \\ Y' & Z \end{pmatrix} \geq 0,$$

where $Y_{11} = A_{02}'P + PA_{02} + \bar{\tau}_1X + Y + Y' + Q + e_1F_0'F_0$, and $Q_{22} = Q - e_2F_1'F_1$.
System 2 defined when $\tau_1 = 0$:

$$\dot{\mathbf{x}}(t) = (A_{01} + E\Delta_{01}F_{01})\mathbf{x}(t) + (A_2 + E\Delta_2F_2)\mathbf{x}(t-\tau_2),$$

where $A_{01} = A_0 + A_1,\ \Delta_{01} = [\Delta_0\ \Delta_1]$, and $F_{01} = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$,
i.e., if there exist $P > 0,\ Q > 0,\ X,\ Y$ and Z and positive scalars e_1 and e_2 such that

$$\begin{pmatrix} Y_{11} & -Y + PA_2 & \bar{\tau}_2A_{01}'Z & PE & PE \\ -Y' + A_2'P & -Q_{22} & \bar{\tau}_2A_2'Z & 0 & 0 \\ \bar{\tau}_2ZA_{01} & \bar{\tau}_2ZA_2 & -\bar{\tau}_2Z & \bar{\tau}_2ZE & \bar{\tau}_2ZE \\ E'P & 0 & \bar{\tau}_2E'Z & -e_1I & 0 \\ E'P & 0 & \bar{\tau}_2E'Z & 0 & -e_2I \end{pmatrix} < 0,$$

$$\begin{pmatrix} X & Y \\ Y' & Z \end{pmatrix} \geq 0,$$

where $Y_{11} = A_{01}'P + PA_{01} + \bar{\tau}_2X + Y + Y' + Q + e_1F_0'F_0$, and $Q_{22} = Q - e_2F_2'F_2$.

Example 3:
$\dot{\mathbf{x}}(t) = (A_0 + E\Delta_0F_0)\mathbf{x}(t) + \sum_{i=1}^{2}(A_i + E\Delta_iF_i)\mathbf{x}(t-\tau_i)$,
$0 \leq \tau_1 \leq 0.75,\ \ 0 \leq \tau_2 \leq 3,$ where
$E = F_i = I_2, A_0 = A_2 = \begin{pmatrix} 0 & 0.5 \\ -0.5 & -0.5 \end{pmatrix}$, and $A_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

- System without uncertainties is stable when $\tau_1 = \tau_2 = 0$ and unstable when $\tau_1 > \pi$ and $\tau_2 > 5.71$ along τ_2 and τ_1 axis, respectively.
- The box $B_1 = \{(\tau_1,\tau_2) :\ 0 \leq \tau_1 \leq 0.75,\ 0 \leq \tau_2 \leq 3\}$ is entirely in the FSR
- The two LMI's are satisfied for $\bar{\tau}_1 = 0.75$, and $\bar{\tau}_2 = 3$.
- The largest possible box within the FSR is $B_2 = \{(\tau_1,\tau_2) :\ 0 \leq \tau_1 \leq 1.15,\ 0 \leq \tau_2 \leq 5.71\}$